J. OPERATOR THEORY
6(1981), 51–54

© Copyright by INCREST, 1981

OPERATORS COMMUTING WITH TOEPLITZ AND HANKEL OPERATORS MODULO THE COMPACT OPERATORS

MICHAEL J. HOFFMAN

For a separable Hilbert space H, let $\mathscr{K}(H)$ be the ideal of compact operators in the algebra $\mathscr{L}(H)$ of all bounded linear operators on H. For $\mathscr{S} \subseteq \mathscr{L}(H)$, the essential commutant is

$$\mathscr{S}^{\#} = \{T \in \mathscr{L}(H): ST - TS \in \mathscr{K}(H) \ \forall\, S \in \mathscr{S}\}.$$

In 1975, K. Davidson showed that the essential commutant of the algebra generated by all Toeplitz operators is exactly the set of compact perturbations of Toeplitz operators with quasi-continuous symbol [2]. Using this and D. Sarason's characterization of quasi-continuous functions as those essentially bounded functions with vanishing mean oscillation [9], S. C. Power has shown that adding the requirement that the operator have compact commutator with all Hankel operators imposes only the additional condition that the symbol have symmetric Fourier series [6, p. 56]. Thus the Hankel and Toeplitz operators together do not generate all operators. This paper presents a proof of Power's result using neither Davidson's theorem nor functions of vanishing mean oscillation.

Let D be the open unit disk in the complex plane and $L^2 = L^2(\partial D, \mathrm{d}\theta/2\pi)$. Let P be the orthogonal projection of L^2 onto the Hardy space H^2 of functions in L^2 whose Fourier coefficients of negative index are 0. Let U be the unitary operator on L^2 given by $U(e_k) = e_{-k}$ where e_k is the basis function $e_k(\mathrm{e}^{it}) = \mathrm{e}^{ikt}$. Equivalently, $(Uf)(\mathrm{e}^{it}) = f(\mathrm{e}^{-it})$. For $f \in L^\infty(\partial D) = L^\infty$, let $f^*(\mathrm{e}^{it}) = \overline{f(\mathrm{e}^{it})}$, and for $V \subseteq L^\infty$, let $V^* = \{f^*: f \in V\}$. Let M_f be the multiplication operator defined on L^2 by $M_f g = fg$; let T_f be the Toeplitz operator on H^2 given by $T_f g = PM_f g$, and let H_f be the Hankel operator on H^2 defined by $H_f g = PUM_f g$. Let $\mathscr{m}(V) = \{M_f: f \in V\}$, $\mathscr{T}(V) = \{T_f: f \in V\}$, and $\mathscr{H}(V) = \{H_f: f \in V\}$. If the difference of two operators T and S is compact, write $T \simeq S$.

We will use several subspace of L^∞. Let $H^\infty = H^2 \cap L^\infty$ and $C = C(\partial D)$. Sarason [7, p. 191] has shown that $H^\infty + C$ is a uniformly closed subalgebra of L^∞. It is not a $*$-algebra, but the space $QC = (H^\infty + C) \cap (H^\infty + C)^*$ of quasi-continuous functions is. We will also need the symmetric quasi-continuous functions

$$SQC = \{f \in QC : \hat{f}(n) = \hat{f}(-n) \ \forall\, n\},$$

where $\hat{f}(n)$ are the Fourier coefficients of f.

For $\mathscr{S} \subseteq \mathscr{L}(H)$, let $C^*(\mathscr{S})$ be the C^*-algebra generated by $\mathscr{S}$ and the identity. If $S^* \in \mathscr{S}$ whenever $S \in \mathscr{S}$, then $\mathscr{S}^\# = (C^*(\mathscr{S}))^\#$. The theorem to be proved is:

THEOREM.

$$(C^*(\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty)))^\# = \mathscr{T}(SQC) + \mathscr{K}(H^2).$$

Since $(T_f)^* = T_{f^*}$, and $(H_f)^* = H_{U(f^*)}$, the remark above shows that it suffices to show that

$$(\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty))^\# = \mathscr{T}(SQC) + \mathscr{K}(H^2).$$

In [3, Th. 7] and [8], R. G. Douglas and Donald Sarason show that $\mathscr{T}(L^\infty) \cap \mathscr{T}(L^\infty)^\# = \mathscr{T}(QC)$. To find the effect of the added assumption about Hankel operators, we use the fact that the compact Hankel operators are precisely those with symbol in $H^\infty + C$, [4].

If $g \in SQC$, then H_g and H_{Ug} are compact since g and Ug are in $H^\infty + C$. So for any $f \in L^\infty$,

$$H_f T_g = H_{fg} - T_{Uf} H_g + T_{Uf} E_0 H_g \simeq H_{fg},$$

and

$$T_g H_f = H_{fUg} - H_{Ug} T_f + H_{Ug} E_0 T_f \simeq H_{fUg}$$

where E_0 is the projection onto the space spanned by e_0. Also $g = Ug$ so that

$$H_f T_g - T_g H_f \simeq H_{fg} - H_{fUg} = H_{(g-Ug)f} = 0.$$

We conclude that

$$\mathscr{T}(SQC) + \mathscr{K}(H^2) \subseteq (\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty))^\#.$$

Conversely, suppose $A \in (\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty))^\#$. The idea is to extend A to an operator A'' on L^2 which has compact commutator with all multiplication operators. Since $\mathscr{M}(L^\infty)$ is a maximal abelian von Neumann algebra, it is equal to its commutant and a theorem of Johnson and Parrott, [5], shows that $\mathscr{M}(L^\infty)^\# = \mathscr{M}(L^\infty) + \mathscr{K}(L^2)$. First extend A to an operator A' on L^2 by letting the action be 0 on $(H^2)^\perp$. Let $A'' = UA'U + A'$. Then A'' may be thought of more or less in the form

$$A'' \simeq \begin{bmatrix} UAU & 0 \\ 0 & A \end{bmatrix}.$$

The "more or less" is a compact perturbation. U does not quite interchange H^2 and $(H^2)^\perp$. It leaves the one-dimensional span of e_0 fixed. So this representation is inaccurate along the center row and column by a compact perturbation. Similarly if f is in L^∞,

$$M_f \simeq \begin{bmatrix} UT_{Uf}U & UH_f \\ H_{Uf}U & T_f \end{bmatrix}.$$

A computation gives

$$A''M_f - M_fA'' \simeq \begin{bmatrix} U(AT_{Uf} - T_{Uf}A)U & U(AH_f - H_fA) \\ (AH_{Uf} - H_{Uf}A)U & (AT_f - T_fA) \end{bmatrix}.$$

The assumptions on A make this compact for every f in L^∞, so $A'' \in \mathfrak{m}(L^\infty)^\#$ and there is a g in L^∞ and an operator K in $\mathscr{K}(L^2)$ such that $A'' = M_g + K$. Thus,

$$\begin{bmatrix} UAU & 0 \\ 0 & A \end{bmatrix} \simeq \begin{bmatrix} UT_{Ug}U & UH_g \\ H_{Ug}U & T_g \end{bmatrix}.$$

This forces $A \simeq T_g$ so $A \in \mathscr{T}(L^\infty) + \mathscr{K}(H^2)$. Furthermore, since U is unitary, $A \simeq T_{Ug}$ and $T_{g-Ug} = T_g - T_{Ug} \simeq 0$. As there are no nonzero compact Toeplitz operators, $g = Ug$, and g is symmetric. Also $H_g \simeq 0$ so $g \in H^\infty + C$. Since $(\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty))^\#$ is a C^*-algebra, the argument may be repeated with A^* to give $A^* \simeq T_h$ with $h \in H^\infty + C$. But $A^* \simeq T_g^* = T_{g^*}$. So $0 \simeq T_{h-g^*}$ and $h = g^*$. Thus $g \in QC$. This completes the proof of the theorem. The computations suggested by the matrices above are readily carried out explicitly using P, $I - P$, and E_0.

COROLLARY. *The Toeplitz and Hankel operators together do not generate $\mathscr{L}(H^2)$ as a C^*-algebra.*

Proof. Since there are non-scalars in $\mathscr{T}(SQC)$ and there are no nonzero compact Toeplitz operators, $C^*(\mathscr{T}(L^\infty) \cup \mathscr{H}(L^\infty))^\#$ is strictly larger than $\mathbf{C}I + \mathscr{K}(H^2)$ which is $\mathscr{L}(H^2)^\#$. See [1].

Suppose D is a diagonal operator defined on H^2 by $De_n = d_ne_n$ and W is the weighted shift $We_n = d_ne_{n+k}$. The function

$$f(e^{it}) = 2\cos t = e^{-it} + e^{it}$$

is in SQC, and

$$(DT_f - T_fD)e_n = (d_{n-1} - d_n)e_{n-1} + (d_{n+1} - d_n)e_{n+1}.$$

If $d_{n+1} - d_n$ does not tend to 0, then $DT_f - T_fD$ is not compact. So D cannot be in the C^*-algebra generated by the Hankel and Toeplitz operators. Since $D = (T_z^*)^kW$, neither is W.

The author would like to thank Professor Donald Sarason for many helpful conversations and the referee for suggestions significantly shortening the proof.

REFERENCES

1. CALKIN, J., Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math.*, **42**(1941), 839–873.
2. DAVIDSON, K. R., On operators commuting with Toeplitz operators modulo the compact operators, *J. Functional Analysis*, **24**(1977), 291–302.
3. DOUGLAS, R. G., *Banach algebra techniques in the theory of Toeplitz operators*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics No. 15, American Mathematical Society, Providence, R. I., 1973.
4. HARTMAN, P., On completely continuous Hankel matrices, *Proc. Amer. Math. Soc.*, **9**(1958), 862–866.
5. JOHNSON, B. E.; PARROTT, S. K., Operators commuting with a von Neumann algebra modulo the set of compact operators, *J. Functional Analysis*, **11**(1972), 39–61.
6. POWER, S. C., C^*-algebras generated by Hankel and Toeplitz operators, *J. Functional Analysis*, **31**(1979), 52–68.
7. SARASON, D., Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127**(1967), 179–203.
8. SARASON, D., On products of Toeplitz operators, *Acta Sci. Math. (Szeged)*, **35**(1973), 7–12.
9. SARASON, D., Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.*, **207**(1975), 391–405.

MICHAEL J. HOFFMAN
Department of Mathematics 253–37,
California Institute of Technology,
Pasadena, California 91125,
U.S.A.

Received July 2, 1980.