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COMPARING FUNCTORS CLASSIFYING EXTENSIONS OF C^*-ALGEBRAS

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1. INTRODUCTION AND STATEMENT OF RESULTS

Suppose that A and B are separable C^*-algebras. We are concerned with the classification of extensions of C^*-algebras of the form

$$(1.1)\qquad 0 \to B \otimes \mathscr{K} \to E \to A \to 0,$$

where $\mathscr{K} = \mathscr{K}(\mathscr{H})$ is the C^*-algebra of compact operators on a complex separable infinite-dimensional Hilbert space $\mathscr{H}$. Of special interest is the case where $B = C_0(Y)$, the continuous functions vanishing at infinity on a (second countable) locally compact space Y. When $B = \mathbf{C}$, the complex numbers (equivalently, $Y = \mathrm{pt}$, the one-point space), then (1.1) reduces to

$$(1.2)\qquad 0 \to \mathscr{K} \to E \to A \to 0.$$

"Essential" extensions of the form (1.2), modulo a suitable equivalence relation, form a group $\mathrm{Ext}(A)$, provided that A is nuclear, by results of Brown, Douglas, and Fillmore [6] (abbreviated hereafter BDF), Choi-Effros [8], Voiculescu [21], and Arveson [2]. This group has been calculated in terms of topological K-theory when A is commutative by Kahn, Kaminker, and Schochet [13], generalizing results of BDF.

When B is non-trivial, there is a great variety of complicated classification problems associated with extensions (1.1) (see [7], [9], and [10] for an idea of the complexity of the subject). Nevertheless, there are two approaches, which we now describe, which identify extensions (1.1) with elements of commutative semigroups or groups having nice functorial properties. Both approaches begin with the observation of Busby [7, Theorem 4.3] that every diagram (1.1) defines canonically a commutative diagram

$$(1.3)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & B \otimes \mathscr{K} & \longrightarrow & E & \xrightarrow{\tilde{\pi}} & A & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & B \otimes \mathscr{K} & \longrightarrow & \mathscr{M}(B) & \xrightarrow{\pi} & Q(B) & \longrightarrow & 0 \end{array}$$

where $\mathcal{M}(B)$ denotes the algebra $M(B \otimes \mathcal{K})$ of two-sided multipliers of $B \otimes \mathcal{K}$, $Q(B) = \mathcal{M}(B)/(B \otimes \mathcal{K})$, and the map $\tilde{\pi}$ is the "pull-back" of τ and the quotient map π. When $B = C_0(Y)$, we write $\mathcal{M}(Y)$ and $Q(Y)$ for $\mathcal{M}(B)$ and $Q(B)$; by [1, Theorem 3.3 and its Corollaries], $\mathcal{M}(Y)$ may be identified with $C^{\mathrm{b}}_{\mathrm{s}*}(Y, \mathcal{B})$, the algebra of norm-bounded, strong-$*$ continuous functions from Y to $\mathcal{B} = \mathcal{B}(\mathcal{H})$, the bounded operators on $\mathcal{H}$. Thus when $Y = \mathrm{pt}$, $\mathcal{M}(Y) = \mathcal{B}$ and $Q(Y) = \mathcal{B}/\mathcal{K} = Q$, the Calkin algebra.

The $*$-homomorphism $\tau: A \to Q(B)$ is called the *Busby invariant* of (1.1), and determines (1.1) up to a very strong notion of equivalence [7, Definition 4.1]. Therefore one often identifies an extension with its Busby invariant. However, it is most convenient to work with a much weaker equivalence relation on extensions. Therefore, if U is a unitary in $\mathcal{M}(B)$ with

$$(\pi U)\tau(\pi U)^* = \tau',$$

we say τ' is *(unitarily) equivalent* to τ. The set of equivalence classes of extensions is given a natural associative, commutative addition operator defined as follows: given $\tau_1, \tau_2: A \to Q(B)$, we define $\tau_1 \oplus \tau_2$ to be the equivalence class of the composite

$$A \xrightarrow{(\tau_1, \tau_2)} Q(B) \oplus Q(B) \hookrightarrow Q(B) \otimes M_2 \xrightarrow{\cong} Q(B),$$

where M_n denotes the n by n complex matrices and we identify $\mathcal{H} \oplus \mathcal{H}$ with $\mathcal{H}$ via any unitary. (This definition is due to Kasparov [14] but agrees with that previously given by BDF when $B = \mathbf{C}$.) An extension $\tau: A \to Q(B)$ *splits* if there is a $*$-homomorphism $\sigma: A \to \mathcal{M}(B)$ with $\tau = \pi \circ \sigma$. (The other reasonable definitions of a split extension coincide with this one, see [7, Proposition 5.3].) To this point the two approaches agree.

Pimsner, Popa, and Voiculescu [16], [17] (hereafter abbreviated PPV) continue as follows. First, they insist that $B = C(Y)$, where Y is a finite-dimensional compact metric space (or equivalently, Y is a closed subset of S^n for some n). Second, they require that A be unital and that $\tau: A \to Q(Y)$ be unital (so that E is also unital and $\tilde{\pi}: E \to A$ is unital). Third, they insist that all extensions be homogeneous. (For any $y \in Y$, the evaluation map $p_y: \mathcal{M}(Y) \to \mathcal{B}$ induces $p_y: Q(Y) \to Q$. An extension $\tau: A \to Q(Y)$ is *homogeneous* if $p_y \circ \tau$ is injective for all $y \in Y$.) Under these restrictions, unitary equivalence classes of homogeneous extensions form a semigroup with unit, denoted $\mathrm{Ext}(Y; A)$. If one assumes in addition that A is nuclear, then essentially the same argument as for the BDF theory shows that $\mathrm{Ext}(Y; A)$ is a group. If $y_0 \in Y$ is a basepoint, then

$$\mathrm{Ext}(Y, y_0; A) = \{[\tau] \in \mathrm{Ext}(Y; A): p_{y_0} \circ \tau \text{ is split}\}$$

is a natural subgroup.

Kasparov [14], on the other hand, puts no restrictions on the algebra B. He does not insist that A be unital, nor does he insist that extensions be "essential" or "homogeneous" in any way (τ need not be injective). Instead, he takes the semigroup of unitary equivalence classes of extensions, divides out the unitary equivalence classes of split extensions, and studies the resulting semigroup with unit. He denotes it $\mathrm{Ext}(A, B)$, but for purposes of comparison with the PPV groups we shall denote it by $\mathrm{Kas}(B; A)$ (or $\mathrm{Kas}(Y; A)$ when $B = C_0(Y)$). As in the PPV theory (again because of the Choi-Effros lifting theorem), $\mathrm{Kas}(B; A)$ is a group if A is nuclear. Kasparov also defines other functors KK^* by quite a different construction; his main result is that $\mathrm{Kas}(B; A) \cong KK^1(A, B)$ for nuclear A.

To emphasize the difference between the Kasparov and PPV approaches, note that for PPV, if $[\tau_1] = [\tau_2]$ and τ_1 splits then τ_2 splits. For Kasparov, however, if $[\tau_1] = [\tau_2]$ and τ_1 splits then τ_2 need not split. The most that can be said is that there exists a split extension τ_3 such that $\tau_2 \oplus \tau_3$ splits. In analogy with vector bundles, we say τ_2 is *stably split*. (For a precise justification of the analogy and the terminology, see [18, §5].) In Section 2 we consider some examples of this phenomenon.

The PPV groups have been calculated when A is commutative by Schochet [19]. The Kasparov groups are not calculated explicitly in [14] unless $A = C_0(X)$, $B = C_0(Y)$, and the one-point compactifications of Y and X are of the homotopy type of finite CW-complexes, although presumably they could be obtained from the KK^* functors in certain other cases. Since many cases of practical interest involve commutative A and B with spectra whose one-point compactifications are *not* finite complexes, we have been led to try to compute the Kasparov groups in some new cases. At the same time, we were tempted to determine the precise relationship between the Kasparov and PPV functors.

We now state our principal results. In Section 2 these are applied to study the C^*-algebra of the Heisenberg group and related matters. In Section 3 the proofs of the main theorems are given. In Section 4 we examine the relationship between the Kasparov and PPV functors and certain other invariants using the more familiar K-groups of a Banach algebra, and discuss the perhaps unexpected effectiveness of K-theory techniques in proving non-splitting theorems about naturally occurring extensions.

If A is a C^*-algebra, A^+ denotes the algebra obtained by adjoining an identity to A. If Y is a locally compact space, Y^+ denotes its one-point compactification and $+$ denotes the point adjoined at infinity. These notations are consistent in that $C(Y^+) \cong C_0(Y)^+$.

THEOREM 1.4. *Let Y be a finite-dimensional compact metric space and let A be a separable nuclear C^*-algebra (not necessarily unital). Then there is a natural isomorphism of groups*

$$\mathrm{Ext}(Y; A^+) \cong \mathrm{Kas}(Y; A).$$

Note that if A is unital then $A^+ = A \oplus \mathbf{C} \neq A$, and (usually) $\mathrm{Ext}(Y; A^+) \neq \neq \mathrm{Ext}(Y; A)$. The difference stems from the fact that PPV insist upon equivalence of extensions via unitaries from $\mathcal{M}(Y)$ (strong equivalence, in the BDF terminology). If $Y = \mathrm{pt}$ then $\mathrm{Ext}(\mathrm{pt}; A)$ is the strong BDF group, whereas $\mathrm{Kas}(\mathrm{pt}; A)$ is the weak BDF group. In the Kasparov theory one can always add a non-unital split extension to any unital extension so that one may assume *ab initio* that all maps are non-unital.

Theorem 1.4 was established by Schochet [19] in the case that A is commutative, and hence is perhaps not a surprising result. What is more interesting is the following theorem relating the Kasparov groups for non-compact Y, which classify primarily "singular" extensions (in the sense of [22]), to the PPV groups for the one-point compactification.

THEOREM 1.5. *Let Y be a locally compact subset of $\mathbb{R}^n$ and let A be a separable nuclear C^*-algebra (not necessarily unital). Then there is a natural isomorphism of groups*

$$\mathrm{Ext}(Y^+, +; A^+) \cong \mathrm{Kas}(Y; A).$$

Note that Y is a locally compact subset of $\mathbb{R}^n$ for some n if and only if Y^+ is a finite-dimensional compact metric space [12], so that the PPV group is defined.

COROLLARY 1.6. *Let X and Y be locally compact subsets of $\mathbb{R}^n$. Then*

$$\mathrm{Kas}(Y; C_0(X)) \cong \tilde{\mathrm{K}}^0(Y^+ \wedge F(X^+))$$

where $F(X^+)$ is the function spectrum of X^+ and $\tilde{\mathrm{K}}^0(-)$ is (reduced) representable topological K*-theory* [13].

Proof. Combine 1.5 and [19]. ▨

Corollary 1.6 implies most of Theorem 5 of Kasparov [14]. Theorem 1.5 allows us to show that $\mathrm{Kas}(Y; A)$ is periodic of period two in each variable, satisfies appropriate homotopy invariance and exactness properties, etc., without appealing to the Kasparov KK^* machine. There is, however, a great deal more in Kasparov's announcement which the PPV work does not imply.

Corollary 1.6 and the standard identification of $F(S^n)$ with the sphere spectrum yields the following corollary.

COROLLARY 1.7. *Let Y be a locally compact subset of Euclidean space. Then*

$$\mathrm{Kas}(Y; C_0(\mathbf{R}^n)) \cong \begin{cases} \mathrm{K}^{-1}(Y^+) & \text{if } n \text{ is even} \\ \tilde{\mathrm{K}}^0(Y^+) & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 1.7 contains many cases of interest, as will be shown in Sections 2 and 4.

2. THE HEISENBERG GROUP EXTENSION

Let G_3 be the Heisenberg group of real dimension 3 and let $C^*(G_3)$ be its group C^*-algebra. Then $C^*(G_3)$ can be written in a canonical manner as an extension of C^*-algebras

$$0 \to C_0(\mathbf{R}-\{0\}) \otimes \mathscr{K} \to C^*(G_3) \to C_0(\mathbf{R}^2) \to 0\,. \tag{2.1}$$

In answer to a problem posed by J.M.G. Fell a number of years ago, Voiculescu [22] and Kasparov [14] have both shown that the extension (2.1) is not a split extension. (The extension (2.1) defines an element in the group

$$\mathrm{Kas}(\mathbf{R} - \{0\};\, C_0(\mathbf{R}^2)) \cong \mathrm{K}^{-1}((\mathbf{R} - \{0\})^+) = \mathrm{K}^{-1}(S^1 \vee S^1) = \mathbf{Z} \oplus \mathbf{Z};$$

Kasparov shows that it corresponds to a non-zero element of this group.)

Voiculescu actually proves something stronger. Let Y be a sequence of points in $\mathbf{R} - \{0\}$ which converges to zero. Then Y together with $\mathbf{R}^2$ defines a closed subset of $\hat{G}_3$ and hence a quotient C^*-algebra E of $C^*(G_3)$, yielding an extension of the form

$$0 \to C_0(Y) \otimes \mathscr{K} \to E \to C_0(\mathbf{R}^2) \to 0\,. \tag{2.2}$$

Voiculescu, using delicate analytic techniques, proves that (2.2) does not split. Note that (2.2) defines an element of $\mathrm{Kas}(Y;\, C_0(\mathbf{R}^2))$; this group is $\cong \mathrm{K}^{-1}(Y^+)$, by Corollary 1.7. If Y is infinite discrete then $\mathrm{K}^{-1}(Y^+) = 0$. Hence (2.2) is stably split; it is possible to add a split extension to (2.2) in such a manner that the resulting extension splits. This makes precise Voiculescu's remark that his non-splitting result for (2.2) "seems not to be obtainable [from Kasparov's methods]" [22]. This also gives a proof of the stable splitting of (2.2) which is independent of the hardest part of Kasparov's results.

The example (2.2) is instructive since it illustrates so dramatically the difference between splitting and stable splitting.

(The analogous behavior for vector bundles is well-known. The tangent bundle $T(S^2)$ of the sphere S^2 is stably trivial, for the sum of $T(S^2)$ with the normal bundle $N(S^2)$ (relative to the standard embedding of S^2 in $\mathbf{R}^3$) is a three-dimensional trivial bundle, and $N(S^2)$ is a trivial line bundle. However, $T(S^2)$ is not trivial. If it were trivial then S^2 would have a nowhere-zero vector field and it would be possible to comb the hair of a coconut.)

3. PROOFS OF THE MAIN THEOREMS

In this section we prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Suppose that $\tau\colon A^+ \to Q(Y)$ represents $[\tau] \in \mathrm{Ext}(Y;\, A^+)$. Let $\Phi(\tau)$ be the restriction of τ to A. Then $\Phi(\tau)\colon A \to Q(Y)$ is a Kasparov extension. It is immediate that

$$\Phi\colon \mathrm{Ext}(Y;\, A^+) \to \mathrm{Kas}(Y;\, A) \tag{3.1}$$

is a well defined homomorphism.

Suppose that $\Phi(\tau) = 0$. Then there exists a split extension $\tau_1 \colon A \to Q(Y)$ such that $\Phi(\tau) \oplus \tau_1 = \pi\sigma$, for some representation $\sigma \colon A \to \mathcal{M}(Y)$. Without loss of generality we may assume that τ_1 is homogeneous. Extend τ_1 and σ to A^+; then

$$0 = [\pi\sigma] = [\tau \oplus \tau_1] = [\tau]$$

in $\mathrm{Ext}(Y; A^+)$ and thus Φ is an injection.

It remains to show that Φ is a surjection. Suppose that $\tau \colon A \to Q(Y)$ represents $[\tau] \in \mathrm{Kas}(Y; A)$. Choose a homogeneous split extension $\mu \colon A^+ \to Q(Y)$ and form $\tau \oplus \mu$, where τ is extended unitally to A^+. This is a homogeneous extension, and

$$\Phi(\tau \oplus \mu) = [\tau \oplus \mu|_A] = [\tau].$$

Thus Φ is an isomorphism. ▨

Proof of Theorem 1.5. If Y is a compact subset of Euclidean space then it is a finite-dimensional compact metric space, and Y^+ is the disjoint union of Y and $+$. Thus

$$\begin{aligned} \mathrm{Ext}(Y^+, +; A^+) = \mathrm{Ext}(Y; A^+) &\cong \\ &\cong \mathrm{Kas}(Y; A) \qquad \text{by (1.4).} \end{aligned}$$

Thus attention may be restricted to the case where Y is dense but not closed in Y^+.

Let $[\tau] \in \mathrm{Ext}(Y^+, +; A^+)$. By adding on a non-unital split extension τ and then restricting to A, one obtains from τ a non-unital extension τ_1 of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(Y^+, \mathcal{K}) & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow \equiv & & \big\downarrow & & \big\downarrow \tau_1 & & \\ 0 & \longrightarrow & C(Y^+, \mathcal{K}) & \longrightarrow & C_{s*}(Y^+, \mathcal{B}) & \longrightarrow & Q(Y^+) & \longrightarrow & 0, \end{array} \tag{3.2}$$

where we have made the identifications $A = E/C(Y^+, \mathcal{K})$, $C(Y^+, \mathcal{K}) \cong C(Y^+) \otimes \mathcal{K}$, and $\mathcal{M}(Y^+) = C_{s*}(Y^+, \mathcal{B})$ (the last of these courtesy of [1, 3.3]). The composite

$$p_+\tau_1 \colon A \to Q$$

splits, since $[p_+\tau] = 0$, so the extension

$$0 \to \mathcal{K} \to E/C_0(Y, \mathcal{K}) \xrightarrow{\pi} A \to 0 \tag{3.3}$$

which is obtained from (3.2) also splits. Choose a splitting σ for (3.3); it is given by a C^*-injection $\sigma \colon A \to E/C_0(Y, \mathcal{K})$ with $\pi\sigma = \mathrm{id}$. (Note that σ is not unique.) The inclusion

$$A \xrightarrow{\sigma} E/C_0(Y, \mathcal{K}) \hookrightarrow C^{\mathrm{b}}_{s*}(Y, \mathcal{B})/C_0(Y, \mathcal{K}) \tag{3.4}$$

is then a candidate for $\Phi[\tau] \in \mathrm{Kas}(Y; A)$. (Note that in 3.4 we have indentified $\mathcal{M}(Y) = C^b_{s*}(Y, \mathcal{B})$ using [1,3.3] again.) We must show that Φ is well defined.

First consider the case when $[\tau] = 0$ in $\mathrm{Ext}(Y^+, +; A^+)$. Then τ_1 in (3.2) is trivial, so let $\sigma_1 : A \to E$ be some splitting of τ_1. The composite

$$A \xrightarrow{\sigma_1} E \to E/C_0(Y, \mathcal{K})$$

is a possible splitting of (3.3), and this leads to a choice for $\Phi[\tau]$ which is obviously split, hence represents 0 in $\mathrm{Kas}(Y; A)$. The problem is to show that any other choice of a splitting of (3.3) also yields a split extension. This may be assured by choosing τ initially to be a "constant" trivial extension, as follows. Take some trivial extension $\sigma: A^+ \to \mathcal{B}(\mathcal{H})$ and define $\sigma_2: A^+ \to C_{s*}(Y^+, \mathcal{B})$ by $\sigma_2(a)(y)(\xi) = \sigma(a)(\xi)$ for $a \in A^+$, $y \in Y^+$, $\xi \in \mathcal{H}$; then let $\tau = \pi \circ \sigma_2$. (Any trivial PPV extension is equivalent to an extension of this form, by [16, Theorem 2.10].) Then the resulting extension (3.3) becomes

$$\begin{array}{ccccccccc} 0 \to \mathcal{K} & \to & C^*\{C(Y^+, \mathcal{K}), \sigma_2(A)\}/C_0(Y, \mathcal{K}) & \to & C^*\{C(Y^+, \mathcal{K}), \sigma_2(A)\}/C(Y^+, \mathcal{K}) & \to 0 \\ \downarrow \equiv & & \downarrow \cong & & \downarrow \cong & \\ 0 \to \mathcal{K} & \longrightarrow & C^*\{\mathcal{K}, \sigma(A)\} & \longrightarrow & A & \longrightarrow 0, \end{array} \tag{3.5}$$

for which any splitting can obviously be extended to a "constant" splitting of (3.2). Thus, if $[\tau] = 0$ in $\mathrm{Ext}(Y^+, +; A^+)$ then $\Phi[\tau] = 0$ in $\mathrm{Kas}(Y; A)$.

It is now easy to show in general that Φ is well defined. Let $[\tau] \in \mathrm{Ext}(Y^+, +; A^+)$. Choose representatives $\tau: A^+ \to Q(Y^+)$ and $\mu: A^+ \to Q(Y^+)$ for $[\tau]$ and for $-[\tau]$ respectively, and let τ_1 and $\mu_1: A \to Q(Y^+)$ be their restrictions as in (3.2). As $p_+\mu$ is split, there is a split extension (as in 3.3)

$$0 \to \mathcal{K} \to E(\mu_1)/C_0(Y, \mathcal{K}) \xrightarrow{\pi} A \to 0. \tag{3.6}$$

Fix some splitting ν for (3.6). Similarly, since $p_+\tau$ is trivial, there is a split extension

$$0 \to \mathcal{K} \to E(\tau_1)/C_0(Y, \mathcal{K}) \xrightarrow{\pi} A \to 0. \tag{3.7}$$

Let $\sigma_1, \sigma_2: A \to E(\tau_1)/C_0(Y, \mathcal{K})$ be two splittings of (3.7). To show that Φ is well defined, we must show that $[\sigma_1] = [\sigma_2]$ in the group $\mathrm{Kas}(Y; A)$. But consider the extensions

$$\sigma_j \oplus \nu: A \to E(\tau_1 \oplus \mu_1)/C_0(Y, \mathcal{K}), \qquad j = 1, 2.$$

These are split extensions by the above, since each corresponds to the split extension $\tau_1 \oplus \mu_1$. Thus

$$[\sigma_1 \oplus \nu] = [\sigma_2 \oplus \nu] = 0$$

in Kas(Y; A), which implies that

$$[\sigma_1] = [\sigma_2]$$

in Kas(Y; A).

Thus Φ: Ext(Y^+, $+$; A^+)$\to$Kas(Y; A) is a well defined function. It is clear from the construction that Φ is a homomorphism which is natural in the obvious senses.

Next we show that Φ is surjective. Let $[\tau] \in$ Kas(Y; A) be an extension represented by

$$\tau\colon A \to Q(Y) = C^{\mathrm{b}}_{\mathrm{s*}}(Y, \mathscr{B})/C_0(Y, \mathscr{K}).$$

Suppose there is a unitary $U \in C^{\mathrm{b}}_{\mathrm{s*}}(Y, \mathscr{B})$ such that $(\pi U)\tau(\pi U)^*$ has image in $C_0^{\mathrm{s*}}(Y,\mathscr{B})/C_0(Y,\mathscr{K})$, where $C_0^{\mathrm{s*}}(Y,\mathscr{B})$ denotes the subalgebra of functions in $C^{\mathrm{b}}_{\mathrm{s*}}(Y, \mathscr{B})$ which converge $*$-strongly to zero at $+ \in Y^+$. Then it is clear that $[\tau] = \Phi([\sigma])$, where σ is the composite of $(\pi U)\,\tau\,(\pi U)^*$ with the injection

$$C_0^{\mathrm{s*}}(Y, \mathscr{B})/C_0(Y, \mathscr{K}) \to C_{\mathrm{s*}}(Y^+, \mathscr{B})/C(Y^+, \mathscr{K})$$

(modified by adding a constant split extension to make it homogeneous). It remains to produce U.

For convenience in this part of the proof, take $\mathscr{H}$ to be the concrete Hilbert space $L^2([0, 1])$ and let $p_t (0 \leqslant t \leqslant 1)$ denote multiplication by the characteristic function of $[0, t]$ on $\mathscr{H}$. Note that p_t is a strong-$*$ continuous function of t. Define unitary operators u_t on $\mathscr{H}$ by

$$u_t(g)\,(s) = \begin{cases} t^{-1/2}g(t^{-1}s) & 0 \leqslant s \leqslant t/2 \\ (2-t)^{-1/2}g((s+1-t)/(2-t)) & t/2 \leqslant s \leqslant 1 \end{cases}$$

for $0 < t \leqslant 1$, $g \in \mathscr{H}$. Note that $u_t p_{1/2} u_t^* = p_{t/2}$.

Returning to $\tau\colon A \to C^{\mathrm{b}}_{\mathrm{s*}}(Y, \mathscr{B})/C_0(Y, \mathscr{K})$, we may assume by adding on a suitable trivial extension that the associated algebra E in the extension

$$0 \to C_0(Y, \mathscr{K}) \to E \to A \to 0$$

has the property that for fixed $f \in E$,

$$f(y) - p_{1/2}f(y)p_{1/2} \in \mathscr{K} \qquad \text{for all } y \in Y$$

(3.8) and

$$\|f(y) - p_{1/2}f(y)p_{1/2}\| \to 0 \qquad \text{as } y \to + \text{ in } Y^+.$$

Let $m\colon Y^+ \to [0, 1]$ be a continuous function with $m^{-1}\{0\} = \{+\}$ (e.g., take m to be the distance to $+$ in a suitable metric). Define a unitary $U \in C^{\mathrm{b}}_{s*}(Y, \mathscr{B})$ by $U(y) = u_{m(y)}$. Then

$$(Up_{1/2}U^*)(y) = u_{m(y)}p_{1/2}u_{m(y)}{}^* =$$
$$= p_{m(y)/2}$$

and hence

$$(3.9) \qquad (Up_{1/2}U^*)(y) \to 0 \qquad *\text{-strongly as } y \to +\,.$$

Combining (3.8) and (3.9), we see that $UEU^* \in C_0^{s*}(Y, \mathscr{B})$ and hence τ is equivalent to an extension of the form

$$A \to C_0^{s*}(Y, \mathscr{B})/C_0(Y, \mathscr{K})$$

which is in the image of Φ. Hence Φ is surjective.

It remains to demonstrate that Φ is injective. Suppose that $[\tau]$ is in the kernel of Φ. Then the extension

$$\sigma\colon A \to E/C_0\,(Y, \mathscr{K})$$

in (3.4) is trivial in $\mathrm{Kas}(Y;\, A)$. If σ were a split extension via $\theta\colon A \to E$, then θ would also split τ and hence $[\tau] = 0$. In general we only know that $\sigma \oplus \sigma_1$ is a split extension, where σ_1 is some split extension. Since Φ is surjective, we may represent σ_1 as an extension

$$\sigma_1\colon A \to C_{s*}(Y^+, \mathscr{B})/C_0(Y, \mathscr{K})\,.$$

Then $\tau \oplus \sigma_1$ splits, and

$$[\tau] = [\tau \oplus \sigma_1] = 0 \qquad \text{in } \mathrm{Ext}(Y^+,\ +;\ A^+).$$

Hence Φ is an isomorphism. ▨

4. THE UNEXPECTED EFFECTIVENESS OF K-THEORY

It is well-known that to every complex Banach algebra with unit (not necessarily commutative), one may associate abelian groups $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$. (The definitions are given in [20, §§ 5 and 6] for abelian A but apply just as well in the noncommutative case. $\mathrm{K}_0(A)$ is the usual K_0-group of algebraic K-theory; $\mathrm{K}_1(A)$ is a modification of the algebraic K_1-group that takes the topology of $\mathrm{GL}(n, A)$ into account.) For compact spaces X, one has $\mathrm{K}_i(C(X)) = \mathrm{K}^{-i}(X)$, the latter as usually defined in algebraic topology using vector bundles. For a Banach algebra B not necessarily with unit, we define

$$\mathrm{K}_i(B) = \widetilde{\mathrm{K}}_i(B^+) = \ker(\mathrm{K}_i(B^+) \to \mathrm{K}_i(\mathbf{C}));$$

this definition agrees with the previous one when B is unital. The K-groups of Banach algebras satisfy Bott periodicity; this implies

$$K_i(B \otimes C_0(\mathbb{R}^n)) \cong \begin{cases} K_i(B), & n \text{ even} \\ K_{1-i}(B), & n \text{ odd}, \end{cases}$$

and any extension of C^*-algebras of the form (1.1) defines a repeating six-term long exact sequence of K-groups. (See [20, Theorem 10.1]; once again, the commutativity is unnecessary.) Furthermore, K-groups are "stable" invariants, so that $K_i(B \otimes \mathscr{K})$ and $K_i(B)$ may be identified.

Consider the connecting homomorphisms

$$(4.1) \qquad \begin{aligned} \delta_0(\tau)&: K_0(A) \to K_1(B \otimes \mathscr{K}) \cong K_1(B) \\ \delta_1(\tau)&: K_1(A) \to K_0(B \otimes \mathscr{K}) \cong K_0(B) \end{aligned}$$

in the long exact K-theory sequence associated with an extension (1.1) with Busby invariant τ. It is easy to check that:

(a) δ_0 and δ_1 depend only on the unitary equivalence class of τ,

(b) δ_0 and δ_1 vanish when τ is split, and

(c) $\delta_i(\tau_1 \oplus \tau_2) = \delta_i(\tau_1) + \delta_i(\tau_2)$.

Thus δ_0 and δ_1 combine to yield a homomorphism

$$\gamma: \mathrm{Kas}(B; A) \xrightarrow{(\delta_0, \delta_1)} \mathrm{Hom}_{\mathbb{Z}}(K_0(A), K_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(K_1(A), K_0(B)).$$

In particular, $\gamma(\tau) \neq 0$ implies that τ is not split, and this provides an easy way to check non-splitting of many C^*-algebra extensions (see [18]). The converse is obviously false, since $\gamma(\tau) = 0$ whenever τ is stably split but not split.

In fact, it is known from the BDF theory that $\gamma(\tau) = 0$ doesn't necessarily imply that $[\tau] = 0$ in $\mathrm{Kas}(B; A)$. It is true, however, that when $\gamma(\tau) = 0$, the long exact K-theory sequence coming from (1.1) breaks up into two short exact sequences of abelian groups

$$(4.2) \qquad S_i(\tau): 0 \to K_i(B \otimes \mathscr{K}) \cong K_i(B) \to K_i(E) \to K_i(A) \to 0,$$

$i = 0, 1$. These will split when τ does and once again depend additively on the unitary equivalence class of τ, so

$$\tau \mapsto ([S_0(\tau)], [S_1(\tau)])$$

defines a group homomorphism

$$\varkappa: \ker \gamma \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), K_0(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(A), K_1(B)).$$

Here "Ext" has its usual meaning from homological algebra. The definition of $\varkappa$ is due to L. G. Brown [4], who also, along with R. G. Douglas and P. A. Fillmore, used the maps γ_1 and γ_∞ which are special cases of γ.

Rosenberg [18] has shown how various invariants of extensions of particular classes of C^*-algebras turn out to be special cases of γ. In the case of AF algebras (for which K_1, and hence also γ, always vanishes), a similarly important role is played by $\varkappa$ (see for instance [15], [11], and [5]). Is it just good fortune that the γ- and $\varkappa$-invariants turn out to be so effective? The answer seems to be that this is not just a matter of luck; we are led to the following definitions.

DEFINITION 4.3. The group $\mathrm{Kas}(B; A)$ is said to be *determined by* K-*theory* if $\varkappa$ is an injection, or equivalently if $\gamma(\tau) = \varkappa(\tau) = 0$ implies τ is stably split.

A somewhat stronger condition frequently holds in practice.

DEFINITION 4.4. The group $\mathrm{Kas}(B; A)$ is said to *satisfy the Universal Coefficient Theorem* (UCT) if γ is surjective and if $\varkappa$ is an isomorphism.

There are analogous definitions for the PPV extension groups. For certain purposes it is sometimes convenient to insist that the UCT hold both for A and for its (non-unital) suspension $A \otimes C_0(\mathbf{R})$.

We shall now review a number of cases where the UCT holds.

EXAMPLE 4.5. Cases where $B = \mathbf{C}$.

The "strong" BDF Ext-groups of compact metric spaces (i.e., abelian separable unital C^*-algebras) were shown by Brown ([3]; cf. [13]) to satisfy the UCT. Since $\mathrm{Kas}(\mathrm{pt}; C_0(X)) = \mathrm{Ext}_s(X^+)$, this means the UCT holds for the Kasparov groups $\mathrm{Kas}(\mathrm{pt}; A)$ when A is abelian. It is also interesting to note in this connection the observation of Brown that when $\mathrm{Kas}(\mathrm{pt}; C_0(X))$ is finitely generated, then the kernel of γ is precisely its torsion subgroup. (This follows from the UCT since if $\mathrm{Ext}(K^0(X), \mathbf{Z})$ is finitely generated, it must be finite, by standard abelian group theory.) There are also extensive results of Brown (largely unpublished) showing that for certain classes of non-commutative C^*-algebras A, $\mathrm{Kas}(\mathrm{pt}; A)$ satisfies the UCT; it is possible, for instance, that this holds for all separable nuclear A.

The following results are to some extent improvements on [17, Proposition 7.6 and Corollary 9.11] and on part of [14, Theorem 5]. [1)]

[1)] After this paper was typed, we received a copy of a preprint by J. Cuntz entitled "A class of C^*-algebras and topological Markov chains. II: Reducible Markov chains and the Ext-functor for C^*-algebras", in which the same results are proved by a slightly different method. See in particular Cuntz's Lemma 3.2 and the following Remark. In his Theorem 3.11, Cuntz proves a version of the UCT for the Kasparov groups $\mathrm{Kas}(B; \mathcal{O}_A)$.

LEMMA 4.6. *Let B be any unital C^*-algebra, and as before let $\mathcal{M}(B)$ be the multiplier algebra of $B \otimes \mathcal{K}$ and let $Q(B)$ be $\mathcal{M}(B)/B \otimes \mathcal{K}$. Then the connecting maps δ of the* K*-theory sequence associated to*

$$0 \to B \otimes \mathcal{K} \to \mathcal{M}(B) \xrightarrow{\pi} Q(B) \to 0$$

map $K_0(Q(B))$ onto $\mathrm{K}_1(B)$ *and $K_1(Q(B))$ onto* $\mathrm{K}_0(B)$. (*Equivalently, the natural maps* $\mathrm{K}_i(B) \to \mathrm{K}_i(\mathcal{M}(B))$, $i = 0, 1$, *are zero.*)

Proof. Note that the spatial C^*-tensor product $B \otimes_{\min} \mathcal{B}(\mathcal{H})$ may be embedded naturally in $\mathcal{M}(B)$. First consider the map δ_1: $\mathrm{K}_1(Q(B)) \to \mathrm{K}_0(B)$. The group $\mathrm{K}_0(B)$ is generated by classes of projections in $B \otimes M_n \subset B \otimes \mathcal{K}$ for various n, and it is enough to show that any such class is in the image of $\mathrm{K}_1(Q(B))$. If we absorb the M_n-factor with the B, we are reduced to considering a projection $e \in B$ and showing that $[e]$ is in the image of δ_1. The following argument is not the shortest possible, but exhibits an explicit class in $\mathrm{K}_1(Q(B))$ mapping under δ_1 to $[e]$. Let S be the unilateral shift on $\mathcal{H}$, so that $S^*S = 1$ and $SS^* = 1 - p$, where p is a projection in $\mathcal{K}(\mathcal{H})$ of rank one. Then $v = \pi(e \otimes S + (1 - e) \otimes 1)$ is a unitary element of $Q(B)$ and

$$\pi\left(e \otimes \begin{pmatrix} S & p \\ 0 & S^* \end{pmatrix} + (1 - e) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$$

in $Q(B) \otimes M_2 \cong Q(B)$. Thus from the description of δ_1 in [20, 8.1], we see

$$\delta_1([v]) = \left[e \otimes \begin{pmatrix} S & p \\ 0 & S^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S^* & 0 \\ p & S \end{pmatrix} + (1 - e) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] =$$

$$= -\left[e \otimes \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right],$$

which shows $\delta_1(-[v]) = [e]$.

To deal with δ_0, first identify $\mathrm{K}_1(B)$ with $\mathrm{K}_0(B \otimes C_0(\mathbf{R})) \hookrightarrow \mathrm{K}_0(B \otimes C(S^1))$. The argument above shows that $\mathrm{K}_1(Q(B) \otimes (C(S^1))$ maps onto $\mathrm{K}_0(B \otimes C(S^1))$, hence $\delta_0(\mathrm{K}_0(Q(B))) = \mathrm{K}_1(B)$. ▨

THEOREM 4.7. *Let B be any unital C^*-algebra. Then* $\mathrm{Kas}(B; \mathbf{C})$ *satisfies the* UCT *and is naturally isomorphic to* $\mathrm{K}_1(B)$.

Proof. Since $\mathrm{K}_0(\mathbf{C}) = \mathbf{Z}$ and $\mathrm{K}_1(\mathbf{C}) = 0$, any extension

$$0 \to B \otimes \mathcal{K} \to E \to \mathbf{C} \to 0 \tag{4.8}$$

with Busby invariant $\tau: \mathbf{C} \to Q(B)$ has $\delta_1(\tau) = 0$ and $\delta_0(\tau) \in \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_1(B)) \cong \mathrm{K}_1(B)$. Thus we must show δ_0: $\mathrm{Kas}(B; \mathbf{C}) \to \mathrm{K}_1(B)$ is an isomorphism. To prove surjectivity, note that by 4.6, any class in $\mathrm{K}_1(B)$ is in the image of $\mathrm{K}_0(Q(B))$. This group is generated by classes of projections $p \in Q(B)$, and we can define a $*$-homomorphism $\tau: \mathbf{C} \to Q(B)$ with δ_0-invariant $\delta_0([p])$ by setting $\tau(1) = p$. Thus δ_0 is surjective.

Next suppose $\delta_0(\tau) = 0$. We must show (4.8) is stably split. Now by assumption, $\delta_0(\tau(1)) = 0$, so $\tau(1)$ is stably liftable to $\mathscr{M}(B)$, i.e., $\tau(1) \oplus 1_r \oplus 0_q$ in $M_{r+q+1}(Q(B)) \cong Q(B)$ can be lifted to a projection in $\mathscr{M}(B)$ for some r and q. This means exactly that the Kasparov sum of (4.8) and of some trivial extension splits. ▨

THEOREM 4.9. *Let B be any unital C^*-algebra. Then* $\mathrm{Kas}(B; C_0(\mathbf{R}))$ *satisfies the* UCT *and is naturally isomorphic to* $\mathrm{K}_0(B)$.

Proof. Since $\mathrm{K}_1(C_0(\mathbf{R})) = \mathbf{Z}$ and $\mathrm{K}_0(C_0(\mathbf{R})) = 0$, any extension

$$0 \to B \otimes \mathscr{K} \to E \to C_0(\mathbf{R}) \to 0 \tag{4.10}$$

with Busby invariant $\tau: C_0(\mathbf{R}) \to Q(B)$ has $\delta_0(\tau) = 0$ and $\delta_1(\tau) \in \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(B)) = \mathrm{K}_0(B)$. Thus we must show $\delta_1: \mathrm{Kas}(B; C_0(\mathbf{R})) \to \mathrm{K}_0(B)$ is an isomorphism. Observe that any unitary $u \in Q(B)$ with non-zero class in $\mathrm{K}_1(Q(B))$ must have spectrum the whole unit circle, hence gives rise to a $*$-monomorphism $C(S^1) \to C^*(u) \subseteq Q(B)$ mapping the generator of $K^1(\mathbf{R}) = \mathrm{K}^1(S^1)$ to the class of u. Surjectivity of δ_1 therefore follows from 4.6 as in the proof of the last theorem.

Next suppose $\delta_1(\tau) = 0$. Adding a split extension to (4.10) if necessary, we may assume τ extends to a unital map $C_0(\mathbf{R})^+ = C(S^1) \to Q(B)$ mapping the identity function on S^1 to a unitary $u \in Q(B)$ with spectrum the whole unit circle. By assumption, $\delta_1([u]) = 0$ and so u is stably liftable to a unitary in $\mathscr{M}(B)$, i.e., $u \oplus 1_n \in M_{n+1}(Q(B)) \cong Q(B)$ is liftable for some n. This then says that (4.10) is stably split. ▨

REMARK 4.11. Theorems 4.7 and 4.9 (which essentially identify $\mathrm{K}_0(B)$ and $\mathrm{K}_1(B)$ with ways of choosing a unitary element or a projection, respectively, in $Q(B)$) are closely related to the usual realizations of K-theory for a compact space X. Indeed, we have $\mathrm{K}^0(X) \cong [X, \mathrm{GL}(Q)]$ and $\mathrm{K}^{-1}(X) \cong [X, \mathscr{F}^1]$, where $\mathscr{F}^1$ denotes the nontrivial connected component of the self-adjoint Fredholm operators on $\mathscr{H}$ and [,] denotes homotopy classes of maps. But a homotopy class of maps $X \to \mathrm{GL}(Q)$ represents a class of unitaries in $Q(X)$, and a homotopy class of maps $X \to \mathscr{F}^1$ represents a class of projections in $Q(X)$.

REMARK 4.12. Theorems 4.7 and 4.9 immediately extend to the case of nonunital algebras B. Thus for any C^*-algebra B, $\mathrm{Kas}(B; \mathbf{C}) \cong \mathrm{K}_1(B)$ and $\mathrm{Kas}(B; C_0(\mathbf{R})) \cong \mathrm{K}_0(B)$.

Proof. Everything goes through as before once one has the analogue of Lemma 4.6 for a non-unital algebra B. To show that the K-groups of $Q(B)$ map onto those of B, note that from the proof of 4.6, we really only need to use the image in $Q(B)$ of $B \otimes_{\min} \mathscr{B}(\mathscr{H})$. Suppose, say, that B is nuclear. Then from the exact sequence

$$0 \to B \to B^+ \to \mathbb{C} \to 0$$

we obtain the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B \otimes \mathscr{K} & \to & B \otimes \mathscr{B} & \to & B \otimes Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B^+ \otimes \mathscr{K} & \to & B^+ \otimes \mathscr{B} & \to & B^+ \otimes Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{B} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Given any class b in $\mathrm{K}_j(B) = \mathrm{K}_j(B \otimes \mathscr{K})$, choose a class in $\mathrm{K}_{j+1}(B^+ \otimes Q)$ mapping onto its image in $\mathrm{K}_j(B^+ \otimes \mathscr{K})$ (by 4.6); by diagram chasing, this comes from a class in $\mathrm{K}_j(B \otimes Q)$, represented by a unitary u or projection e in $Q(B)$. Then $\delta_1([u])$ or $\delta_0([e]) = b$. If B is not nuclear, this doesn't exactly work (because of possible non-exactness of $\otimes_{\min}$), but note that in the proof of 4.6, we didn't really need all of $\mathscr{B}$, only the subalgebra $C^*(S)$, which is even type I. Thus the same idea goes through with $\mathscr{B}$ replaced by a suitable nuclear subalgebra. ▨

If we are willing to make use of periodicity of the Kasparov groups in the A-variable [14, Theorem 3], Remark 4.12 above immediately implies

THEOREM 4.13. *Let B be any C^*-algebra with countable approximate unit. Then*

$$\mathrm{Kas}(B; C_0(\mathbb{R}^n)) \cong \begin{cases} \mathrm{K}_1(B), & n \text{ even}, \\ \mathrm{K}_0(B), & n \text{ odd}. \end{cases}$$

Note that 4.7 and 4.9 are special cases of 4.13, and that Corollary 1.7 already establishes 4.13 for $B = C_0(Y)$, Y a locally compact subset of Euclidean space, without our needing to use any of Kasparov's more difficult results. Similarly, Corollary 1.6 can be used to quickly check the UCT in many other cases of practical interest.

We shall establish the UCT for $\mathrm{Kas}(B; A)$ when A and B are separable C^*-algebras with A in a certain large subclass of the nuclear algebras in a future paper. However, the techniques required are considerably more sophisticated than those discussed here, and require the full power of Kasparov's machinery plus some homological algebra.

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