J. OPERATOR THEORY
6(1981), 75–86

ON THE FORM SUM AND THE FRIEDRICHS EXTENSION OF SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

HANS L. CYCON

1. INTRODUCTION

Let Ω be an open set in $\mathbf{R}^n$. Consider the formal Schrödinger operator

(1.1) $$\tau := -\Delta + V,$$

where V is a real-valued function and

(1.2) $$V \in L^1_{\text{loc}}(\Omega).$$

The problem we are interested in is to determine a Hamiltonian associated with τ as a self-adjoint operator in the Hilbert space $L^2(\Omega)$. Since, regarding the high singularities allowed to V, there exists in general no "minimal" operator (i.e., a symmetric operator with the space of test functions $C_0^\infty(\Omega)$ as domain), Kato 1974 [11] suggested considering instead the "maximal" operator given by

(1.3) $$T_{\max} u := \tau u,$$

with domain

$$D(T_{\max}) := \{u \in L^2(\Omega) \mid Vu \in L^1_{\text{loc}}(\Omega),\ \tau u \in L^2(\Omega)\},$$

where τu is taken in the distribution sense. If $T_{\max}$ is not self-adjoint, the question arises whether there are self-adjoint restrictions of $T_{\max}$.

If we assume for a moment that

(1.4) $$V \in L^2_{\text{loc}}(\Omega)$$

instead of (1.2), then $T_{\min} := \tau \restriction C_0^\infty(\Omega)$ exists, $T_{\max} = T^*_{min}$ and the question is equivalent to that of asking for self-adjoint extensions of $T_{\min}$ [11]. The answer to this is highly ambiguous if $T_{\min}$ is not essentially self-adjoint, since then there are infinitely many such extensions [15].

Therefore one usually selects extensions which arise in a natural "physical" way [17,7]. [1]

In the case (1.2) we will proceed analogously. Let $V = V_0 + V_1$, and assume

(C_1) $V_0 \in L^2_{loc}(\Omega)$ real-valued such that $(-\Delta + V_0) \upharpoonright C_0^\infty(\Omega)$ is bounded from below

and

(C_2) $V_1 \in L^1_{loc}(\Omega)$ real-valued and bounded from below.

If T_0 denotes the Friedrichs extension of $(-\Delta + V_0) \upharpoonright C_0^\infty(\Omega)$, then we can use quadratic forms to define at least two self-adjoint restrictions of T_{max}. (Note that there is a one-to-one correspondence between semibounded closed quadratic forms and semibounded self-adjoint operators [9, Ch. VI]).

The first is the form sum $T_0 \dotplus V_1$, which is the self-adjoint operator associated with the sum of the semibounded closed forms of T_0 and V_1.

The second is the self-adjoint operator T_F associated with the closure of the minimal form, which is the restriction of the sum of the two forms of T_0 and V_1 to $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Obviously in the case (1.4) T_F is the Friedrichs extension of T_{min}; we therefore might call T_F the *Friedrichs restriction* of T_{max} in the general case (1.2).

If $V_0 = 0$, then the domains of both the form sum and the Friedrichs restriction are contained in the form domain of the "free" Hamiltonian $H_0 := (-\Delta) \upharpoonright C_0^\infty(\Omega)$ [2]. This can be interpreted physically to mean that both operator domains contain only states with finite kinetic energy. In this sense both T_F and $T_0 \dotplus V_1$ can be considered as possible "physical" Hamiltonians associated with τ.

But as long as T_F and $T_0 \dotplus V_1$ do not coincide we are still left with an ambiguity.

For special cases, i.e. if $\Omega = \mathbf{R}^n \setminus S$ (where S is either a finite set of points or of balls in $\mathbf{R}^n$), V_0 is either zero or has r^{-2}-singularities in S, and various restrictions are made on V_1 and n, Robinson et al. [17,4] and Semenov [19] have shown the coincidence of T_F and $T_0 \dotplus V_1$; a very general result (among others) of this type has also been obtained by Combescure-Moulin and Ginibre [1] [3] for N-particle systems, when V_0 has certain r^{-2}-singularities in the set of hyperplanes where the particles collide. For $V_0 = 0$ and $\Omega = \mathbf{R}^n$ the result has also been obtained by Simon [21, Th. 2.1].

---

[1] Note that we are considering the semibounded case only.

[2] In fact $D(T)$ is contained in the form domain $Q(T)$ for any T and $Q(T_F) \subseteq Q(T_0 \dotplus V_1) = Q(T_0) \cap Q(V_1)$ (cf. Section 3).

[3] Our paper has in part been stimulated by [1].

In the present paper we prove the following general result:
Assume (C_1) and (C_2); then

$$T_F = T_0 \dot{+} V_1 \tag{1.5}$$

for $n \in \mathbf{N}$ and any open Ω in $\mathbf{R}^n$.

We add several remarks:

1. Note that there is a third possibility of defining a Hamiltonian by physical considerations, i.e. by the limit in the strong resolvent sense of a monotone increasing cut-off procedure for V_1. But since it is well-known that this limit coincides with the form sum (see [3, Th. 7.12; 18]), this creates no additional ambiguity.

2. (1.5) is a Schrödinger operator result. For general semibounded self-adjoint operators the form sum and the Friedrichs extension of the sum may not coincide (see [9, Chap. VI, 2.5] for a counterexample).

3. We make no use of any explicit form of V_0 and V_1. Thus, if one chooses Jacobian coordinates, the Schrödinger operator for N-particle systems with center of mass removed is of the form (1.1) (i.e. no Hughes-Eckart terms occur [16, p. 78 ff.]), and our result includes also the N-particle case.

4. There would be no difficulty in replacing τ by a more general second-order differential operator of elliptic type, such as one involving a magnetic potential.

5. Note that (1.5) means commutativity between closure and summation of forms, i.e.

$$\overline{t_0 + t_{V_1}} = \bar{t}_0 + \overline{t_{V_1}},$$

where t_0, t_{V_1} denote the quadratic forms associated with T_0 and V_1 on $C_0^\infty(\Omega)$.

The proof of our main theorem is based on Kato's inequality [10], in fact the method of the proof itself is a variant of Kato's technique, which has been used very successfully in tackling self-adjointness problems [10, 11, 20, 1]. But, rather than looking for an "approximative" positive ground state for T_0 (an idea which goes back to Simon [20]), we show that the cone of a.e. positive C_0^∞-functions is dense (in the sense of the form norm of T_0) in the cone of a.e. positive elements in the form domain of T_0. This will be referred to as T_0 having a *positive form core*.

The key result of this paper is that any semibounded Schrödinger operator has a positive form core. This is an analytical property of Schrödinger operators which might be of some mathematical interest in its own right.

To prove this property we use a positivity preserving "approximative identity" which commutes "approximately" with T_0, a technique which we already used to deal with self-adjointness problems [2].

The paper is organized as follows. In Section 2 we show some stability properties for Schrödinger operators with positive form core and finally that all semibounded operators have positive form core. Section 3 contains the main theorem and some additional uniqueness results concerning the Friedrichs extension.

In conclusion we remark that in the proofs we always assume T_0 and V_1 to be positive. We can do this without loss of generality, since one can always add a suitable constant, without changing domains, form topologies, etc.

Aknowledgements: I owe thanks to R. Wüst for very enlightning and stimulating discussions.

2. SCHRÖDINGER OPERATORS WITH POSITIVE FORM CORE

We first give some notations and recall some basic properties of quadratic forms (for an extensive treatment of quadratic forms see [9, Ch. VI; 3; 14]). Given $D \subseteq L^2(\Omega)$, we denote by D^+ the set of a.e. positive elements in D. Let t be a semibounded quadratic form with lower bound α defined on $D \times D$; then we denote the form domain of t by $Q(t):=D$. t is closed if $Q(t)$ together with the inner product $(\cdot,\cdot)_t$ is a Hilbert space, where

$$(u, v)_t := t[u, v] + (1 - \alpha)(u, v); \quad u, v \in Q(t).$$

The associated norm will be denoted by $\|\cdot\|_t$.

There is a one-to-one correspondence between semibounded closed quadratic forms and semibounded self-adjoint operators. The domain of the closed form associated with the operator T is called the form domain of T and is denoted by $Q(T)$, and the associated norm in $Q(T)$ will be called the form norm of T. For a semibounded symmetric operator T we denote by T_F the Friedrichs extension (this is the self-adjoint extension of T associated with the closure of the quadratic form

$$t[u, v] := (Tu, v); \quad u, v \in D(T)).$$

DEFINITION. Let T be a symmetric operator bounded from below in $L^2(\Omega)$ with $D(T) = C_0^\infty(\Omega)$. Then we say that T has a *positive form core* (p.f.c.) if, for any $u \in Q(T_F)^+$ there exists a sequence $\{u_m\}_{m\in\mathbf{N}}$ in $C_0^\infty(\Omega)^+$ with $\|u_m - u\|_t \to 0$ $(m \to \infty)$, where t denotes the form associated with T_F.

The following lemma states that the positive form core property is stable under relatively bounded form perturbations with bound < 1. We will make no explicit use of it in this paper, but it might be useful for possible extensions of Theorem 2 (cf. Remark 4).

LEMMA 1. *Let T be a symmetric operator bounded from below with $D(T) = C_0^\infty(\Omega)$, A be a symmetric operator such that $D(T) \subseteq D(A)$ and suppose that there exist $0 \leqslant a < 1$ and $b \geqslant 0$ with*

$$|(Aw, w)| \leqslant a(Tw, w) + b(w, w), \quad w \in C_0^\infty(\Omega).$$

If T has p.f.c., then so has $T + A$.

Proof. Since we know that $Q(T_F) = Q((T + A)_F)$ and the associated form norms for T_F and $(T + A)_F$ are equivalent [15, Th. X. 17], the assertion follows immediately.

If we denote the Laplacian on $C_0^\infty(\Omega)$ by

$$H_0 := (-\Delta) \upharpoonright C_0^\infty(\Omega), \tag{2.1}$$

then H_0 has p.f.c. and this is not changed by the addition of a semibounded $L^2_{\mathrm{loc}}(\Omega)$-potential, as we will show in the next lemma.

LEMMA 2. *Let H_0 be as in* (2.1) *and $V_b \in L^2_{\mathrm{loc}}(\Omega)$ real-valued and bounded from below. Then $H_0 + V_b$ has p.f.c.* .

Proof. Without loss of generality assume $V_b \geqslant 0$. Denoting by t the quadratic form associated with $T := (H_0 + V_b)_F$, we have

$$\|u\|_t^2 = \|u\|^2 + \|\nabla u\|^2 + \|V_b^{1/2} u\|^2, \quad u \in Q(H_0) \cap Q(V_b)\ ^{4)}.$$

Let $w \in C_0^\infty(\Omega)$. Then using Friedrichs mollifiers we obtain a sequence $\{w_k\}_{k \in \mathbf{N}}$ in $C_0^\infty(\Omega)^+$ such that

$$\|w_k - |w|\,\|_t \to 0 \quad (k \to \infty), \tag{2.2}$$

and it follows that $|w| \in Q(T)$. (For this and the following compare [6, proof of Lemma 1].)

Now, let $u \in Q(T)^+$. Then there is a sequence $\{\tilde{u}_m\}_{m \in \mathbf{N}}$ in $C_0^\infty(\Omega)$ such that

$$\|\tilde{u}_m - u\|_t \to 0 \quad (m \to \infty).$$

Then

$$\|\,|\tilde{u}_m| - u\|_t^2 \leqslant \|\tilde{u}_m - u\|^2 + \{\|\nabla(\tilde{u}_m - u)\| + |(\operatorname{sign}\tilde{u}_m - 1)|\,\|\nabla u\|\}^2 + \\ + \|V_b^{1/2}(\tilde{u}_m - u)\|^2.$$

Together with $\partial_i u = 0$ a.e. on $\{x \in \Omega \mid u(x) = 0\}$ for $i \in \{1, \ldots, n\}$ [5, Lemma 7.7], we can use Riesz Lemma and then the dominated convergence theorem to pass to a subsequence $\{u_m\}_{m \in \mathbf{N}}$ such that

$$\|\,|u_m| - u\,\|_t \to 0 \quad (m \to \infty). \tag{2.3}$$

Consequently, by an $\varepsilon/2$ argument, we can obtain from (2.2) and (2.3) a sequence in $C_0^\infty(\Omega)^+$ which converges to u in the form norm of T. This proves Lemma 2.

---

4) We denote

$$\|\nabla u\|^2 := \sum_{i=1}^{n} \|\partial_i u\|^2, \quad u \in Q(H_0).$$

Assuming (C_1), we will use the following notation in this and the next section (in addition to (2.1)):

$$T_0 := \text{Friedrichs extension of } (-\Delta + V_0)\restriction C_0^\infty(\Omega);$$

$$V_0^+ := \max\{0, V_0\},\ V_0^- := \max\{0, -V_0\};$$

(2.4)

$$T_+ := (H_0 + V_0^+)_F;$$

$$V_0^k := \min\{k, V_0^-\},\ T_k := (H_0 + V_0^+ - V_0^k)_F \text{ for } k \in \mathbf{N};$$

by t_0, t_+ and t_k we denote the quadratic forms associated with T_0, T_+ and T_k respectively ($k \in \mathbf{N}$).

What follows is a more technical lemma, which we need for the main theorem in this section.

LEMMA 3. *Assume* (C_1). *Then* $T_k \to T_0$ $(k \to \infty)$ *in the strong resolvent sense and* $Q(T_+) \subseteq Q(T_0)$.

Proof. Since, for $k \in \mathbf{N}$, we know $T_0 \leqslant T_k$, $C_0^\infty(\Omega) \subseteq Q(T_k)$ and $(w, T_k w) \to (w, T_0 w)$ $(k \to \infty)$, $w \in C_0^\infty(\Omega)$, the first part of Lemma 3 follows from [3, Th. 7.9]. The second part is obvious, because of $T_+ \geqslant T_0$.

The following theorem is the main result of this section. It states that any semi-bounded Schrödinger operator with $L^2_{\text{loc}}(\Omega)$-potential has p.f.c.. The proof runs roughly parallel to that of Theorem 1 in [2].

THEOREM 1. *Assume* (C_1). *Then* T_0 *has p.f.c.*.

Proof. Assume $T_0 \geqslant 0$.

We know from Lemma 2 and Lemma 3 that

(2.5) $$T_+ \text{ has p.f.c.}$$

and

(2.6) $$Q(T_k) = Q(T_+) \subseteq Q(T_0).$$

(2.5), (2.6) and

(2.7) $$\|\cdot\|_{t_0} \leqslant \|\cdot\|_{t_k} \leqslant \|\cdot\|_{t_+} \quad (k \in \mathbf{N})$$

imply that there exists for any $\tilde{u} \in Q(T_+)^+$ a sequence $\{\tilde{u}_m\}_{m \in \mathbf{N}}$ in $C_0^\infty(\Omega)^+$ with

$$\|\tilde{u}_m - \tilde{u}\|_{t_0} \to 0 \quad (m \to \infty).$$

Therefore by an $\varepsilon/2$ argument, it suffices to show that for any $u \in Q(T_0)^+$ there exists a sequence $\{u_j\}_{j \in \mathbf{N}}$ in $Q(T_+)^+$ such that

(2.8) $$\|u_j - u\|_{t_0} \to 0 \quad (j \to \infty).$$

Thus, let $u \in Q(T_0)^+$ and $m, k \in \mathbf{N}$; set

$$u_m := \left(\frac{1}{m} T_0 + 1\right)^{-1} u$$

and

$$u_m^k := \left(\frac{1}{m} T_k + 1\right)^{-1} u \cdot$$

(Note that we assumed $T_0 \geqslant 0$ and thus $T_k \geqslant 0$.)

Since $(T_k + 1)^{-1}$ is positivity preserving [6], together with (2.6) we have

$$u_m^k \in Q(T_+)^+ . \tag{2.9}$$

Since $(T_0 + 1)^{1/2}$ and $\left(\frac{1}{m} T_0 + 1\right)^{-1}$ commute, we have

$$\|u_m - u\|_{t_0} = \left\|\left(\left(\frac{1}{m} T_0 + 1\right)^{-1} - 1\right)(T_0 + 1)^{1/2} u\right\|.$$

Therefore, since $\left(\frac{1}{m} T_0 + 1\right)^{-1} \to 1$ $(m \to \infty)$ strongly, it follows that

$$\|u_m - u\|_{t_0} \to 0 \quad (m \to \infty). \tag{2.10}$$

Now, let $m \in \mathbf{N}$. Then for $k \in \mathbf{N}$, using (2.7) and the Schwarz inequality we get the following estimates:

$$\|(T_k + m)^{-1}u - (T_0 + m)^{-1}u\|^2_{t_0+m} \leqslant$$

$$\leqslant \|(T_k + m)^{-1}u\|^2_{t_k+m} + \|(T_0 + m)^{-1}u\|^2_{t_0+m} - 2((T_k + m)^{-1}u, u) \leqslant \tag{2.11}$$

$$\leqslant \|(T_0 + m)^{-1}u - (T_k + m)^{-1}u\| \, \|u\|.$$

From Lemma 3 we know $(T_k + m) \to (T_0 + m)$ $(k \to \infty)$ in the strong resolvent sense. Therefore, the identity $\left(\frac{1}{m} T + 1\right)^{-1} = m(T + m)^{-1}$ for self-adjoint operators T and (2.11) yields

$$\|u_m^k - u_m\|_{t_0+m} \to 0 \quad (k \to \infty),$$

which is equivalent to

$$\|u_m^k - u_m\|_{t_0} \to 0 \quad (k \to \infty). \tag{2.12}$$

Bearing in mind (2.9), we can use (2.10) and (2.12) to choose a subsequence $\{u_j\}_{j \in \mathbf{N}}$ out of $\{u_m^k\}_{m \in \mathbf{N},\, k \in \mathbf{N}}$ with the desired properties (2.8). Thus Theorem 1 is proved.

3. FORM SUM AND FRIEDRICHS RESTRICTION

Assume (C_1), (C_2) and define $T_{\max}$ and T_0 by (1.3) and (2.4) respectively. Denote by t_q the semibounded closed quadratic form given by the sum of the two forms associated with T_0 and V_1. Then the self-adjoint operator associated with t_q is called the *form sum* of T_0 and V_1. We denote it by $T_0 \dotplus V_1$. Obviously $T_0 \dotplus V_1$ is a self-adjoint restriction of $T_{\max}$. Note that $Q(T_0 \dotplus V_1) = Q(T_0) \cap Q(V_1)$. If we denote by $t_{\min}$ the semibounded quadratic form which is the restriction of t_q to $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, then the self-adjoint operator associated with the closure of $t_{\min}$ is also a self-adjoint restriction of $T_{\max}$. We denote it by T_F and call it the *Friedrichs restriction* of $T_{\max}$. Note that $Q(T_F)$ is the closure of $C_0^\infty(\Omega)$ in the sense of the form norm of $T_0 \dotplus V_1$.

We now prove our main result. The proof is very similar to that of Theorem 4.1 in [1].

THEOREM 2. *Assume* (C_1), (C_2) *and define* T_0 *by* (2.4). *Then* $T_F = T_0 \dotplus V_1$.

Proof. Assume $T_0 \geqslant 0$ and $V_1 \geqslant 0$. We denote here $T_q := T_0 \dotplus V_1$. Since $t_{\min}$ and t_q coincide on $C_0^\infty(\Omega)$, it is sufficient to show that $C_0^\infty(\Omega)$ is dense in the Hilbert space $Q(T_q)$ with the inner product $(\cdot,\cdot)_{t_q}$. Let $v \in Q(T_q)$ be orthogonal to $C_0^\infty(\Omega)$ in the sense of the inner product $(\cdot,\cdot)_{t_q}$. This is equivalent to

$$\Delta v = (V_0 + V_1 + 1)v \tag{3.1}$$

in the distribution sense.

Now $(V_0 + V_1)v \in L^1_{\text{loc}}(\Omega)$, since $v \in Q(V_1)$, $V_1 \in L^1_{\text{loc}}(\Omega)$ and $V_0 \in L^2_{\text{loc}}(\Omega)$ by hypothesis. Thus the right hand side of (3.1), and therefore also the left hand side, is in $L^1_{\text{loc}}(\Omega)$. Therefore we can apply Kato's inequality [10]

$$\Delta|v| \geqslant \operatorname{Re}\{(\operatorname{sign} v)\Delta v\}$$

and since $V_1 \geqslant 0$, we arrive at

$$((T_0 + 1)w, |v|) \leqslant 0 \quad (w \in C_0^\infty(\Omega)^+). \tag{3.2}$$

Since $v \in Q(T_0)$, we know $|v| \in Q(T_0)$ [6]; it follows that we can write (3.2) in terms of the Hilbert space $Q(T_0)$ with inner product $(\cdot,\cdot)_{t_0}$, i.e.

$$(w, |v|)_{t_0} \leqslant 0 \quad (w \in C_0^\infty(\Omega)^+). \tag{3.3}$$

Now let $u := (T_0 + 1)^{-1}|v|$. Thus $u \in Q(T_0)^+$, since $(T_0 + 1)^{-1}$ is positivity preserving [6]. From Theorem 1 we know that T_0 has p.f.c.; thus there exists a sequence $\{u_m\}_{m\in\mathbf{N}}$ in $C_0^\infty(\Omega)^+$ such that

$$\lim_{m\to\infty} (u_m, |v|)_{t_0} = ((T_0 + 1)^{-1}|v|, |v|)_{t_0} = \|v\|^2.$$

From (3.3) $v = 0$ follows. Thus $C_0^\infty(\Omega)$ is dense in the Hilbert space $Q(T_q)$ and the theorem is proved.

REMARK. If we replace (C_2) by the stronger condition

(C_2)′ $V_1 \in L^2_{\mathrm{loc}}(\Omega)$, V_1 real-valued and bounded from below,

then

$$T_{\min} u := \tau u, \quad D(T_{\min}) := C_0^\infty(\Omega) \tag{3.4}$$

is well defined as a symmetric semibounded operator in $L^2(\Omega)$ and the Friedrichs restriction of $T_{\max}$ coincides with the Friedrichs extension of $T_{\min}$. In this case Theorem 2 states that the form sum of T_0 and V_1 coincides with the Friedrichs extension of $T_{\min}$.

We now give some uniqueness results which show the distinguished role of the Friedrichs extension (compare with [7]).

The proof of the following lemma is analogous to that of Theorem 4.4 in [1].

LEMMA 4. *Assume* (C_1), (C_2)′ *and define* $T_{\min}$ *and* T_0 *by* (3.4) *and* (2.4). *Then the Friedrichs extension* T_F *is the only self-adjoint extension of* $T_{\min}$ *that has its domain in* $Q(T_0)$.

Proof. Assume $T_0 \geqslant 0$ and $V_1 \geqslant 0$. Let T_1 be a self-adjoint extension of $T_{\min}$ with domain contained in $Q(T_0)$. It is sufficient to show that this implies $D(T_1) \subseteq D(T_F)$ since T_1 and T_F are self-adjoint extensions of $T_{\min}$.

Let $u \in D(T_1)$ and

$$u' := (T_F + 1 + \mathrm{i})^{-1}(T_1 + 1 + \mathrm{i})u.$$

Thus for any $w \in C_0^\infty(\Omega)$

$$0 = ((T_F + 1 + \mathrm{i})u' - (T_1 + 1 + \mathrm{i})u, w) = (u' - u, (T_{\min} + 1 - \mathrm{i})w)$$

which yields

$$\Delta v = (V_0 + V_1 + 1 + \mathrm{i})v \qquad \text{for } v := u' - u$$

in the distribution sense. From this one deduces $v = 0$ by the same argument as in the proof of Theorem 2. Since $u' \in D(T_F)$, $D(T_1) \subseteq D(T_F)$ and Lemma 4 is proved.

If instead of (C_1) we take the stronger condition that V_0 is a bounded H_0-form with relative bound < 1, then T_0 is semibounded and $Q(T_0) = Q(H_0)$ [15, Th. X. 17]. We can immediately deduce the following theorem from Lemma 4:

THEOREM 3. *Assume* (C_2)′ *and define* $T_{\min}$ *and* T_0 *by* (3.4) *and* (2.4). *Let* $V_0 \in L^2_{\mathrm{loc}}(\Omega)$ *be real-valued and there exist* $0 \leqslant a < 1$ *and* $b \geqslant 0$ *such that*

$$|(V_0 w, w)| \leqslant a(H_0 w, w) + b(w, w); \quad w \in C_0^\infty(\Omega).$$

Then the Friedrichs extension of $T_{\min}$ *is the only self-adjoint extension of* $T_{\min}$ *with domain contained in* $Q(H_0)$.

Theorem 3 is analogous to a result of Nenciu [13] and Wüst and Klaus [12] in the Dirac operator case. It can be interpreted physically as meaning that the minimal operator has a unique self-adjoint extension whose domain contains only states with finite kinetic energy.

A similar result for the potential energy can be obtained for the case $\Omega = \mathbf{R}^n$. But we first proof a technical lemma.

LEMMA 5. *Let* $\Omega = \mathbf{R}^n$. *Assume* (C_1), $(C_2)'$ *and define* $T_{\min}$ *and* T_0 *by* (3.4) *and* (2.4). *If* T_1 *is a self-adjoint extension of* $T_{\min}$, *then* $D(T_1) \subseteq Q(V_0^-)$ *implies* $D(T_1) \subseteq Q(T_0)$.

Proof. We use a lemma due to Karlsson [8, Lemma 1.1] in which he proves essentially that for $u \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$ with

$$\tau u \in L^2_{\mathrm{loc}}(\mathbf{R}^n) \qquad \text{and} \qquad (V_0^-)^{1/2} u \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$$

and for a $\varphi \in C_0^\infty(\mathbf{R}^n)^+$ the following inequality holds (cf. [8, (1.10)]):

$$\int \varphi\{|\nabla u|^2 + V_0^+ |u|^2\} \leqslant \mathrm{Re}\left\{\int \varphi \tau u \bar{u}\right\} + \int \varphi V_0^- |u|^2 - \frac{1}{2}\int (-\Delta\varphi)|u|^2. \tag{3.5}$$

Let T_1 be a self-adjoint extension of $T_{\min}$ with $D(T_1) \subseteq Q(V_0^-)$. Let $u \in D(T_1)$ and for $k \in \mathbf{N}$ define φ_k by

$$\varphi_k := \Phi\left(\frac{x}{k}\right),$$

where Φ is a function in $C_0^\infty(\mathbf{R}^n)^+$ which takes the value 1 in the ball with radius 1 and 0 outside of the ball with radius 2. Then (3.5) holds if we replace φ by φ_k for $k \in \mathbf{N}$. Since $u \in Q(V_0^-)$, we can use the dominated convergence theorem for replacing φ by 1 in (3.5). Thus we conclude $u \in Q(H_0)$ and $u \in Q(V_0^+)$. Therefore we have $u \in Q(T_0)$ and Lemma 5 is proved.

THEOREM 4. *Let* $\Omega = \mathbf{R}^n$. *Assume* (C_1), $(C_2)'$ *and define* $T_{\min}$ *and* T_0 *by* (3.4) *and* (2.4). *Then there is at most one self-adjoint extension of* $T_{\min}$ *with domain contained in* $Q(V_0)$. *This coincides then with the Friedrichs extension of* $T_{\min}$.

Proof. Let T_1 be a self-adjoint extension of $T_{\min}$ with $D(T_1)$ contained in $Q(V_0)$. Obviously $D(T_1)$ is contained in $Q(V_0^-)$ and, using Lemma 5, also in $Q(T_0)$. But Lemma 4 shows us then that $T_1 = T_F$, which proves the theorem.

Note that Theorem 4 does not mean that a self-adjoint extension of $T_{\min}$ with domain contained in $Q(V_0)$ always exists.

If V_0 is the negative part and V_1 the positive part of V, and V_0 is a H_0-form bounded with relative bound < 1, then the domain of the Friedrichs extension of $T_{\min}$ is contained in $Q(V)$ (and therefore in $Q(V_0)$). This is then the only extension with this property.

We prove this in the following

COROLLARY. *Let* $\Omega = \mathbf{R}^n$. *Assume* $V \in L^2_{loc}(\mathbf{R}^n)$ *real-valued with* $V = V_0 + V_1$, *where* $V_0 = \min\{0, V\}$, $V_1 = \max\{0, V\}$ *and define* $T_{\min}$ *and* T_0 *by* (3.4) *and* (2.4). *If there exist* $0 \leqslant a < 1$ *and* $b \geqslant 0$ *such that*

$$|(V_0 w, w)| \leqslant a(H_0 w, w) + b(w, w), \qquad w \in C_0^\infty(\Omega),$$

then the Friedrichs extension T_F *of* $T_{\min}$ *is the only self-adjoint extension of* $T_{\min}$ *with domain contained in* $Q(V)$.

Proof. Because of Theorem 2 and the H_0-form boundedness of V_0 with bound < 1 we have

$$Q(T_F) = Q(H_0) \cap Q(V_0) \cap Q(V_1) \subseteq Q(V).$$

If T_1 is another self-adjoint extension of $T_{\min}$ with $D(T_1)$ contained in $Q(V)$, then (since $Q(V) \subseteq Q(V_0)$) $T_1 = T_F$ follows from Theorem 4.

The corollary can be interpreted physically as meaning that the Friedrichs extension is the unique self-adjoint extension of $T_{\min}$ having only states with finite potential energy in its domain. Note that in this case this is also the only extension with finite kinetic energy states in its domain (cf. Theorem 3).

Supplementary note.

B. Simon has pointed out [5)] that Theorem 2 may be considered as a positive solution of the "form analog" of Jörgens conjecture, i.e. the stability of self-adjointness under positive potential perturbations (see [2]).

REFERENCES

1. COMBESCURE-MOULIN, M.; GINIBRE, J., Essential self-adjointness of many particle Schrödinger operators with singular two body potentials, *Ann. I.H.P.*, **23**(1975), 211–234.
2. CYCON, H. L., On the stability of self-adjointness of Schrödinger operators under positive perturbations, *Proc. Roy. Soc. Edinburgh*, A **68**(1980), 165–173.
3. FARIS, W. G., *Self-adjoint operators*, Lect. Notes in Math, **433** (1975), Springer.
4. FERRERO, P.; DE PAZZIS, O.; ROBINSON, D. W., Scattering theory with singular potentials. II. The N-body problem and hard cores, *Ann. I.H.P.*, **21**(1974), 217–231.
5. GILBARG, D.; TRUDINGER, N. S., *Elliptic partial differential equations of second order*, New York, Springer, 1977.
6. GOELDEN, H.-W., On non-degeneracy of the ground state of Schrödinger operators, *Math. Z.*, **155**(1977), 239–247.
7. KALF, H.; SCHMINCKE, U.-W.; WALTER, J.; WÜST, R., On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials, *Symposium Dundee 1974*, Lect. Notes in Math., **448**(1975), 182–226, Springer.

[5)] private communication.

8. KARLSSON, B., *Self-adjointness of Schrödinger operators*, Inst. Mittag Leffler, Report Nr. 6 (1976).
9. KATO, T., *Perturbation theory for linear operators*, New York, Springer, 1966.
10. KATO, T., Schrödinger operators with singular potentials, *Israel J. Math.*, **13**(1972), 135–148.
11. KATO, T., A second look at the essential self-adjointness of Schrödinger operators, *Physical reality and mathematical description*, C. P. Enz and J. Mehra eds., D. Reichel Dortrecht (1974).
12. KLAUS, M.; WÜST, R., Characterization and uniqueness of distinguished self-adjoint extensions of Dirac operators, *Commun. Math. Phys.*, **64**(1979), 171–176.
13. NENCIU, G., Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms, *Commun. Math. Phys.*, **48**(1976), 235–247.
14. REED, M.; SIMON, B., *Methods of modern mathematical physics I: Functional analysis*, Academic Press, New York, 1972.
15. REED, M.; SIMON, B., *Methods of modern mathematical physics II: Fourier analysis, Self-adjointness;* New York, Academic Press, 1975.
16. REED, M.; SIMON, B., *Methods of modern mathematical physics III: Scattering theory*, New York, Academic Press, 1979.
17. ROBINSON, D. W., Scattering theory with singular potentials. I: The two body problem, *Ann. I.H.P.*, **21**(1974), 185–215.
18. SCHECHTER, M., Cut-off potentials and form extensions, *Letters in Math. Phys.*, **1**(1976), 265–273.
19. SEMENOV, YU. A., On the problem of convergence of a bounded below sequence of symmetric forms for the Schrödinger operator, *Stud. Math.*, **64**(1979), 77–88.
20. SIMON, B., Essential self-adjointness of Schrödinger operators with singular potentials, *Arch. Rat. Mech. Anal.*, **52**(1973), 44–48.
21. SIMON, B., Maximal and minimal Schrödinger forms, *J. Operator Theory*, **1**(1979), 37–47.

HANS L. CYCON
Technische Universität Berlin,
Fachbereich Mathematik,
West Berlin.

Received July 18, 1980.