J. OPERATOR THEORY
6(1981), 13–23

POSITIVE CONES AND L^p-SPACES ASSOCIATED WITH A VON NEUMANN ALGEBRA

HIDEKI KOSAKI

0. INTRODUCTION

By using the Tomita-Takesaki theory, Araki, [2], introduced a one parameter family of positive cones P^α, $\alpha \in [0,1/2]$, associated with a von Neumann algebra $\mathscr{M}$ with a cyclic and separating vector ξ_0. On the other hand, Connes, [5], Haagerup, [9], and Hilsum, [10], developed a theory of non-commutative L^p-spaces for an arbitrary von Neumann algebra.

In the present paper, first we represent the algebra in question on a Hilbert space $L^2(\mathscr{M})$ (see § 1.1), consisting of certain operators, and obtain simple relations between the positive cones and non-commutative L^p-spaces. Based on this observation, we consider certain problems concerning the positive cones. Among other results, we obtain the necessary and sufficient conditions for a normal state to admit a representative vector in P^α, $\alpha \in]1/4, 1/2]$.

We shall freely use the standard results and notations of the relative modular theory as well as the Tomita-Takesaki theory, which are found in [3], [7], and [17].

Some results in the paper were obtained while the author stayed at the Centre de Physique Théorique, C.N.R.S., Marseille (France), University of Odense (Denmark), and University of Trondheim (Norway), and he would like to express his sincere gratitudes to Professor D. Kastler, for his invitation and warm hospitality, and Professors U. Haagerup and C. Skau for stimulating discussions on the present materials. At last, but his most sincere gratitude goes to Professor M. Takesaki for discussions and constant encouragement. Also, the author is indebted to the referee for improvement of the presentation of the paper.

1. PRELIMINARIES

In this section, we fix notations and collect some basic facts for later use.

1.1 STANDARD FORM. ([8], [9])

Let $\mathcal{M}$ be a von Neumann algebra. By making use of a crossed product and a dual action, [18], we can construct a pair $(\mathcal{M}_0, \theta)$, which is unique up to isomorphism, consisting of a semi-finite von Neumann algebra $\mathcal{M}_0$ and an automorphism group θ_s $(s \in \mathbf{R})$ of $\mathcal{M}_0$, whose fixed point subalgebra $\mathcal{M}_0^\theta$ is exactly $\mathcal{M}$. The algebra $\mathcal{M}_0$ admits a faithful semifinite normal trace τ satisfying $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbf{R}$, and, for any semi-finite normal weight φ on $\mathcal{M}$, we denote the Radon-Nikodym derivative of the dual weight $\hat{\varphi}$ on $\mathcal{M}_0$ with respect to τ by h_φ. Haagerup, [9], showed that h_φ is τ-measurable in the sense of Segal, [15], if and only if $\varphi \in \mathcal{M}_*^+$, and constructed L^p-spaces, $p \in [1, \infty]$, associated with $\mathcal{M}$. He defined $L^p(\mathcal{M})$ as a set of all τ-measurable operators h (affiliated with $\mathcal{M}_0$) satisfying $\theta_s(h) = e^{-s/p}h$, $s \in \mathbf{R}$, so that we can freely add and multiply elements by using the concepts of strong sums and strong products, [15]. Among L^p-spaces, $L^2(\mathcal{M})$ turns out to be a Hilbert space and $\{\pi_l(\mathcal{M}), L^2(\mathcal{M}), J, L^2(\mathcal{M})_+\}$ is a standard form, [8], where a representation of $\mathcal{M}$ on $L^2(\mathcal{M})$ is given by $\pi_l(x)h = xh$, $x \in \mathcal{M}$, $h \in L^2(\mathcal{M})$, and a conjugation J is given by $Jh = h^*$, $h \in L^2(\mathcal{M})$.

Throughout the paper, we identify $\mathcal{M}$ with $\pi_l(\mathcal{M})$, and fix a standard form $\{\mathcal{M}, L^2(\mathcal{M}), J, L^2(\mathcal{M})_+\} = \{\mathcal{M}, \mathcal{H}, J, \mathcal{P}^\natural\}$ and a unit cyclic and separating vector $h_0^{1/2} = h_{\varphi_0}^{1/2}$ in $\mathcal{P}^\natural$, whose vector state is φ_0.

1.2 RELATIVE MODULAR OPERATORS. ([3], [7], [17])

For each $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{1/2}$ is a unique representative vector in $\mathcal{P}^\natural$ for φ. A densely defined (conjugate linear) operator $S_{\varphi\varphi_0}: xh_0^{1/2} \in \mathcal{M}h_0^{1/2} \to x^*h_\varphi^{1/2} \in \mathcal{M}h_\varphi^{1/2}$ is closable and its closure admits the polar decomposition $S_{\varphi\varphi_0} = J\Delta_{\varphi\varphi_0}^{1/2}$. Here $\Delta_{\varphi\varphi_0}$ is a positive self-adjoint operator on $\mathcal{H}$ (called a relative modular operator) whose support is $[\mathcal{M}'h_\varphi^{1/2}] = [h_\varphi^{1/2}\mathcal{M}] = p\mathcal{H}$, where p is the support of φ as a linear functional. We notice that $\Delta_{\varphi\varphi_0}^\alpha$ is realized by

$$\mathcal{D}(\Delta_{\varphi\varphi_0}^\alpha) = \{h \in L^2(\mathcal{M});\ h_\varphi^\alpha h h_0^{-\alpha} \in L^2(\mathcal{M})\},$$
$$\Delta_{\varphi\varphi_0}^\alpha h = h_\varphi^\alpha h h_0^{-\alpha}, \quad h \in \mathcal{D}(\Delta_{\varphi\varphi_0}^\alpha).$$

(See § 2, [11], for the meaning of $h_\varphi^\alpha h h_0^{-\alpha}$.)

Since φ_0 is kept fixed throughout the paper, we write $\Delta = \Delta_{\varphi_0\varphi_0}$ (the usual modular operator associated with $(\mathcal{M}, h_0^{1/2})$) and denote its associated modular automorphism group on $\mathcal{M}$ by σ_t, $t \in \mathbf{R}$. A set of all elements $x \in \mathcal{M}$ such that the Fourier transform (as a distribution) of a map: $t \in \mathbf{R} \to \sigma_t(x) = h_0^{it}xh_0^{-it} \in \mathcal{M}$ is supported by a compact set is denoted by $\mathcal{A}$. Thus $\mathcal{A}$ is a σ-weakly dense $*$-subalgebra of $\mathcal{M}$, and, for each $x \in \mathcal{A}$, a map: $t \in \mathbf{R} \to \sigma_t(x) \in \mathcal{M}$ extends (uniquely) to an $\mathcal{M}$-valued entire function whose value at $z \in \mathbf{C}$ is denoted by $\sigma_z(x)$.

We notice that $\Delta_{\varphi\varphi_0}^{it}\Delta^{-it}$, $t \in \mathbf{R}$, is nothing but the Radon-Nikodym cocycle $(D\varphi: D\varphi_0)_t$, [3]. Originally $(D\varphi: D\varphi_0)_t$ was defined and studied for a faithful φ. However, almost all properties of $(D\varphi: D\varphi_0)_t$ remain valid for a non faithful φ with natural modification.

1.3 POSITIVE CONES. ([2], [11])

The following one parameter family of closed positive cones in $\mathscr{H}$ is the fundamental object of the paper.

DEFINITION 1.1. (Araki, [2]). For each $\alpha \in [0,1/2]$, P^α is *the closure of a positive cone* $\Delta^\alpha \mathscr{M}_+ h_0^{1/2}$, where $\mathscr{M}_+$ is the set of all positive elements in $\mathscr{M}$.

We notice that $P^0 = \mathscr{P}^\#$, $P^{1/4} = \mathscr{P}^\natural = L^2(\mathscr{M})_+$, and $P^{1/2} = \mathscr{P}^\flat$, [4], [17], and Araki, [2], observed that

$$P^{1/2-\alpha} = JP^\alpha = (P^\alpha)',$$

the dual cone of P^α.

We showed the following Radon-Nikodym type theorem in [11]:

THEOREM 1.2. *For each* $\alpha \in [0, 1/4]$ *($\alpha \in [0, 1/2]$ if $\mathscr{M}$ is finite), the map* $\pi_\alpha : k \in P^\alpha \to \omega_k \in \mathscr{M}_*^+$ *is bijective. Here* $\omega_k \in \mathscr{M}_*^+$ *means* $\omega_k(x) = (xk \mid k)$, $x \in \mathscr{M}$.

1.4. CONNES-HILSUM'S L^p-SPACES, $p \in [1, \infty]$. ([5], [10])

We briefly describe Connes-Hilsum's L^p-spaces in our frame. In our realization of a standard form, $\mathscr{M}'$ is exactly $\pi_r(\mathscr{M})$, where π_r is an anti-representation of $\mathscr{M}$ on $\mathscr{H}$ given by $\pi_r(x)h = hx$, $x \in \mathscr{M}$, $h \in \mathscr{H}$. We write the vector state on $\mathscr{M}'$ given by $h_0^{1/2} \in \mathscr{P}^\natural$ simply by φ_0', that is, $\varphi_0'(\pi_r(x)) = (h_0^{1/2}x \mid h_0^{1/2})$, $x \in \mathscr{M}$.

DEFINITION 1.3. (Connes, [5]). Let α be a real number. An (unbounded) operator T on $\mathscr{H}$ is *α-homogeneous with respect to* φ_0', if T is densely defined and closed, and its polar decomposition $T = u \mid T \mid$ satisfies:

(i) the phase part u belongs to $\mathscr{M}$,

(ii) $\mid T \mid^{it} \pi_r(x) = \pi_r(\sigma_{-\alpha t}(x)) \mid T \mid^{it}$, $x \in \mathscr{M}$, $t \in \mathbf{R}$.

A (-1)-homogeneous positive self-adjoint operator T with respect to φ_0' is *φ_0'-integrable*, if $h_0^{1/2}$ belongs to the domain of $T^{1/2}$.

Connes-Hilsum's $L^p(\mathscr{M}; \varphi_0')$, $p \in [1,\infty]$, consists of all $(-1/p)$-homogeneous operators $T = u\,|T|$ with respect to φ_0' such that $|T|^p$ is φ_0'-integrable. Therefore, it is reasonable to say that T is affiliated with $L^p(\mathscr{M}; \varphi_0')$ if T is $(-1/p)$-homogeneous with respect to φ_0'.

It is easily shown that $L^p(\mathscr{M}; \varphi_0')_+ = \{\Delta_{\varphi\varphi_0}^{1/p}; \varphi \in \mathscr{M}_*^+\}$ and the correspondence:

$$T = u\Delta_{\varphi\varphi_0}^{1/p} = uh_\varphi^{1/p} \cdot h_0^{-1/p} \in L^p(\mathscr{M}; \varphi_0') \to uh_\varphi^{1/p} \in L^p(\mathscr{M}),$$

is indeed an isomorphism between two Banach spaces. For any elements in $L^p(\mathscr{M})$ with $p > 0$, (not necessarily $p \geq 1$) one can perform a strong product. Thus, the product of $\Delta_{\varphi\varphi_0}^\alpha$ and $\Delta_{\psi\varphi_0}^\beta$ ($\varphi, \psi \in \mathscr{M}_*^+$; $\alpha, \beta \geq 0$) is densely defined closable operator on $\mathscr{H}$. Furthermore, we have

$$((\Delta_{\varphi\varphi_0}^\alpha \Delta_{\psi\varphi_0}^\beta)^- \Delta_{\chi\varphi_0}^\gamma)^- = (\Delta_{\varphi\varphi_0}^\alpha (\Delta_{\psi\varphi_0}^\beta \Delta_{\chi\varphi_0}^\gamma)^-)^-,$$

for $\varphi, \psi, \chi \in \mathscr{M}_*^+$; $\alpha, \beta, \gamma > 0$. Therefore, whenever α and β are positive, one can omit the closure sign of $(\Delta_{\varphi\varphi_0}^\alpha \Delta_{\psi\varphi_0}^\beta)^-$ without making any confusion.

2. TECHNICAL LEMMAS

In this section, we prepare some technical lemmas.

LEMMA 2.1. *Let α be a real number and T be a densely defined closed operator on $\mathscr{H}$. If the domain of T is invariant under any $\pi_r(x)$, $x \in \mathscr{A}$, and*

$$(1) \qquad T\pi_r(x)h = \pi_r(\sigma_{\alpha i}(x))Th, \qquad h \in \mathscr{D}(T), \ x \in \mathscr{A},$$

then T is α-homogeneous with respect to φ_0'.

Proof. Let $T = u|T|$ be the polar decomposition of T. Then (1) is easily shown to be valid for T^* and for $T^*T = |T|^2$ with 2α instead of α. Thus, by the proof of Lemma 3.2, [11], we know that the entire function

$$f_1(z) = (\pi_r(x)k | \, |T|^{2\bar{z}}h)$$

is identical to another entire function

$$f_2(z) = (\pi_r(\sigma_{2\alpha i z}(x))T^{2z}h \mid k),$$

where h, $k \, (\in \mathscr{H})$ belong to a bounded spectral subspace of $|T|$. Thus, by putting $z = it/2$, $t \in \mathbf{R}$, we conclude that

$$(2) \qquad |T|^{it}\pi_r(x) = \pi_r(\sigma_{-\alpha t}(x)) \, |T|^{it}, \qquad x \in \mathscr{A}, \ t \in \mathbf{R}.$$

Since $\mathscr{A}$ is σ-weakly dense in $\mathscr{M}$, (2) is valid for any $x \in \mathscr{M}$, that is, $|T|$ is α-homogeneous with respect to φ_0'. Finally, u belongs to $\mathscr{M}$ due to Lemma 3.4, [11].

(Q.E.D.)

LEMMA 2.2. *Let p be a real number in* $[1, \infty]$. *If a positive self-adjoint operator T on $\mathscr{H}$ is affiliated with $L^p(\mathscr{M}; \varphi_0)$, and $h_0^{1/2}$ belongs to the domain of T, then $Th_0^{1/2}$ belongs to $P^{1/2p}$.*

Proof. Take any $x \in \mathscr{A}$ and h in a bounded spectral subspace of T and fix them. A function $f_1(z)$ (resp. $f_2(z)$)defined by

$$f_1(z) = (\pi_r(x)h_0^{1/2} | \, T^{\bar{z}}h)$$

$$(\text{resp. } f_2(z) = (T^z h_0^{1/2} | \, \pi_r(\sigma_{i\bar{z}/p}(x^*)) \, h))$$

is entire (resp. uniformly bounded and continuous on $\mathrm{Re} z \in [0,1]$, and analytic on $\mathrm{Re} z \in]0,1[$). We then notice that, for any pure imaginary $z = it$,

$$f_1(it) = (\pi_r(x)h_0^{1/2} | \, T^{-it}h) =$$

$$= (T^{it}\pi_r(x)h_0^{1/2} \mid h) =$$

$$= (\pi_r(\sigma_{1/p}(x)) \, T^{it}h_0^{1/2} \mid h) =$$

$$= (T^{it}h_0^{1/2} \mid \pi_r(\sigma_{1/p}(x^*)) \, h).$$

so that $f_1 = f_2$ by the uniqueness of analytic extensions. In particular, we have, with $z = 1$,

$$(\pi_r(x)h_0^{1/2} \mid Th) = (Th_0^{1/2} \mid \pi_r(\sigma_{i/p}(x^*))h) =$$
$$= (\pi_r(\sigma_{-i/p}(x))Th_0^{1/2} \mid h).$$

Since the union of all bounded spectral subspaces of T forms a core of T, we conclude that $T\pi_r(x)h_0^{1/2} = \pi_r(\sigma_{-i/p}(x))Th_0^{1/2}$. Thus, by the argument given in the proof of Lemma 3.5 (ii), [11], we have $Th_0^{1/2} \in P^{1/2p}$.

(Q.E.D.)

LEMMA 2.3. *Assume that a densely defined closed operator T, whose polar decomposition is $T = u \mid T \mid$, is affiliated with $L^p(\mathcal{M}; \varphi_0')$ for some $p \in [1, \infty]$, and that $h_0^{1/2}$ belongs to $\mathcal{D}(T) = \mathcal{D}(|T|)$. If $h = JTh_0^{1/2}$ and $k = J|T|h_0^{1/2}$, then $\omega_h = \omega_k$.*

Proof. For any $x \in \mathcal{M}$, we simply compute

$$\omega_k(x) = (xk \mid k) = (xJ \mid T \mid h_0^{1/2} \mid J \mid T \mid h_0^{1/2}) =$$
$$= (\mid T \mid h_0^{1/2} \mid JxJ \mid T \mid h_0^{1/2}) = (u^*u \mid T \mid h_0^{1/2} \mid JxJ \mid T \mid h_0^{1/2}) =$$
$$= (u \mid T \mid h_0^{1/2} \mid JxJu \mid T \mid h_0^{1/2}) = \qquad , u \in \mathcal{M},\ JxJ \in \mathcal{M}',$$
$$= (Th_0^{1/2} \mid JxJTh_0^{1/2}) =$$
$$= (xJTh_0^{1/2} \mid JTh_0^{1/2}) = \omega_h(x).$$

(Q.E.D.)

3. RELATIONS BETWEEN POSITIVE CONES AND L^p-SPACES

In this section, we establish simple relations between positive cones P^α, $\alpha \in [0, 1/2]$, and $L^p(\mathcal{M}, \varphi_0')$, $p \in [1, \infty]$. We begin with a characterization of the "bounded part" of $\mathcal{P}^\natural$.

PROPOSITION 3.1. *Let h be a vector in $\mathcal{P}^\natural$. Then there exists a positive element a in $\mathcal{M}$ with $h = aJaJh_0^{1/2} = ah_0^{1/2}a$ if and only if $\varphi_h \leqslant l\varphi_0$ (1/4) in the sense of Connes-Takesaki, § II, 4, [6], with some $l \geqslant 0$, where $\varphi_h \in \mathcal{M}_*^+$ is defined by $\varphi_h(x) = (\Delta^{1/4}xh_0^{1/2} \mid h)$. Furthermore, if the above condition is satisfied, then a is uniquely determined and $0 \leqslant a \leqslant l^{1/2}$.*

Proof. Since $\Delta^{1/4}$ is given by $h_0^{1/4} \cdot h_0^{-1/4}$, we compute, for any $x \in \mathcal{M}$,

$$\varphi_h(x) = (h_0^{1/4}xh_0^{1/4} \mid h) =$$
$$= \mathrm{tr}(hh_0^{1/4}xh_0^{1/4}) = \qquad \text{(see [9])}$$
$$= \mathrm{tr}(xh_0^{1/4}hh_0^{1/4}) =$$
$$= (x(h_0^{1/4}hh_0^{1/4})^{1/2} \mid (h_0^{1/4}hh_0^{1/4})^{1/2}),$$

with $(h_0^{1/4}hh_0^{1/4})^{1/2} \in \mathscr{P}^\natural$. Thus, by Lemma 3.13, [4], we notice that $\varphi_h \leqslant l\varphi_0(1/4)$ if and only if $(h_0^{1/4}hh_0^{1/4})^{1/2} \leqslant l^{1/2}h_0^{1/2}$ (as an operator).

At first, assume that $h = ah_0^{1/2}a$ with $a \in \mathscr{M}_+$. Then, by (a generalization of) [13] (for measurable operators, Proposition 7.6, [14]), we have

$$(h_0^{1/4}hh_0^{1/4})^{1/2} \leqslant l^{1/2}h_0^{1/2}, \qquad 0 \leqslant a \leqslant l^{1/2},$$

and a is uniquely determined.

Conversely, assume that $(h_0^{1/4}hh_0^{1/4})^{1/2} \leqslant l^{1/2}h_0^{1/2}$. Then, by [13] again, there exists a unique $a \in (\mathscr{M}_0)_+$ such that $h = ah_0^{1/2}a$. It suffices to show that $a \in \mathscr{M}_+$. However, for $s \in \mathbf{R}$, we compute

$$h = e^{s/2}\theta_s(h) = e^{s/2}\theta_s(ah_0^{1/2}a) =$$

$$= e^{s/2}\theta_s(a)\theta_s(h_0^{1/2})\theta_s(a) =$$

$$= \theta_s(a)\, h_0^{1/2}\theta_s(a),$$

so that $a = \theta_s(a)$ by the uniqueness of a.

(Q.E.D).

The following proposition generalizes a result due to Araki, [2]:

PROPOSITION 3.2. *Let h be a vector in $\mathscr{P}^\natural$. If $\omega_h \leqslant l\varphi_0(1/4)$ in the sense of Connes-Takesaki for some $l \geqslant 0$, then there exists a unique $a \in \mathscr{M}_+$ such that $h = aJaJh_0^{1/2}$. Furthermore, we have $0 \leqslant a \leqslant l^{1/4}$.*

Proof. The assumption on h yields that $h^{1/2}(l^{1/4}h_0^{1/4})^{-1}$ is bounded, that is, $h \leqslant l^{1/2}h_0^{1/2}$ as an operator. Thus, we have the result by [13].

(Q.E.D.)

Now we exhibit relations between the positive cones P^α, $\alpha \in [0,1/2]$, and $L^p(\mathscr{M}; \varphi_0')$, $p \in [1, \infty]$. At first, we assume that $p \in [2, \infty]$. Any $T \in L^p(\mathscr{M}; \varphi_0')_+$ has a form $T = \Delta_{\varphi\varphi_0}^{1/p}$, $\varphi \in \mathscr{M}_*^+$, so that $h_0^{1/2} \in \mathscr{D}(T)$ because $0 \leqslant 1/p \leqslant 1/2$. It follows from Lemma 2.2 that $Th_0^{1/2} \in P^{1/2p}$. Thus, $L^p(\mathscr{M}; \varphi_0')_+ h_0^{1/2} \subseteq P^{1/2p}$, however, actually we have more:

PROPOSITION 3.3. *For each $p \in [2, \infty]$, we have*

$$\{L^p(\mathscr{M}; \varphi_0')_+ h_0^{1/2}\}^- = P^{1/2p}.$$

Proof. It is enough to show that $\Delta^{1/2p}\mathscr{M}_+ h_0^{1/2} \subseteq L^p(\mathscr{M}; \varphi_0')_+h_0^{1/2}$. For each $x \in \mathscr{M}_+$, we compute

$$\Delta^{1/2p}x\, h_0^{1/2} = h_0^{1/2p}x\, h^{1/2-1/2p} =$$

$$= h_0^{1/2p}x\, h_0^{1/2p}h_0^{1/2-1/p}$$

with $1/2-1/p \geqslant 0$.

We notice that $h_0^{1/2p} x h_0^{1/2p} \in L^p(\mathcal{M}; \varphi_0')_+$, so that we conclude that $h_0^{1/2p} x h_0^{1/2p} = h_\varphi^{1/p}$ with unique $\varphi \in \mathcal{M}_*^+$. Thus, we conclude that

$$\Delta^{1/2p} x h_0^{1/2} = h_\varphi^{1/p} h_0^{1/2-1/p} = \Delta_{\varphi\varphi_0}^{1/p} h_0^{1/2}$$

belongs to $L^p(\mathcal{M}; \varphi_0')_+ h_0^{1/2}$.

(Q.E.D.)

Next, we consider the case that $p \in [1,2]$. Then $h_0^{1/2}$ does not necessarily belong to the domain of an operator in $L^p(\mathcal{M}; \varphi_0')_+$. We denote the set of all elements $T \in L^p(\mathcal{M}; \varphi_0')_+$ with $h_0^{1/2} \in \mathcal{D}(T)$ by $L^p(\mathcal{M}; \varphi_0')_+^\sim$. Then we have:

PROPOSITION 3.4. *For each $p \in [1, 2]$, the map:*

$$T \in L^p(\mathcal{M}; \varphi_0')_+^\sim \to T h_0^{1/2} \in P^{1/2p}$$

is bijective.

Proof. By Lemma 2.2, $Th_0^{1/2}$ certainly belongs to $P^{1/2p}$. To show the injectivity, assume that $T_1 h_0^{1/2} = T_2 h_0^{1/2}$ with $T_1 = \Delta_{\varphi_1\varphi_0}^{1/p} (\varphi_1 \in \mathcal{M}_*^+)$ and $T_2 = \Delta_{\varphi_2\varphi_0}^{1/p} (\varphi_2 \in \mathcal{M}_*^+)$. We then have

$$h_{\varphi_1}^{1/p} = (T_1 h_0^{1/2})\, h_0^{1/p-1/2} = (T_2 h_0^{1/2}) h_0^{1/p-1/2} = h_{\varphi_2}^{1/p},$$

that is, $T_1 = T_2$.

To show the surjectivity, take an arbitrary $h \in P^{1/2p}$ and set $k = Jh \in P^{1/2-1/2p}$ and $\varphi = \omega_k \in \mathcal{M}_*^+$. Since $T = \Delta_{\varphi\varphi_0}^{1/2} \Delta^{1/p-1/2} \in L^p(\mathcal{M}; \varphi_0')$ and $JTh_0^{1/2} = Jh_\varphi^{1/2} = h_\varphi^{1/2}$, we have $\omega_k = \varphi = \omega_l$ with $l = J|T|h_0^{1/2}$, by Lemma 2.3. Since k and l belong to $P^{1/2-1/2p}$, $1/2 - 1/2p \in [0,1/4]$, we have $k = l$ due to the injectivity of Theorem 1.2, that is,

$$h = |T|\, h_0^{1/2} \quad \text{with } |T| \in L^p(\mathcal{M}; \varphi_0')_+^\sim.$$

(Q.E.D.)

REMARK 3.5. For each $\xi \in \mathcal{P}^b$, it is known that there exists a positive self-adjoint operator A', affiliated with $\mathcal{M}'$ such that $A'\xi_0 = \xi$, [17]. However, unfortunately, this A' is not unique, unless $\mathcal{M}$ is finite, [12], [16]. The above proposition $(p = 1)$ is in great contrast to this ambiguity.

4. REPRESENTATIVE VECTORS FOR STATES IN POSITIVE CONES

A state always admits a representative vector in P^α, $\alpha \in [0,1/4]$, (Theorem 1.2). The following theorem gives us criteria for the state in question to admit a representative vector in P^α, $\alpha \in]1/4, 1/2]$:

THEOREM 4.1. *For a state $\varphi \in \mathcal{M}_*^+$ and a real number $p \in]2, \infty]$, the following four conditions are all equivalent:*

(i) *The (densely defined) operator $\Delta_{\varphi\varphi_0}^{1/2} \Delta^{1/p-1/2}$ is closable* $(1/p - 1/2 \leqslant 0)$.

(ii) *There exists an operator T affiliated with $L^p(\mathcal{M}; \varphi_0')$ such that*

$$h_0^{1/2} \in \mathcal{D}(T) \quad \textit{and} \quad h_\varphi^{1/2} = Th_0^{1/2}.$$

(iii) *There exists a vector $h \in P^{1/2-1/2p}$ such that $\varphi = \omega_h$.*

(iv) *There exists a vector $h \in \mathcal{D}(\Delta^{1/p-1/2})$ such that $\varphi = \omega_h$.*

Proof. ((i) $\Rightarrow$ (ii)) One can show that $\mathcal{D}(\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2})$ is invariant under $\pi_r(x)$, $x \in \mathcal{A}$, and

$$\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2}\pi_r(x)k = \pi_r(\sigma_{-i/p}(x))\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2}k, \quad k \in \mathcal{D}(\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2}).$$

Thus, passing to the closure $T = (\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2})^-$, we know that the above statement remains valid for T so that T is affiliated with $L^p(\mathcal{M}; \varphi_0')$ by Lemma 2.1 and

$$Th_0^{1/2} = \Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2}h_0^{1/2} = \Delta_{\varphi\varphi_0}^{1/2}h_0^{1/2} = h_\varphi^{1/2}.$$

((ii) $\Rightarrow$ (iii)) Since $JTh_0^{1/2} = Jh_\varphi^{1/2} = h_\varphi^{1/2}$, the vector $h = J|T|h_0^{1/2}$ belongs to $P^{1/2-1/2p}$ (Lemma 2.2) and $\varphi = \omega_h$ (Lemma 2.3).

((iii) $\Rightarrow$ (iv)) This is obvious because $P^{1/2-1/2p} \subseteq \mathcal{D}(\Delta^{1/p-1/2})$, [2].

((iv) $\Rightarrow$ (i)) Since $\varphi = \omega_h$ with $h \in \mathcal{D}(\Delta^{1/p-1/2})$, there exists a partial isometry $u' \in \mathcal{M}'$ such that

$$h_\varphi^{1/2} = u'h,$$

$$u'u'^* = \text{the projection onto } [\mathcal{M}h_\varphi^{1/2}],$$

$$u'^*u' = \text{the projection onto } [\mathcal{M}h].$$

Set $u = Ju'J$. Thus u is a partial isometry in $\mathcal{M}$ such that

$$h_\varphi^{1/2} = uJh,$$

$$uu^* = \text{the projection onto } [\mathcal{M}'h_\varphi^{1/2}] = p, \text{ the support projection of } \varphi,$$

$$u^*u = \text{the projection onto } [\mathcal{M}'Jh].$$

For $x = uy^*$, $y \in \mathcal{A}$, we compute

$$\Delta_{\varphi\varphi_0}^{1/2}xh_0^{1/2} = Jx^*h_\varphi^{1/2} = J(uy^*)^*uJh =$$

$$= Jyu^*uJh = JyJh,$$

so that $xh_0^{1/2}$ belongs to $\mathcal{D}(\Delta^{1/p-1/2}\Delta_{\varphi\varphi_0}^{1/2})$ since h belongs to $\mathcal{D}(\Delta^{1/p-1/2})$, which is invariant under JyJ. By noting that $\Delta^{1/p-1/2}\Delta_{\varphi\varphi_0}^{1/2}$ vanishes on $(1-p)\mathcal{H}$ and that

$\{xh_0^{1/2};\ x = uy^*,\ y \in \mathscr{A}\}$ is dense in $p\mathscr{H}$, we conclude that $\Delta^{1/p-1/2}\Delta_{\varphi\varphi_0}^{1/2}$ is a densely defined operator on $\mathscr{H}$, that is, $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/p-1/2}$ is closable.

(Q.E.D.)

As immediate consequences of the above theorem, we have three corollaries. The second one is a more precised version of Araki's one parameter family of Radon-Nikodym theorems, [1], while the third one can be considered as a "T-theorem" for L^p-spaces, $p \in]\,2, \infty]$.

COROLLARY 4.2. *Let α be a real number in* $[1/4, 1/2]$. *If a state φ admits a representative vector in P^α, then φ admits a representative vector in any P^β, $\beta \in [0, \alpha]$.*

Proof. It follows from the above theorem ((iii) $\Rightarrow$ (iv)) and Theorem 1.2.

(Q.E.D.)

COROLLARY 4.3. *Let λ be a real number in* $[0,1/2]$. *If a state $\varphi \in \mathscr{M}_*^+$ satisfies $\varphi \leqslant l\varphi_0(\lambda)$ in the sense of Connes-Takesaki with some $l \geqslant 0$, then φ admits a representative vector in any P^α, $\alpha \in [0,1/4 + \lambda/2]$.*

Proof. Set $T_\alpha = \Delta_{\varphi\varphi_0}^{1/2-\alpha}(\mathrm{D}\varphi : \mathrm{D}\varphi_0)_{-\alpha i}$, $\alpha \in [0, \lambda]$. By the uniqueness of analytic extensions, we easily show that $(\mathrm{D}\varphi : \mathrm{D}\varphi_0)_{-\alpha i} h_0^{1/2} = \Delta_{\varphi\varphi_0}^{\alpha} h_0^{1/2}$, that is, $T_\alpha h_0^{1/2} = \Delta_{\varphi\varphi_0}^{1/2} h_0^{1/2} = h_\varphi^{1/2}$. Since $T_\alpha \in L^{2/1-2\alpha}(\mathscr{M}; \varphi_0')$, the corollary follows from the above theorem ((ii) $\Leftrightarrow$ (iii)) and Theorem 1.2.

(Q.E.D.)

COROLLARY 4.4. *When $\mathscr{M}$ is a finite von Neumann algebra, for any $p \in [2, \infty]$ and any $h \in \mathscr{H}$, there exists an operator T affiliated with $L^p(\mathscr{M}; \varphi_0')$ such that $h_0^{1/2} \in \mathscr{D}(T)$ and $h = Th_0^{1/2}$.*

Proof. Theorem 1.2 and the above theorem ((ii) $\Leftrightarrow$ (iii)) yield that the statement in the corollary is valid for any k in $\mathscr{P}^\natural$. However, any element h in $\mathscr{H}$ can be written as $h = uk$ with a partial isometry u in $\mathscr{M}$ and k in $\mathscr{P}^\natural$.

(Q.E.D.)

For each $\alpha \in [0,1/4]$, the map $(\pi_\alpha)^{-1}$ from $\mathscr{M}_*^+$ onto P^α is bijective (Theorem 1.2). Furthermore, we have $(\pi_\alpha)^{-1}(\varphi) = J\,|\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2-2\alpha}|\,h_0^{1/2}$, where $|\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2-2\alpha}|$ denotes the absolute value part of the operator $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2-2\alpha}$. In fact, $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2-2\alpha}h_0^{1/2} = h_\varphi^{1/2}$ and $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2-2\alpha} \in L^{1/1-2\alpha}(\mathscr{M}; \varphi_0')$ so that we have the above explicit expression for $(\pi_\alpha)^{-1}$ by combining Lemma 2.2 and Lemma 2.3.

If $\alpha \geqslant 0$, the product of $\Delta_{\varphi\varphi_0}^{1/2}$ and Δ^α makes sense. (The last part of § 1.4.). Thus, the above argument also shows that $J\,|\,\Delta_{\varphi\varphi_0}^{1/2}\Delta^\alpha\,|\,h_0^{1/2}$ is a representative vector for φ.

THEOREM 4.5. *For any $\varphi \in \mathscr{M}_*^+$ and any positive number α, there exists a representative vector for φ in $\mathscr{D}(\Delta^\alpha)$.*

Proof. If suffices to show that $J|\Delta_{\varphi\varphi_0}^{1/2}\Delta^{\alpha}|\, h_0^{1/2}$ belongs to $\mathscr{D}(\Delta^{\alpha})$. Since $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{\alpha}$ is nothing but $h_{\varphi}^{1/2}h_0^{\alpha}\cdot h_0^{-1/2-\alpha}$, the absolute value $|\Delta_{\varphi\varphi_0}^{1/2}\Delta^{\alpha}|$ is given by $|h_{\varphi}^{1/2}h_0^{\alpha}|\cdot h_0^{-1/2-\alpha}$, where $h_{\varphi}^{1/2}h_0^{\alpha}=u\,|h_{\varphi}^{1/2}h_0^{\alpha}|$ denotes the polar decomposition of a τ-measurable operator $h_{\varphi}^{1/2}h_0^{\alpha}$. Thus, we compute

$$h=J\,|\,\Delta_{\varphi\varphi_0}^{1/2}\Delta_0^{\alpha}\,|\,h_0^{1/2}=$$

$$=(|\,h_{\varphi}^{1/2}h_0^{\alpha}\,|\,h_0^{1/2}h_0^{-1/2-\alpha})^*=$$

$$=(u^*h_{\varphi}^{1/2}h_0^{\alpha}h_0^{1/2}h_0^{-1/2-\alpha})^*=$$

$$=(u^*h_{\varphi}^{1/2})^*=h_{\varphi}^{1/2}u.$$

Since $|h_{\varphi}^{1/2}h_0^{\alpha}|=u^*h_{\varphi}^{1/2}h_0^{\alpha}\geqslant 0$,

$$h_0^{\alpha}h_{\varphi}^{1/2}u=|\,h_{\varphi}^{1/2}h_0^{\alpha}\,|=u^*h_{\varphi}^{1/2}h_0^{\alpha},$$

so that we have

$$h_0^{\alpha}hh_0^{-\alpha}=h_0^{\alpha}h_{\varphi}^{1/2}uh_0^{-\alpha}=$$

$$=u^*h_{\varphi}^{1/2}h_0^{\alpha}h_0^{-\alpha}=$$

$$=u^*h_{\varphi}^{1/2}\in L^2(\mathscr{M};\varphi_0').$$

Thus, h belongs to $\mathscr{D}(\Delta^{\alpha})$.

(Q.E.D.)

This work is supported in part by the National Science Foundation.

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HIDEKI KOSAKI
Department of Mathematics,
University of Kansas, Lawrence,
Kansas 66045,
U.S.A.

Received December 30, 1979.