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THE SPECTRAL FLAVOUR OF SCOTT BROWN'S TECHNIQUES

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The aim of this paper is to prove that some restrictions to invariant subspaces of operators with "good" spectral properties, have proper invariant subspaces. The operators with "good" spectral properties are unconditionally decomposable operators, as defined in § 1 below, and this means decomposable operators, in the sense of C. Foiaş [11], enjoying an additional property. The tool will be Scott Brown's techniques [4] combined with a sharpening of the techniques of [3]. The paper is divided in two sections. In § 1 we give a brief account on the spectral theory we need. To make the results of § 2 more relevant we include some examples of unconditionally decomposable operators. In § 2 we prove Theorems 2.6, 2.7, and 2.8, the main results on invariant subspaces.

1. SPECTRAL THEORY

Let X be a Banach space over the complex field $\mathbf{C}$, and let $\mathscr{L}(X)$ denote the algebra of all bounded linear operators acting on X. Let $T \in \mathscr{L}(X)$ be given. We shall say that T has the *single-valued extension property* if the equation $(\lambda - T)f(\lambda)=0$ has only the null X-valued analytic solution in any given domain. If T has the single-valued extension property, then for any $x \in X$ there exists a unique maximal open set $\rho_T(x)$ $(\supset \rho(T))$ and a unique X-valued analytic function $x_T(\cdot)$ defined in $\rho_T(x)$ such that

$$(\lambda - T)x_T(\lambda) = x, \qquad \lambda \in \rho_T(x).$$

Moreover, if $\sigma \subset \mathbf{C}$ is a closed set and $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$, then

$$X_T(\sigma) = \{x \in X \colon \sigma_T(x) \subset \sigma\}$$

is a linear manifold, hyperinvariant for T (i.e. invariant for the commutant of T).

Let Y be a subspace (= closed linear manifold) in X. We call Y *a spectral maximal space* of T if $TY \subset Y$ and for any subspace $Z \subset X$ such that $TZ \subset Z$, $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ we have $Z \subset Y$. We shall say that T is a decomposable operator if for any open covering $\{G_j\}_{j=1}^n$ of $\sigma(T)$ there exists a system $\{X_j\}_{j=1}^n$ of spectral maximal spaces of T such that

$$\sum_{j=1}^{n} X_j = X, \qquad \sigma(T \mid X_j) \subset G_j.$$

All the above definitions are taken from the monograph [7]. Recall also that any decomposable operator T has the single-valued extension property ([7], Ch. 2, Corollary 1.4), and for any closed set $\sigma \subset \mathbf{C}$, $X_T(\sigma)$ is a spectral maximal space of T ([7], Ch. 2, Theorem 1.5) and $\sigma(T \mid X_T(\sigma)) = \sigma(T) \cap \sigma$ ([7], Ch. 1, Proposition 3.8).

Suppose T is decomposable and let $\sigma \subset \mathbf{C}$ be a closed set. We shall denote by X_T^σ the quotient space $X/X_T(\sigma)$, and for any $x \in X$, the symbol x^σ will denote the coset of x in X_T^σ. The symbol T_σ will denote the restriction of T to $X_T(\sigma)$, and $T^\sigma \in \mathscr{L}(X_T^\sigma)$ will be defined by the equation

$$T^\sigma x^\sigma = (Tx)^\sigma, \qquad x \in X.$$

For our convenience we introduce a new definition. We shall say that T is *unconditionally decomposable* if it is decomposable and for any system $\{\sigma_k\}_{k=1}^n$ of disjoint closed subsets in $\mathbf{C}$ we have

$$\sup\left\{\left\|\sum_{k=1}^{n} e^{i\theta_k} x_k\right\| : 0 \leqslant \theta_k < 2\pi\right\} \leqslant a_T \left\|\sum_{k=1}^{n} x_k\right\|,$$

where $x_k \in X_T(\sigma_k)$ and $a_T < \infty$ depends only on T. In the following the symbol a_T will be always associated with an unconditionally decomposable operator T by the above relation.

1.1. Lemma. *Suppose $T \in \mathscr{L}(X)$ is decomposable, $\sigma \subset \mathbf{C}$ is a closed set, $\mu \in \operatorname{int}\sigma$ and $\{x_n\}_{n=1}^\infty \subset X$ is a sequence such that $\lim_{n\to\infty} \|(T-\mu)\, x_n\| = 0$. Then we have $\lim_{n\to\infty} \|x_n^\sigma\| = 0$.*

Proof. Because by [1], Proposition 3.3 we have

$$\mu \notin (\sigma(T) \setminus \sigma(T_\sigma))^- = \sigma(T^\sigma)$$

and $\lim_{n\to\infty} \|(T^\sigma - \mu)x_n^\sigma\| = 0$ it follows $\lim_{n\to\infty} \|x_n^\sigma\| = 0$.

1.2. Lemma. *Suppose $T \in \mathscr{L}(X)$ is unconditionally decomposable and let $G_0 \subset \mathbf{C}$, $\sigma \subset \mathbf{C} \setminus G_0$, $Z \supset X_T(\sigma)$ be given, where G_0 is open, σ is closed and Z is a subspace in X such that $\dim Z^\sigma < \infty$. Then for any $\varepsilon > 0$ there exists a closed set δ, $\delta \subset G_0$ such that*

$$\|x\| \leqslant (2a_T + \varepsilon)\, \|x + z\|, \quad z \in Z,\ \sigma_T(x) \subset G_0,\ \sigma_T(x) \cap \delta = \emptyset.$$

Proof. Let Y_0 be a finite-dimensional subspace such that $Z = Y_0 + X_T(\sigma)$, $Y_0 \cap X_T(\sigma) = \{0\}$. If our lemma is false, we can find $\{x_n\}_{n=1}^{\infty} \subset X$, $\{y_n\}_{n=1}^{\infty} \subset Y_0$, $\{z_n\}_{n=1}^{\infty} \subset X_T(\sigma)$ such that

$$\|x_n\| = 1,\ \sigma_T(x_n) \subset G_0,\ \sigma_T(x_m) \cap \sigma_T(x_n) = \emptyset, \qquad m \neq n,$$

$$\lim_{n\to\infty} y_n = y_0,\ (2a_T + \varepsilon)\| x_n + y_n + z_n\| < 1.$$

If we put $u_{m,n} = x_m - z_n + z_m$, we derive

$$2 \leqslant \|x_n - u_{m,n}\| + \|x_n + u_{m,n}\| \leqslant (1 + a_T)\| x_n - u_{m,n}\| \leqslant$$
$$\leqslant (1 + a_T)(\|x_n + y_n + z_n\| + \|x_m + y_n + z_m\|) \leqslant$$
$$\leqslant (1 + a_T)(\|x_n + y_n + z_n\| + \|x_m + y_m + z_m\| + \|y_m - y_n\|) \leqslant$$
$$\leqslant (1 + a_T)(2(2a_T + \varepsilon)^{-1} + \|y_m - y_n\|),$$

and this is a contradiction because $\lim_{m,n\to\infty} \|y_m - y_n\| = 0$, $a_T \geqslant 1$.

1.3. EXAMPLES OF UNCONDITIONALLY DECOMPOSABLE OPERATORS

a) Let $T \in \mathscr{L}(X)$ be a spectral operator in the sense of N. Dunford [8]. Then T is decomposable and $X_T(\sigma) = E(\sigma)X$, where $\sigma \subset \mathbf{C}$ is closed and E denotes the spectral measure of T (see [7], Ch. 2, Example 1.6). The fact that T is unconditionally decomposable follows from the properties of E ([8], § 3).

b) Let G be a bounded open connected set in $\mathbf{C}$, $G \neq \emptyset$ and let X denote the Banach space of n-times continuously differentiable bounded complex functions defined in G. The norm of $x \in X$ will be

$$\|x\| = \max_{0\leqslant j+k\leqslant n} \sup_{\lambda\in G} \left| \frac{(\partial^{j+k} x)(\lambda)}{\partial(\mathrm{Re}\lambda)^j\, \partial(\mathrm{Im}\lambda)^k} \right|.$$

If we define $T \in \mathscr{L}(X)$ by the equation

$$(Tx)(\lambda) = \lambda x(\lambda), \qquad x \in X,$$

then T is a generalized scalar operator in the sense of C. Foiaş [10]. Since by [7], Ch. 3, Corollary 1.15, T is decomposable, using [7], Ch. 3, Theorem 1.4 it is easy to show that we have

$$X_T(\sigma) = \{x \in X : \mathrm{supp}\, x \subset \sigma\}.$$

If $\{\sigma_k\}_{k=1}^m$ are disjoint closed sets in $\mathbf{C}$ we have

$$\left\| \sum_{k=1}^m e^{i\theta_k} x_k \right\| = \left\| \sum_{k=1}^m x_k \right\|, \quad x_k \in X_T(\sigma_k), \quad 0 \leqslant \theta_k < 2\pi,$$

thus T is unconditionally decomposable and in fact $a_T = 1$.

2. INVARIANT SUBSPACES

Throughout this section $T \in \mathscr{L}(X)$ will be a fixed unconditionally decomposable operator, Y will be an invariant subspace of T, and A will denote the restriction to Y of T. If $\sigma \subset \mathbf{C}$ is a compact set, then $\mathrm{R}(\sigma)$ will denote the uniform closure in $\mathscr{C}(\sigma)$ of all rational functions with poles off σ. As in [3], we denote by G a nonempty, bounded, open, connected subset of $\mathbf{C}$, and $H^\infty(G)$ will denote the algebra of all bounded analytic functions defined in G. If $L^1(G)$ denotes the space of all Lebesgue integrable complex functions, then $M'(G) = L^1(G)/(H^\infty(G)^\perp \cap L^1(G))$ is a separable Banach space, and $H^\infty(G)$ can be canonically identified with the dual of $M'(G)$ (see [14], § 4). For any $\mu \in G$, the evaluation at μ in $H^\infty(G)$ will be denoted by $\mathscr{E}_\mu$. The weak*-topology in $H^\infty(G)$ will be the $M'(G)$-topology, and by [14], Theorem 4.1 the functional $\mathscr{E}_\mu$ is w*-continuous, or equivalently $\mathscr{E}_\mu \in M'(G)$. Following [14], § 4 we call a subset $\sigma \subset G$ dominating in G if

$$\|f\|_\infty = \sup_{\lambda \in \sigma} |f(\lambda)|; \qquad (\forall) f \in H^\infty(G).$$

Let $\mathscr{F}_G$ denote the family of all closed sets $\sigma \subset \mathbf{C}$ such that $(\mathbf{C} \setminus \sigma)^- \subset G$. If $\sigma \in \mathscr{F}_G$ is given, then for any $x^* \in X_T(\sigma)^\perp$ we define the functional $x^*_\sigma \in (X^\sigma_T)^*$ by the equation

$$x^*_\sigma(x^\sigma) = x^*(x), \qquad x \in X.$$

It is well known that the map

$$x^* \to x^*_\sigma, \qquad x^* \in X_T(\sigma)^\perp,$$

is an isometry onto $(X^\sigma_T)^*$. By [1], Proposition 3.3. we have

$$\sigma(T^\sigma) \subset (\sigma(T) \setminus \sigma)^- \subset G,$$

for any $x \in X$, $x^* \in X_T(\sigma)^\perp$, and we can therefore define the bounded linear functional $x \overset{\sigma}{\otimes} x^*$ acting on $H^\infty(G)$ by the equation

$$(x \otimes x^*)(f) = x^*_\sigma(f(T^\sigma)\, x^\sigma), \qquad f \in H^\infty(G),$$

where $f(T^\sigma)$ is defined by the Riesz-Dunford functional calculus with analytic functions. It is an easy exercise to check that for any $\delta \in \mathscr{F}_G$, $\delta \subset \sigma$, $x^* \in X_T(\sigma)^\perp$, $x \in X$, we have

$$x^* \in X_T(\delta)^\perp, \qquad x \overset{\delta}{\otimes} x^* = x \overset{\sigma}{\otimes} x^*.$$

2.1. LEMMA. *For any $\sigma \in \mathscr{F}_G$, $x \in X$, $x^* \in X_T(\sigma)^\perp$ we have $x \overset{\sigma}{\otimes} x^* \in M'(G)$. If $\{x_n\}_{n=1}^\infty \subset X$ is a bounded sequence such that $\lim_{n\to\infty} \|(T-\mu)\, x_n\| = 0$, for some*

$\mu \neq (\sigma(T) \setminus \sigma)^-$, *then we have*

$$\lim_{n\to\infty} | x^*(x_n) \mathscr{E}_\mu - x_n \overset{\sigma}{\otimes} x^* | = 0,$$

uniformly with respect to x^* *in bounded sets.*

Proof. Let $\{f_n\}_{n=1}^\infty \subset H^\infty(G)$ be a w*-convergent to 0 sequence. Because $\{f_n\}_{n=1}^\infty$ tends uniformly to 0 on compact sets and $\sigma(T^\sigma) \subset G$ we derive that $\{f_n(T^\sigma)\}_{n=1}^\infty$ is norm-convergent to 0 and $x \overset{\sigma}{\otimes} x^*$ is sequentially w*-continuous. Now using [5], Theorem 2.3 we deduce $x \overset{\sigma}{\otimes} x^* \in M'(G)$.

Let $f \in H^\infty(G)$ be given and define $f_\mu \in H^\infty(G)$ by the equation

$$f(\lambda) - f(\mu) = f_\mu(\lambda)(\lambda - \mu), \qquad \lambda \in G.$$

Since the map $f \to f_\mu$ is bounded, we deduce

$$| x^*(x_n) f(\mu) - x^*(f(T^\sigma) x_n^\sigma) | = | x_\sigma^*(f_\mu(T^\sigma)(\mu - T)x_n^\sigma) | \leqslant$$

$$\leqslant \|x^*\| \|f_\mu(T^\sigma)\| \|(\mu - T) x_n\|,$$

and this concludes the proof.

2.2. LEMMA. *Let* $\sigma \in \mathscr{F}_G$, $y \in Y$, $x^* \in X_T(\sigma)^\perp$ *be given. Then for any* $\varepsilon > 0$ *there exist two subspaces* $Y' \subset Y$, $X_0^* \subset X_T(\sigma)^\perp$ *such that* $\dim(Y/Y') < \infty$, $\dim(X_T(\sigma)^\perp / X_0^*) < \infty$, X_0^* *is* w*-*closed and*

$$\|y' \overset{\sigma}{\otimes} x^*\| < \varepsilon \|y'\|, \quad y' \in Y',$$

$$\|y \overset{\sigma}{\otimes} x_0^*\| < \varepsilon \|x_0^*\|, \quad x_0^* \in X_0^*.$$

Proof. It is plain that both the sets

$$M = \{f(T^\sigma) y^\sigma : f \in H^\infty(G),\ \|f\|_\infty = 1\},$$

$$M' = \{f(T^\sigma)^* x_\sigma^* : f \in H^\infty(G),\ \|f\|_\infty = 1\},$$

are compact, thus we can find two subspaces $Z \subset X$, $Z' \subset X^*$ such that

$$Z \supset X_T(\sigma),\ Z' \subset X_T(\sigma)^\perp, \quad \dim Z^\sigma < \infty,\ \dim Z' < \infty,$$

and

$$\operatorname{dist}(x, Z) < \varepsilon,\ x \in M, \quad \operatorname{dist}(z_\sigma^*, Z_\sigma') < \varepsilon,\ z_\sigma^* \in M'.$$

Now we can take

$$Y' = \{y' \in Y : z^*(y') = 0,\ z^* \in Z'\}, \quad X_\sigma^* = Z^\perp.$$

2.3. LEMMA. *Suppose* $\sigma(A)$ *is connected,* $\sigma(A)$ *is not a singleton,* $\sigma_p(A) = \sigma_p(A^*) = \emptyset$ *and* $\sigma \in \mathscr{F}_G$, $y \in Y$, $x^* \in X_T(\sigma)^\perp$, $\{\mu_k\}_{k=1}^m \subset \sigma(A) \cap G \cap \operatorname{int}\sigma$,

$\{c_k\}_{k=1}^m \subset \mathbf{C}$ *are given. Then for any* $\varepsilon > 0$, *we can find* $\delta \in \mathscr{F}_G$, $\delta \subset \sigma$ *and* $u \in Y$, $u^* \in X_T(\delta)^\perp$ *such that*

$$\|y - u\| \leqslant \left(\sum_{k=1}^m |c_k|\right)^{1/2}, \quad \|x^* - u^*\| \leqslant 4a_T^2\left(\sum_{k=1}^m |c_k|\right)^{1/2},$$

$$\left\| u \overset{\delta}{\otimes} u^* - y \overset{\sigma}{\otimes} x^* - \sum_{k=1}^m c_k \mathscr{E}_{\mu_k} \right\| < \varepsilon.$$

Proof. Without loss of generality we may suppose that we have $\sum_{k=1}^m |c_k| = 1$. Fix $r > 0$ and put

$$D(\mu_k) = \{\lambda \in \mathbf{C}: 0 < |\lambda - \mu_k| < r\}, \quad G_0 = \bigcup_{k=1}^m D(\mu_k), \quad \delta = \sigma \setminus \bigcup_{k=1}^m D(\mu_k).$$

We shall assume that r is enough small to have

$$D(\mu_k)^- \subset G \cap \operatorname{int}\sigma, \quad D(\mu_k)^- \cap D(\mu_j)^- = \emptyset \qquad k \neq j$$

$$\left\| \sum_{k=1}^m c_k(\mathscr{E}_{\mu_k} - \mathscr{E}_{\zeta_k}) \right\| < \varepsilon/4, \quad \zeta_k \in D(\mu_k)^-.$$

Applying Lemma 2.2 we can find two subspaces $Y' \subset Y$, $X_0^* \subset X_T(\delta)^\perp$ such that $\dim(Y/Y') < \infty$, $\dim(X_T(\delta)^\perp / X_0^*) < \infty$ and

$$\|y' \overset{\sigma}{\otimes} x^*\| < \varepsilon/4\|y'\|, \ y' \in Y', \quad \|y \overset{\delta}{\otimes} x_0^*\| < (\varepsilon/4a_T^2)\|x_0^*\|, \ x_0^* \in X_0^*.$$

If we put

$$Z = \{z \in X: \ x_0^*(z) = 0, \ x_0^* \in X_0^*\},$$

we have $Z \supset X_T(\delta)$ and $\dim Z^\delta < \infty$. Thus by Lemma 1.2 we can find a closed set $\delta_0 \subset G_0$ such that

$$\|x\| \leqslant (5/2)\, a_T\|x + z\|, \quad z \in Z, \quad \sigma_T(x) \subset G_0, \quad \sigma_T(x) \cap \delta_0 = \emptyset.$$

The assumptions on A allow us to pick

$$\zeta_k \in (\sigma(A) \cap D(\mu_k)) \setminus \delta_0, \quad y_k(\eta) \in Y', \quad \eta > 0$$

such that

$$\|y_k(\eta)\| = 1, \quad \|(T - \zeta_k)\, y_k(\eta)\| < \eta.$$

Taking η enough small and applying Lemma 1.1, we may suppose that $\operatorname{dist}(y_k(\eta), X_T(D(\mu_k)^-))$ is arbitrarily small, consequently

$$\left\| \sum_{k=1}^m e^{i\theta_k} \alpha_k y_k(\eta) \right\| \leqslant (4/3)\, a_T \left\| \sum_{k=1}^m \alpha_k y_k(\eta) \right\|.$$

If Y_η denotes the finite-dimensional subspace generated by $\{y_k(\eta)\}_{k=1}^m$, then by [2], Lemma 4.3 we can find $y_\eta^* \in Y_\eta^*$ and $\{\alpha_k(\eta)\}_{k=1}^m \subset \mathbf{C}$ such that

$$\left\| \sum_{k=1}^m \alpha_k(\eta)\, y_k(\eta) \right\| \leqslant 1, \quad \|y_\eta^*\| \leqslant (4/3)\, a_T, \quad \alpha_k(\eta)\, y_\eta^*(y_k(\eta)) = c_k.$$

Let $\delta' \subset G_0 \setminus \delta_0$ be a closed set such that $\zeta_k \in \operatorname{int}\delta'$. Applying again Lemma 1.1 we get

$$\lim_{\eta\to\infty} \sup\{\operatorname{dist}(y_\eta, X_T(\delta')) : y_\eta \in Y_\eta,\ \|y_\eta\| \leqslant 1\} = 0.$$

Thus we may suppose further that we have

$$\|y_\eta\| \leqslant 3a_T \|y_\eta + z\|, \quad z \in Z,\ \forall\, y_\eta \in Y.$$

Let $z_\eta^* \in (Y_\eta + Z)^*$ be defined by the equation

$$z_\eta^*(y_\eta + z) = y_\eta^*(y_\eta), \quad y_\eta \in Y_\eta, \quad z \in Z,$$

and extend z_η^* to an element $v_\eta^* \in X^*$ by the Hahn-Banach Theorem. It is easy to see that we have $\|z_\eta^*\| \leqslant 4a_T^2$, $z_\eta^*(Z) = \{0\}$, hence $\|v_\eta^*\| \leqslant 4a_T^2$, $v_\eta^* \in X_0^*$. Finally setting $v(\eta) = \sum_{k=1}^m \alpha_k(\eta) y_k(\eta)$, we observe that by Lemma 2.1 we may suppose that we have $\left\| \sum_{k=1}^m c_k \mathscr{E}_{\zeta_k} - v(\eta) \overset{\delta}{\otimes} v_\eta^* \right\| < \varepsilon/4$. Thus if we put $u = y + v(\eta)$, $u^* = x^* + v_\eta^*$ we derive

$$\left\| u \overset{\delta}{\otimes} u^* - y \overset{\sigma}{\otimes} x^* - \sum_{k=1}^m c_k \mathscr{E}_{\mu_k} \right\| \leqslant$$

$$\leqslant \|v(\eta) \overset{\sigma}{\otimes} x^*\| + \|x \overset{\delta}{\otimes} v_\eta^*\| + \left\| v(\eta) \overset{\delta}{\otimes} v_\eta^* - \sum_{k=1}^m c_k \mathscr{E}_{\zeta_k} \right\| + \left\| \sum_{k=1}^m c_k (\mathscr{E}_{\mu_k} - \mathscr{E}_{\zeta_k}) \right\| < \varepsilon,$$

and this concludes the proof.

2.4. Lemma. *Suppose $\sigma(A) \cap G$ is dominating in G, $\sigma(A)$ is connected and $\sigma_p(A) = \sigma_p(A^*) = \emptyset$ and let $\mu \in G$, $0 < b < 1$ be given. Then there exist $\{y_n\}_{n=0}^\infty \subset Y$, $\{\sigma_n\}_{n=0}^\infty \subset \mathscr{F}_G$, $x_\eta^* \in X_T(\sigma_n)^\perp$ such that*

$$\|y_{n+1} - y_n\| < b^{n-1}, \|x_{n+1}^* - x_n^*\| < 4a_T^2 b^{n-1}, \ \|\mathscr{E}_\mu - y_n \overset{\sigma_n}{\otimes} x_n^*\| < b^{2(n-1)}.$$

Proof. Proceeding by induction assume that $\{y_j\}_{j=0}^n$, $\{\sigma_j\}_{j=0}^n$, $\{x_j^*\}_{j=0}^n$ are determined, with $y_0 = 0$, $\sigma_0 = \mathbf{C}$, $x_0^* = 0$. Since $\sigma' = \sigma(A) \cap G \cap \operatorname{int}\sigma_n$ is obviously dominating in G, using [4], Lemma 4.4 (or [5], Proposition 2.8), we can find

$\{c_k\}_{k=1}^m \subset \mathbf{C}$, $\{\mu_k\}_{k=1}^m \subset \sigma'$ such that

$$\sum_{k=1}^m |c_k| < b^{2(n-1)}, \quad \left\| \mathscr{E}_\mu - y_n \overset{\sigma_n}{\otimes} x_n^* - \sum_{k=1}^m c_k \mathscr{E}_{\mu_k} \right\| < b^{2n}.$$

Now we can determine y_{n+1}, σ_{n+1}, x_{n+1}^* by Lemma 2.3.

2.5 PROPOSITION. *If $\sigma(A) \cap G$ is dominating in G, then there exists a proper subspace in Y invariant for $(\lambda - A)^{-1}$, $\lambda \notin \sigma(A) \cup G^-$.*

Proof. We may assume that T has no proper hyperinvariant subspace, consequently, $\sigma(A)$ is connected and $\sigma_p(A) = \sigma_p(A^*) = \emptyset$. Let $\mu \in G$, $0 < b < 1$ be given and let $\{y_n\}_{n=0}^\infty \subset Y$, $\{\sigma_n\}_{n=0}^\infty \subset \mathscr{F}_G$, $x_n^* \in X_T(\sigma_n)^\perp$ be determined by Lemma 2.4. If we put

$$y' = \lim_{n\to\infty} y_n, \quad y = (A - \mu)y', \quad x^* = \lim_{n\to\infty} x_n^*$$

we have

$$1 = \mathscr{E}_\mu(1) = \lim_{n\to\infty} x_n^*(y_n) = x^*(y'),$$

thus $y \neq 0$, $x^* \neq 0$. Let Y_0 denote the invariant subspace of A generated by $(\lambda - A)^{-1}(y)$, $\lambda \notin \sigma(A) \cup G^-$. If f is a rational function with poles off $\sigma(A) \cup G^-$ and if we put $f^\mu(\lambda) = f(\lambda)(\lambda - \mu)$ we have $f \in H^\infty(G)$, $f^\mu \in H^\infty(G)$ and

$$x^*(f(A)y) = x^*(f^\mu(A)\,y') = \lim_{n\to\infty} x_n^*(f^\mu(A)y_n) = f^\mu(\mu) = 0.$$

Consequently, $y \in Y_0$, $y' \notin Y_0$ and this shows that Y_0 is a proper subspace.

2.6. THEOREM. *If $\operatorname{int}\sigma(A) \neq \emptyset$ then there exists a proper subspace in Y invariant for $(\lambda - A)^{-1}$, $\lambda \notin \sigma(A)$.*

Proof. We apply Proposition 2.5 with G a connected component of $\operatorname{int}\sigma(A)$.

2.7. THEOREM. *If G is simply connected,* $\mathrm{R}((\sigma(A) \cap G)^-) \neq \mathscr{C}((\sigma(A) \cap G)^-)$, *then there exists a proper subspace in Y invariant for $(\lambda - A)^{-1}$, $\lambda \notin \sigma(A) \cup G^-$.*

Proof. Using Theorem 2.6 we shall assume $\operatorname{int}\sigma(A) = \emptyset$. If we put $K = (\sigma(A) \cap G)^-$, we have $K \neq \emptyset$, $\operatorname{int}K = \emptyset$, and by [13], Theorem 9.3, $\mathrm{R}(K)$ is not a Dirichet algebra in the sense of [12], II, § 3. Let $\mathscr{F}$ denote the family of all compact sets $\sigma \subset \mathbf{C}$ such that

(i) σ is a union of K with a union of bounded connected components of $\mathbf{C} \setminus K$;

(ii) $\mathrm{R}(\sigma)$ is a Dirichlet algebra.

As in the proof of [3], Theorem 1 or [15], Section 5, we can find a minimal element $\delta \in \mathscr{F}$. Let Ω denote a connected component of $\operatorname{int}\delta$, $\Omega \neq \emptyset$. We shall prove that $\sigma(A) \cap \Omega$ is dominating in Ω, or equivalently $K \cap \Omega$ is dominating in Ω.

Indeed, in the contrary case we can find $f \in H^\infty(\Omega)$ such that

$$\sup_{\lambda \in K \cap \Omega} |f(\lambda)| < \sup_{\lambda \in \Omega \setminus K} |f(\lambda)| = \|f\|_\infty$$

and this obviously implies that $\Omega \setminus K$ has a connected component Ω_0 such that $(\partial \Omega_0) \cap (\partial \Omega) \neq \emptyset$. It is easy to check that Ω_0 is a connected component of $\mathbf{C} \setminus K$, thus $F = \delta \setminus \Omega_0$ is a member of $\mathscr{F}$, and by [13], Corollary 9.7, $\mathrm{R}(F)$ is a Dirichlet algebra, contradicting the minimality of δ. The conclusion is that $\sigma(A) \cap \Omega$ is dominating in Ω and because obviously $\Omega \subset \mathrm{int} G^-$, we terminate the proof applying Proposition 2.5.

Our last result is an improvement of [3], Theorem 1. Recall that T is called *quasiscalar* if there exists a continuous linear multiplicative extension

$$\mathscr{V} : \mathscr{C}(\sigma(T)) \to \mathscr{L}(X),$$

of the Riesz-Dunford functional calculus of T with analytic functions. Quasiscalar operators are unconditionally decomposable as seen in the proof of [3], Lemma 3.

2.8. THEOREM. *Suppose T is quasiscalar and Ω is an open disk such that $\sigma(A) \cap \Omega \neq \emptyset$. Then there exists a proper subspace in Y, invariant for $(\lambda - A)^{-1}$, $\lambda \notin \sigma(A) \cup \Omega^-$.*

Proof. If $\mathrm{R}((\sigma(A) \cap \Omega)^-) \neq \mathscr{C}((\sigma(A) \cap \Omega)^-)$, we can apply Theorem 2.7. Thus we shall assume $\mathrm{R}((\sigma(A) \cap \Omega)^-) = \mathscr{C}((\sigma(A) \cap \Omega)^-)$. If $\sigma(A) \subset \Omega$, then A is a quasiscalar operator and obviously A has proper rationally invariant subspaces. If $\sigma(A) \not\subset \Omega$, then proceeding as in the proof of [6], Theorem 1 we can find a proper hyperinvariant subspace for A.

REMARK. In particular, Theorem 2.8 will give the following generalization of Scott Brown's Theorem [4]: "Let S be a subnormal non-normal operator and let Ω be an open disk such that $\sigma(S) \cap \Omega \neq \emptyset$. Then there exists a proper subspace invariant for $(\lambda - S)^{-1}$, $\lambda \notin \sigma(S) \cup \Omega^-$".

REFERENCES

1. APOSTOL, C., Spectral decompositions and functional calculus, *Rev. Roumaine Math. Pures Appl.*, **13**(1968), 1481–1528.
2. APOSTOL, C., Functional calculus and invariant subspaces, *J. Operator Theory*, **4**(1980), 159–190.
3. APOSTOL, C., Invariant subspaces for subquasiscalar operators, *J. Operator Theory*, **3** (1980), 159–164.
4. BROWN, S., Some invariant subspaces for subnormal operators, *J. Integral Eq. and Operator Theory*, **1**(1979), 310–333.

5. BROWN, S.; CHEVREAU, B.; PEARCY, C. M., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
6. CLANCEY, K.; PUTNAM, C. R., Normal parts of certain operators, *J. Math. Soc. Japan*, **24**(1972), 198–203.
7. COLOJOARĂ, I.; FOIAŞ, C., *Generalized Spectral Operators*, Gordon and Breach, 1968.
8. DUNFORD, N., Spectral operators, *Pacific J. Math.*, **4**(1954), 321–354.
9. DUNFORD, N.; SCHWARTZ, J. T., *Linear Operators*. I, Interscience Publishers, New York, 1958.
10. FOIAŞ, C., Une application de distributions vectorielles à la théorie spectrale, *Bull. Sci. Math.*, **84**(1960), 147–158.
11. FOIAŞ, C., Spectral maximal spaces and decomposable operators, *Archiv der Math.*, **14**(1963), 341–349.
12. GAMELIN, T., *Uniform Algebras*, (Russian translation), 1973.
13. GAMELIN, T.; GARNETT, J., Pointwise bounded approximation and Dirichlet algebras, *J. Functional Analysis*, **8**(1971), 360–404.
14. RUBEL, L. A.; SHIELDS, A. L., The space of bounded holomorphic functions on a region, *Ann. Inst. Fourier (Grenoble)*, **16**(1966), 235–277.
15. CHAUMAT, J., Adherence faible-étoilé d'algèbres de functions rationelles, *Ann. Inst. Fourier, (Grenoble)*, **24**(1974), 93–120.

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