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A TOEPLITZ-HAUSDORFF THEOREM FOR MATRIX RANGES

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1. INTRODUCTION

Given a bounded linear operator T defined on a complex Hilbert space $\mathscr{H}$, the spatial numerical range [1] of T is the set

$$\mathscr{W}_1(T) = \{\langle Tx, x\rangle : x \in \mathscr{H},\ \|x\| = 1\}. \tag{1.1}$$

The Toeplitz-Hausdorff Theorem states that $\mathscr{W}_1(T)$ is convex ([9]; [14]); the closure of $\mathscr{W}_1(T)$ is the numerical range,

$$W_1(T) = \{\varphi(T) : \varphi \in S_1\}, \tag{1.2}$$

where S_1, the state space, is the set of norm-one positive linear functionals defined on the set of all bounded linear operators on $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$.

To simplify matters, restrict $\mathscr{H}$ to be finite dimensional; note that $\mathscr{W}_1(T)$ is closed in this case, so $\mathscr{W}_1(T) = W_1(T)$. The Toeplitz-Hausdorff Theorem may be viewed as a *consequence* of the equality between $\mathscr{W}_1(T)$ and $W_1(T)$. This equality comes about by virtue of the fact that not all functionals in S_1 are needed to sweep out $W_1(T)$; only those of the form $\langle(\cdot)x, x\rangle$, where $x \in \mathscr{H}$ and $\|x\| = 1$ — all of which are extreme points in S_1 — are needed. Thus to prove the Toeplitz-Hausdorff Theorem, it suffices to show that the numerical range is swept out by the extreme points of S_1. Such an approach will yield both a geometric proof of the Toeplitz-Hausdorff Theorem, thus answering a question of P. R. Halmos [7; 8, p. 110], and a generalization of the theorem to matrix ranges.

Using completely positive maps, Arveson [2, p. 300] generalized the concept of numerical range in defining matrix ranges. Recall that if $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras and M_m is the set of complex $m \times m$ matrices with identity I_m, a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is said to be *completely positive* (see [1]) if the associated maps

$$\varphi \otimes I_m : \mathscr{A} \otimes M_m \to \mathscr{B} \otimes M_m, \quad m \geqslant 1,$$

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[1] Usage differs. Some authors use "numerical range" and "algebra numerical range" for what are here termed "spatial numerical range" and "numerical range", respectively.

are all positive. The set of all such maps is denoted by $\mathrm{CP}[\mathscr{A}, \mathscr{B}]$. If $\mathscr{A}$ has identity I, then the subset of $\mathrm{CP}[\mathscr{A}, \mathscr{B}]$ consisting of all φ such that $\varphi(I) = K$, K fixed and positive in $\mathscr{B}$, is denoted by $\mathrm{CP}[\mathscr{A}, \mathscr{B}; K]$; in addition, $\mathrm{CP}[\mathscr{A}, M_n; I_n] \equiv S_n$ is termed the n^{th} state space. (S_1 consists of norm-one positive linear functionals on $\mathscr{A}$ — this agrees with previous usage.) Finally, the n^{th} matrix range for $T \in \mathscr{A}$ is

$$W_n(T) \equiv \{p \in M_n : p = \varphi(T),\ \varphi \in S_n\}. \tag{1.3}$$

Each $W_n(T)$ is a compact, convex subset of M_n. For $\mathscr{A} = \mathscr{B}(\mathscr{H})$, $n = 1$, $W_1(T)$ is the numerical range of T given in (1.2).

One possible generalization of the spatial numerical range was given by Bonsall [3], who defined spatial matrix ranges: If $\mathbf{C}^n$ is complex n dimensional space, then

$$\mathscr{W}_n(T) \equiv \{p \in M_n : p = V^*TV,\ V^*V = I_n,\ V \in \mathscr{B}(\mathbf{C}^n, H)\}, \tag{1.4}$$

where in general $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ denotes all bounded linear maps from $\mathscr{H}_1$ into $\mathscr{H}_2$. For $n = 1$, this is the spatial numerical range of T. If $n \geqslant 2$, $\mathscr{W}_n(T)$ lacks two important properties of $\mathscr{W}_1(T)$: it is not convex [4, p. 142] and its closure is not $W_n(T)$ [5, Remark 4.12]. As the work of Bunce and Salinas [5, Sec. 3] demonstrates, the matricial convex hull of $\mathscr{W}_n(T)$ (denoted by $\mathrm{mconv}(\mathscr{W}_n(T))$; see Section 3), which is both convex and has $W_n(T)$ as its closure, is a better analogue to the spatial numerical range.

The main purpose of this paper is to provide for $\mathrm{mconv}(\mathscr{W}_n(T))$ a theorem analogous to the Toeplitz-Hausdorff Theorem; that is, to find a class of maps in $\mathrm{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n]$ which sweep out $\mathrm{mconv}(\mathscr{W}_n(T))$ in the same way that norm-one positive linear functionals of the form $\langle(\cdot)x, x\rangle$ sweep out the spatial numerical range. In particular, it will be shown that the necessary maps all have the form $\sum_{m=1}^{N} V_j^*(\cdot)V_j$, where $V_j \in \mathscr{B}(\mathbf{C}^n, \mathscr{H})$, $\sum_{j=1}^{N} V_j^*V_j = I_n$, and $N < \sqrt{3}n$.

Outline and summary. In Section 2, an extremal problem similar to that discussed by Choi [6] is solved; a special case of the solution to this problem is then used to provide a new proof for the Toeplitz-Hausdorff Theorem. Because this new proof relies on a special case of a complicated theorem, a second, direct geometric proof of the Toeplitz-Hausdorff Theorem — one which does not use results involving completely positive maps — will also be given. The section closes with a corollary which describes the structure of the matrix range when $\mathscr{H}$ is finite dimensional; this result, which is another consequence of having solved the extremal problem mentioned earlier, is used to motivate the definition of the n^{th} *vector matrix range* for T, $w_n(T)$.

Section 3 begins with a theorem which states that the n^{th} vector matrix range for T is precisely the matricial convex hull of $\mathscr{W}_n(T)$; this theorem is the matrix analogue of the Toeplitz-Hausdorff Theorem. The section continues with a discussion of the topological properties of matrix ranges and vector matrix ranges. In particular it is shown that if $W_1(T)$ contains a closed disk of radius r, then $W_n(T)$ contains a closed ball of radius r (see [13], Theorem 4.6). Finally, relationships among essential matrix ranges, matrix ranges, and vector-matrix ranges are given and then used to show that vector-matrix ranges and matrix ranges coincide for a compact operator with 0 in the interior of its numerical range.

Closing remarks are made and a few questions are raised in Section 4.

2. AN EXTREMAL PROBLEM

In [6], Choi gives the following elegant characterization of $\mathrm{CP}[M_r, M_n]$: $\varphi \in \mathrm{CP}[M_r, M_n]$ if and only if there exist $r \times n$ matrices V_j such that

$$\varphi(T) = \sum_{j=1}^{N} V_j^* T V_j, \quad T \in M_r, \; N < \infty. \tag{2.1}$$

The expression for φ given in (2.1) is said to be *canonical* if the set $\{V_j\}$ is linearly independent. In addition, Choi characterizes the extreme points of $\mathrm{CP}[M_r, M_n; K]$ as those $\varphi \in \mathrm{CP}[M_r, M_n; K]$ whose canonical decomposition is such that $\{V_j^* V_k\}$ forms a linearly independent set.

The extremal problem solved by Choi may be viewed as follows: Let $\mathscr{S}_0$ be the subspace of M_r spanned by I_r and let $\varphi \in \mathrm{CP}[M_r, M_n]$. What are the extreme points of the set of all $\Psi \in \mathrm{CP}[M_r, M_n]$ such that $\Psi|_{\mathscr{S}_0} = \varphi|_{\mathscr{S}_0}$? Put this way, there is a natural generalization. Let $\mathscr{S}$ be a self-adjoint subspace of M_r and let $\varphi \in \mathrm{CP}[M_r, M_n]$. The set

$$A(\varphi, \mathscr{S}) \equiv \{\Psi \in \mathrm{CP}[M_r, M_n] \colon \Psi|_{\mathscr{S}} = \varphi|_{\mathscr{S}}\} \tag{2.2}$$

is clearly convex — what are its extreme points? The answer, which will yield the Toeplitz-Hausdorff Theorem as a special case and give rise to the generalization of this theorem, is given below:

THEOREM 2.1. *Let $\mathscr{S}$ be a self-adjoint subspace of M_r and let $A_1, \ldots, A_L$ be self-adjoint matrices which form a basis for $\mathscr{S}$. A completely positive map Ψ having the canonical decomposition*

$$\Psi(T) = \sum_{j=1}^{N} V_j^* T V_j, \qquad T \in M_r, \tag{2.3}$$

is an extreme point of $A(\varphi, \mathscr{S})$ if and only if the set of matrices $\{G_{jk}\}$,

$$G_{jk} = \bigoplus_{l=1}^{L} V_j^* A_l V_k, \quad j, k = 1, \ldots, N, \tag{2.4}$$

is linearly independent in $M_n \oplus \ldots \oplus M_n$ (L copies).

Proof. Suppose that $\Psi(T)$ is an extreme point and that $\{G_{jk}\}$ is a linearly dependent set. The latter assumption implies the existence of complex numbers $\{\alpha_{jk}\}$, such that

$$\sum \alpha_{jk} G_{jk} = 0, \quad \sum |\alpha_{jk}|^2 \neq 0. \tag{2.5}$$

By virtue of the fact that

$$G_{jk}^* = G_{kj}, \quad j, k = 1, \ldots, N, \tag{2.6}$$

these numbers may be chosen so that

$$\alpha_{jk} = \bar{\alpha}_{kj}, \quad j, k = 1, \ldots, N. \tag{2.7}$$

In addition, multiplication by a non-zero real scale factor allows the additional restriction that the two $N \times N$ matrices with components $\delta_{jk} \pm \alpha_{jk}$ be non-negative. As such, there exist self-adjoint $N \times N$ matrices with components $\beta_{jk}^{\pm}$ such that

$$\delta_{jk} \pm \alpha_{jk} = \sum_{\nu=1}^{N} \beta_{j\nu}^{\pm} \beta_{\nu k}^{\pm}, \quad j, k = 1, \ldots, N. \tag{2.8}$$

Define linear maps $\Psi_{\pm} : M_r \to M_n$ by

$$\Psi_{\pm}(T) = \sum_{j,k=1}^{N} \{\delta_{jk} \pm \alpha_{jk}\} V_j^* T V_k, \quad T \in M_r. \tag{2.9}$$

Using (2.8), the fact that $[\beta_{jk}^{\pm}]$ is self-adjoint, manipulating sums, and setting

$$Z_\nu^{\pm} = \sum_{k=1}^{N} \beta_{\nu k}^{\pm} V_k, \tag{2.10}$$

transforms (2.9) into

$$\Psi_{\pm}(T) = \sum_{\nu=1}^{N} (Z_\nu^{\pm})^* T Z_\nu^{\pm}, \quad T \in M_r. \tag{2.11}$$

By (2.11) and the result of Choi mentioned earlier, both Ψ_+ and Ψ_- are completely positive. Furthermore, taking components in (2.5) yields

$$\sum_{j,k=1}^{N} \alpha_{jk} V_j^* A_l V_k = 0, \quad l = 1, \ldots, L. \tag{2.12}$$

Since $\{A_1, \ldots, A_L\}$ is a basis for $\mathscr{S}$, (2.12) gives

$$\sum_{j,k=1}^{N} \alpha_{jk} V_j^* T V_k = 0, \quad \text{if } T \in \mathscr{S}. \tag{2.13}$$

In particular, if $T \in \mathcal{S}$, (2.3), (2.9) and (2.13) imply that $\Psi(T) = \Psi_+(T) = \Psi_-(T)$; the completely positive maps $\Psi_\pm$ thus belong to $A(\varphi, \mathcal{S})$. Manipulating (2.9) and (2.3) gives:

(2.14) $$\Psi = \frac{1}{2}(\Psi_+ + \Psi_-).$$

The extremality of Ψ implies that

(2.15) $$\Psi(\cdot) = \Psi_+(\cdot) = \Psi_-(\cdot).$$

This last equation and a remark of Choi [6, Remark 4] imply that the self-adjoint matrices $[\beta^\pm_{jk}]$ are also isometries:

(2.16) $$\sum_{\nu=1}^{N} \overline{\beta^\pm_{\nu j}} \beta^\pm_{\nu k} = \delta_{jk}.$$

The self-adjointness of $\beta^\pm_{jk}$, (2.8) and (2.16) reveal that $\alpha_{jk} = 0$, $j, k = 1, \ldots, N$. This contradicts (2.5); the set $\{G_{jk}\}$ is linearly independent.

Conversely, suppose $\{G_{jk}\}$ is a linearly independent set; in addition, suppose Ψ satisfies

(2.17) $$\Psi = \frac{1}{2}(\Psi_1 + \Psi_2),$$

where Ψ_1, $\Psi_2 \in A(\varphi, \mathcal{S})$ and have the canonical decompositions,

(2.18) $$\Psi_1(T) = \sum_{j=1}^{N'} X_j^* T X_j, \quad \Psi_2(T) = \sum_{j=1}^{N''} Y_j^* T Y_j,$$

where $T \in M_r$. Substitute (2.18) into (2.17) and replace Ψ by its canonical decomposition (2.3):

(2.19) $$\sum_{j=1}^{N} V_j^* T V_j = \frac{1}{2} \sum_{j=1}^{N'} X_j^* T X_j + \frac{1}{2} \sum_{j=1}^{N''} Y_j^* T Y_j.$$

Choi's remark [6, Remark 4] and (2.19) imply that both the X_j's and Y_j's are linear combinations of the V_j's. In particular,

(2.20) $$X_j = \sum_{k=1}^{N} \mu_{jk} V_k, \quad j = 1, \ldots, N'.$$

Define the set of matrices

(2.21) $$H_{jk} = \bigoplus_{l=1}^{L} X_j^* A_l X_k, \quad j,k = 1, \ldots, N';$$

substitute (2.20) into (2.21) and collect terms:

$$H_{jk} = \sum_{p,q=1}^{N} \bar{\mu}_{jp}\mu_{kq}G_{pq}. \tag{2.22}$$

Because both Ψ and Ψ_1 are in $A(\varphi, \mathscr{S})$, $\sum V_j^* A_l V_j = \sum X_j^* A_l X_j = \varphi(A_l)$. Setting $j = k$ in (2.21), (2.22), (2.4) and summing then gives

$$\sum_{j=1}^{N'} H_{jj} = \sum_{j=1}^{N} G_{jj} = \bigoplus_{l=1}^{L} \varphi(A_l). \tag{2.23}$$

Put the expression for H_{jj} gotten from (2.22) into (2.23) and manipulate the sums:

$$\sum_{p,q=1}^{N} \left\{ \sum_{j=1}^{N'} \bar{\mu}_{jp}\mu_{jq} - \delta_{pq} \right\} G_{pq} = 0. \tag{2.24}$$

The linear independence of $\{G_{pq}\}$ and (2.24) reveals that

$$\sum_{j=1}^{N'} \bar{\mu}_{jp}\mu_{jq} = \delta_{pq}. \tag{2.25}$$

Substitute (2.20) into the expression for Ψ_1 in (2.18), collect terms, and use (2.25):

$$\Psi_1(T) = \Psi(T), \qquad T \in M_r. \tag{2.26}$$

From (2.26), (2.17) it follows that $\Psi = \Psi_1 = \Psi_2$; hence Ψ is extreme in $A(\varphi, \mathscr{S})$. This completes the proof.

REMARK. The proof of Theorem 2.1 was motivated by various proofs used by Choi [6, Remark 4 and Theorem 5].

In the canonical decomposition (2.3) for an extreme point Ψ, how large can N be?

COROLLARY 2.1. *Let Ψ be an extreme point of $A(\varphi, \mathscr{S})$ and let N be as in the canonical decomposition for Ψ given in (2.3). Then N satisfies*

$$N \leqslant \sqrt{L}\, n, \tag{2.27}$$

where L is the dimension of $\mathscr{S}$.

Proof. From (2.4), there are N^2 elements in $\{G_{jk}\}$. On the other hand, the space $M_n \oplus \ldots \oplus M_n$ (L-copies) has dimension Ln^2. For $\{G_{jk}\}$ to be linearly independent, it is necessary that $N^2 \leqslant Ln^2$; (2.27) then immediately follows.

REMARK. The referee has pointed out that it is possible to prove Corollary 2.1 without using Theorem 2.1. His elegant proof follows along lines similar to the proof used later in Remark 2.1.

An interesting application of the results given above is a non-computational, geometric proof of the Toeplitz-Hausdorff Theorem.

COROLLARY 2.2. (Toeplitz-Hausdorff Theorem). *The spatial numerical range of a bounded linear operator defined on a complex Hilbert space is convex.*

Proof. As P. R. Halmos notes [7], it suffices to give a proof for the case in which the Hilbert space is $\mathbf{C}^2$ and the operators are 2×2 matrices (M_2). In this case; the spatial numerical range and numerical range of the operator coincide. Let $v_1, v_2 \in \mathbf{C}^2$ have $\|v_j\| = 1$, $j = 1,2$, and define the (completely) positive maps φ_1, φ_2, and φ by

$$\varphi_j(T) = \langle Tv_j,\ v_j\rangle, \quad j = 1,2; \tag{2.28}$$

$$\varphi(T) = \alpha\varphi_1(T) + (1-\alpha)\varphi_2(T), \quad 0 < \alpha < 1. \tag{2.29}$$

Here $T \in M_2$. Fix $T \in M_2$ and put $\mathscr{S} = \operatorname{span}(I_2, T, T^*)$. The convex set $A(\varphi, \mathscr{S})$ consisting of all positive linear functionals on M_2 which agree with φ on the space $\mathscr{S}$ is clearly compact. It thus has at least one extreme point — say, $\Psi(\cdot)$. The integer N in the decomposition for Ψ given in (2.3) is less than $\sqrt{3}$, because $n = 1$, $L = 3$; thus $N = 1$. The map Ψ then has the form

$$\Psi(S) = v^*Sv = \langle Sv, v\rangle, \quad S \in M_2. \tag{2.3}$$

Since $\Psi \in A(\varphi, \mathscr{S})$, it satisfies the following: $\Psi(I_2) = 1$, $\Psi(T) = \varphi(T)$ (and $\Psi(T^*) = \varphi(T^*)$). The first of these equations implies that $\|v\| = 1$, the second that

$$\langle Tv, v\rangle = \alpha\langle Tv_1, v_1\rangle + (1-\alpha)\langle Tv_2, v_2\rangle.$$

Thus the spatial numerical range is convex.

REMARK 2.1. The proof of the Toeplitz-Hausdorff Theorem given in Corollary 2.2 relies heavily on Theorem 2.1 and Corollary 2.1. For the case needed (i.e. $T \in M_2$; positive linear functionals rather than the more general completely positive maps), a direct route following the lines of Theorem 2.1 is available: Fix $T \in M_2$; let $\varphi \in S_1$ (= norm-one positive linear functionals on M_2), $\mathscr{S} = \operatorname{span}\{I_2, T, T^*\}$, and $A = \{\Psi \in S_1 : \Psi|_{\mathscr{S}} = \varphi|_{\mathscr{S}}\}$. Since A is clearly a compact, convex set, it has an extreme point Ψ. If Ψ is also an extreme point of S_1, then, as is well-known, $\Psi(\cdot) = \langle(\cdot)x, x\rangle$, $\|x\| = 1$; hence, $\Psi(T) = \varphi(T) = \langle Tx, x\rangle$ and the Toeplitz-Hausdorff Theorem holds. So suppose Ψ is not an extreme point of S_1. Using the identification $M_2^* \approx M_2$, Ψ may be uniquely represented as $\Psi(\cdot) = \operatorname{tr}((\cdot)P)$, where $P \in M_2$, $P \geqslant 0$, $\mathrm{tr}P = 1$. The fact that Ψ is not extreme in S_1 means that the eigenvalues of P satisfy $0 < \lambda_j < 1$, $j = 1,2$. Since the complex dimension of $\mathscr{S}$ is at most 3, and since $\mathscr{S}$ is a self-adjoint subspace of M_2, its orthogonal complement relative to the trace inner product is also a self-adjoint subspace and contains a self-adjoint matrix $Q \neq 0$. Obviously Q satisfies $\mathrm{tr}Q = \mathrm{tr}TQ = \mathrm{tr}T^*Q = 0$. In addition, it is clearly possible to choose $\varepsilon > 0$ so small that $P \pm \varepsilon Q$ is positive as well as self-adjoint. The corresponding functionals $\Psi_\pm(\cdot) \equiv \mathrm{tr}((\cdot)(P \pm \varepsilon Q))$ are, however, in A; moreover, $\Psi = \frac{1}{2}(\Psi_+ + \Psi_-)$ — which contradicts Ψ being an extreme point in A. Thus Ψ is extreme in S_1 and the proof is complete.

A more elementary proof with a different flavor was suggested by the referee: Define $P \in M_2$ by $P_{jk} = \varphi(E_{kj})$, where E_{kj} is a 2×2 matrix with one in the $j - k$ position and zeros elsewhere. Note that $\varphi(X) = \mathrm{tr}(XP)$ for all $X \in M_2$, so P is positive and has trace one. Construct Q as above. For $t \in \mathbf{R}$, let $F(t) = P + tQ$. Since $\mathrm{tr}Q = 0$, the signs of the eigenvalues of Q differ; for large t, $F(t)$ is not semi-definite. By continuity of the determinant, there is a $t_0 \geqslant 0$ with $F(t_0)$ having determinant zero and trace one. Thus $F(t_0)$ is a rank one projection. Choosing a unit vector $x_0 \in \mathrm{Range}(F(t_0))$ then gives $\varphi(T) = \mathrm{tr}(TF(t_0)) = \langle Tx_0, x_0 \rangle$, and the result is again obtained.

The proof of Corollary 2.2 and the proof in Remark 2.1 answer a question raised by P. R. Halmos in [7; 8, p. 110].

REMARK 2.2. The bound for N given in Corollary 2.1 does not depend on the dimension of M_r; moreover, the bound is of order n. Is this bound for N the best possible? While the answer is not known for the general case, the referee pointed out the following: *There are examples in which N is at least $[\sqrt{L}]n$. (Here $[x]$ is the greatest integer less than or equal to x). In particular, if L is a perfect square, the bound in Corollary 2.1 is best possible.*

In constructing such examples, the following facts are needed: Let E_{jk} be the elementary matrix with one in the $j - k$ position. Choi [6, Theorems 1 and 2] has shown that φ is in $\mathrm{CP}[M_r, M_n]$ if and only if the $nr \times nr$ matrix $\hat{\varphi} \equiv \sum_{j,k=1}^{r} \varphi(E_{jk}) \otimes E_{jk}$ is positive. Moreover, a careful reading of the proof used by Choi reveals that the integer N in the canonical decomposition for φ given in (2.1) is precisely the rank of $\hat{\varphi}$.

To construct the examples, choose $r = [\sqrt{L}] + 1$ and let $\varphi: M_r \to M_n$ be defined by $\varphi(T) = \mathrm{tr}(T)I_n$. Take $\mathscr{S}$ to be an L-dimensional self-adjoint subspace of M_r which contains all matrices whose last row and column vanish. If $\Psi \in A(\varphi, \mathscr{S})$, it is clear that $\Psi(E_{jk}) = \varphi(E_{jk}) = \delta_{jk}I_n$, where $j, k \leqslant r - 1$. The matrix $\hat{\Psi}$ thus contains $I_{n(r-1)}$ as a block; the rank of $\hat{\Psi}$ is then at least $n(r - 1) = n[\sqrt{L}]$, so

$N \geqslant n[\sqrt{L}]$. In particular, if $\hat{\Psi}$ is extreme and L is a perfect square, this and Corollary 2.1 imply that $N = \sqrt{L}n$.

A second consequence of Theorem 2.1 and its first corollary is a new characterization for the matrix ranges of an operator $T \in M_r$.

COROLLARY 2.3. *If* $T \in M_r$, *then* $p \in W_n(T)$ *if and only if there exists* $\Psi \in \mathrm{CP}[M_r, M_n; I_n]$ *such that* $p = \Psi(T)$ *and* Ψ *has the canonical decomposition*

$$(2.31) \qquad \Psi(S) = \sum_{j=1}^{N} V_j^* S V_j, \quad \sum_{j=1}^{N} V_j^* V_j = I_n, \quad N < \sqrt{3}\, n,$$

where $S \in M_r$.

Proof. An $n \times n$ matrix p is in $W_n(T)$ if and only if there exists a state $\varphi \in \mathrm{CP}[M_r, M_n; I_n]$ such that $p = \varphi(T)$. The set of all completely positive maps whose restrictions to $\mathscr{S} = \operatorname{span}\{I_r, T, T^*\}$ agree with φ is the convex set $A(\varphi, \mathscr{S})$. This is easily seen to be a compact subset of $\mathscr{B}(M_r, M_n)$ and therefore has an extreme point Ψ. Corollary 2.1 implies that Ψ has the form (2.31). Since $\Psi \in A(\varphi, \mathscr{S})$, $\Psi(T) = \varphi(T) = p$. This completes the proof.

The characterization given in the corollary allows $W_n(T)$ to be defined in terms of the subset of completely positive maps having the form (2.31), instead of all maps in $\mathrm{CP}[M_r, M_n; I_n]$. Since the maximum integer N needed in (3.21) is of order n and since this bound is independent of r, it is natural to try to replace M_r by $\mathscr{B}(\mathscr{H})$. To this end, define the following: The *length* of a unital completely positive map $\varphi: \mathscr{B}(\mathscr{H}) \to M_n$ is the smallest integer N such that

$$(2.32) \qquad \varphi(\cdot) = \sum_{j=1}^{N} V_j^*(\cdot) V_j, \quad \sum V_j^* V_j = I_n,$$

where $V_j \in \mathscr{B}(\mathbf{C}^n, \mathscr{H})$; denote the length of φ by $\ell(\varphi)$. If such a representation does not exist, put $\ell(\varphi) = +\infty$. The n^{th} *vector-matrix range* for $T \in \mathscr{B}(\mathscr{H})$ is the set

$$(2.33) \qquad w_n(T) \equiv \{p \in M_n : p = \varphi(T),\ \varphi \in \mathrm{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n],\ \ell(\varphi) < \sqrt{3}\, n\}.$$

In words, this set consists of the image of T under all unital completely positive maps from $\mathscr{B}(\mathscr{H})$ to M_n whose length is less than $\sqrt{3}\, n$. (The term "vector-matrix range" was chosen to reflect the close relationship between the completely positive maps used and the underlying Hilbert space.)

REMARK 2.3. In case $\mathscr{H} \approx \mathbf{C}^r$ ($\mathscr{B}(\mathscr{H}) \approx M_r$), Corollary 2.3 implies that the n^{th} matrix range and n^{th} vector matrix range coincide; that is,

$$(2.34) \qquad w_n(T) = W_n(T), \quad T \in \mathscr{B}(\mathscr{H}) \approx M_r.$$

REMARK 2.4. In case $n = 1$, the first vector-matrix range and the spatial numerical range coincide. For $n > 1$, it is clear that $\mathscr{W}_n(T) \subseteq w_n(T) \subseteq W_n(T)$.

Properties of vector-matrix ranges will be explored in the next section.

3. VECTOR-MATRIX RANGES

Vector-matrix ranges share with spatial matrix ranges the property of being defined in terms of the underlying Hilbert space $\mathscr{H}$. ($\mathscr{H}$ will be assumed infinite dimensional throughout this section.) As will be seen, vector-matrix ranges are closely related to corresponding spatial matrix ranges: the n^{th} vector-matrix range is the matricial convex hull of the n^{th} spatial matrix range; as such, it is convex and has the n^{th} matrix range as closure (Bunce and Salinas [5], Theorem 3.5). The relationship between the n^{th} vector-matrix range and the n^{th} matrix range is thus quite analogous to that between the spatial numerical range and the numerical range. The purpose of this section is to investigate this analogy. To do this, the following lemma is essential:

LEMMA 3.1. *Let $T \in \mathscr{B}(\mathscr{H})$ and suppose $p \in W_n(T)$. If $p = \varphi(T)$, where $\varphi \in \mathrm{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n]$, and if the length of φ is finite, then $p \in w_n(T)$.*

Proof. Put $\nu = \ell(\varphi)$; by assumption, $\nu < \infty$ and the map φ has the form

$$\varphi(\cdot) = \sum_{j=1}^{\nu} V_j^*(\cdot)V_j, \quad \sum_{j=1}^{\nu} V_j^* V_j = I_n, \tag{3.1}$$

where $V_j \in \mathscr{B}(\mathbf{C}^n,\mathscr{H})$. Take $\xi_1, \ldots, \xi_n$ to be the standard basis for $\mathbf{C}^n$; that is, $\xi_k = \mathrm{col}\,(0, \ldots, 0,1,0, \ldots, 0)$, where one is in the k^{th} position. Define Q to be the orthogonal projection in $\mathscr{H}$ onto $\mathrm{span}\{V_j\xi_k\}$. Since this space is finite dimensional, Q is a finite rank projection; moreover, $Q\mathscr{H} \approx \mathbf{C}^r$, where $r = \mathrm{rank}Q$. In addition, it is clear that

$$\varphi(S) = \varphi(QSQ), \quad S \in \mathscr{B}(\mathscr{H}); \tag{3.2}$$

hence the map $\varphi' \colon Q\mathscr{H} \to M_n$ given by

$$\varphi'(R) = \varphi(QRQ), \quad R \in \mathscr{B}(Q\mathscr{H}), \tag{3.3}$$

is completely positive and unital. Set

$$T' = QTQ \in \mathscr{B}(Q\mathscr{H}), \tag{3.4}$$

and apply (3.3) to get

$$p = \varphi'(T') \in W_n(T'). \tag{3.5}$$

From Remark 2.3 and the finite dimensionality of $Q\mathscr{H}$, (3.5) implies that $p \in w_n(T')$. Thus there exists $\Psi' \in \mathrm{CP}[\mathscr{B}(Q\mathscr{H}), M_n; I_n]$ such that

$$p = \Psi'(T'), \quad \ell(\Psi') < \sqrt{3}\, n. \tag{3.6}$$

Set $\Psi(S) = \Psi'(QSQ)$, $S \in \mathscr{B}(\mathscr{H})$. Inspection reveals that $\Psi \in \mathrm{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n]$, $p = \Psi(T)$, and $\ell(\Psi) < \sqrt{3}\, n$: $p \in w_n(T)$, and the proof is done.

REMARK. Since there exist operators with non-closed spatial numerical range, the hypothesis that φ has finite length cannot be omitted from Lemma 3.1.

The matricial convex hull of a subset $\Gamma \subseteq M_n$, denoted by mconv(Γ), is the set of all $n \times n$ matrices having the form $\sum_{j=1}^{J} X_j^* g_j X_j$, $\sum_{j=1}^{J} X_j^* X_j = I_n$, $J < \infty$, where $X_j \in M_n$, $g_j \in \Gamma$. The following theorem relates vector-matrix ranges and spatial matrix ranges; it is the analogue of the Toeplitz-Hausdorff Theorem.

THEOREM 3.1. *The n^{th} vector-matrix range for $T \in \mathscr{B}(\mathscr{H})$ is the matricial convex hull of the corresponding spatial matrix range; that is,*

$$w_n(T) = \text{mconv}(\mathscr{W}_n(T)). \tag{3.7}$$

Proof. If $p \in \text{mconv}(\mathscr{W}_n(T))$, there exist $n \times n$ matrices $X_1, \ldots, X_J$ and isometries $V_j \colon \mathbf{C}^n \to \mathscr{H}$ such that

$$p = \sum_{j=1}^{J} X_j^* V_j^* T V_j X_j, \quad \sum_{j=1}^{J} X_j^* X_j = I_n. \tag{3.8}$$

From the form given in (3.8), it is clear that $p = \varphi(T)$, $\varphi \in \text{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n]$, $\ell(\varphi) < \infty$; by Lemma 3.1, $p \in w_n(T)$:

$$\text{mconv}(\mathscr{W}_n(T)) \subseteq w_n(T). \tag{3.9}$$

If $p \in w_n(T)$, then p is given by

$$p = \sum_{j=1}^{N} Z_j^* T Z_j, \quad \sum_{j=1}^{N} Z_j^* Z_j = I_n, \quad N < \sqrt{3}\, n, \tag{3.10}$$

where $Z_j \in \mathscr{B}(\mathbf{C}^n, \mathscr{H})$. Define K_j to be the positive square root of the matrix $Z_j^* Z_j$; thus,

$$K_j^2 = Z_j^* Z_j, \quad K_j \geqslant 0, \quad j = 1, \ldots, N. \tag{3.11}$$

Next consider the map V_j: Range $(K_j) \to \mathscr{H}$ defined by

$$V_j(K_j x) = Z_j x, \quad x \in \mathbf{C}^n. \tag{3.12}$$

It is easy to check that $\|V_j(K_j x)\| = \|Z_j x\|$, and so V_j may be extended to all of $\mathbf{C}^n$ as an isometry. In addition, inspection of (3.12) yields

$$Z_j = V_j K_j, \quad Z_j^* = K_j V_j^*. \tag{3.13}$$

Equations (3.10), (3.11), (3.13) and the fact that each V_j is an isometry imply that p has the form

$$p = \sum_{j=1}^{N} K_j (V_j^* T V_j) K_j, \quad \sum K_j^2 = I_n, \quad V_j^* V_j = I_n; \tag{3.14}$$

hence $p \in \text{mconv}(\mathscr{W}_n(T))$:

(3.15) $$w_n(T) \subseteq \text{mconv}(\mathscr{W}_n(T)).$$

Combine (3.9) and (3.15) to complete the proof.

The following consequence of Theorem 3.1 stresses the analogy between vector-matrix range and spatial numerical range.

COROLLARY 3.1. *The n^{th} vector-matrix range for $T \in \mathscr{B}(\mathscr{H})$ is matricially convex and has the n^{th} matrix range as its closure.*

Proof. In view of Theorem 3.1, the matricial convexity is obvious; that $w_n(T)^- = W_n(T)$ follows from a theorem of Bunce and Salinas [5, Theorem 3.5].

If C is a convex subset of a finite dimensional real vector space V, then the *relative interior* of C (ri(C)) is the interior of C when C is regarded as a subset of its affine hull (aff(A)) (Rockafellar [12], Section 6). Viewing M_n as a $2n^2$-dimensional real vector space yields the following:

COROLLARY 3.2. *If $T \in \mathscr{B}(\mathscr{H})$, then the relative interiors of $w_n(T)$ and $W_n(T)$ are the same; that is,*

(3.16) $$\text{ri}(w_n(T)) = \text{ri}(W_n(T)).$$

Proof. Convex subsets of a finite dimensional real vector space which have the same closure also have the same relative interior (Rockafellar, [12], Corollary 6.3.1, p. 46). By Corollary 3.1, $w_n(T)$ is convex and has the convex set $W_n(T)$ as its closure, so $\text{ri}(w_n(T)) = \text{ri}(W_n(T))$.

In studying the topological properties of a convex set, only its affine hull needs to be considered; the rest of the underlying space may be dispensed with. Given the importance of the affine hull of a convex set, it is natural to ask what the affine hulls are for the matrix ranges of an operator. The answer to this question is easily obtained from the following strengthened version of a theorem found in [13, Theorem 4.6] and the subsequent remark:

THEOREM 3.2. *Let $r > 0$, $T \in \mathscr{B}(\mathscr{H})$, and $z_0 \in W_1(T)$. If $W_1(T)$ contains the closed disk $B_1(z_0, r) \equiv \{z \in \mathbf{C}: |z - z_0| \leqslant r\}$, then $W_n(T)$ contains the closed ball $B_n(z_0, r) = \{\mu \in M_n: \|z_0 I_n - \mu\| \leqslant r\}$.*

Proof. Because $W_n(T) = z_0 I_n + W_n(T - z_0 I_n)$, there is no loss of generality in assuming that $z_0 = 0$.

Two results found in [11] are needed:

(1) If $H_n(\cdot)$ is the support function for $W_n(T)$, then $p \in W_n(T)$ if and only if

(3.17) $$\text{Re}[\text{tr}(\lambda^* p)] \leqslant H_n(\lambda)$$

for every $\lambda \in M_n$ [11, Proposition, 2.4].

(2) If $z_1, \ldots, z_n$ are arbitrary points in $W_1(T)$, and if $\{v_j\}_{j=1}^n$ is an orthonormal basis for $\mathbf{C}^n$, then

(3.18) $$H_n(\lambda) \geqslant \sum_{j=1}^{n} \langle \text{Re}(z_j \bar{\lambda}) v_j, v_j \rangle,$$

where $\bar{\lambda}$ is the matrix whose components are complex conjugates of those in λ [11, Proposition 3.3].

To obtain the result, first put $\bar{\lambda}$ into its polar form; that is, write $\bar{\lambda} = \rho u$ where $\rho \geqslant 0$, u is unitary. Next, take $v_1, \ldots, v_n$ to be the orthonormal set of eigenvectors corresponding to $e^{i\theta_1}, \ldots, e^{i\theta_n}$ — the eigenvalues of u — and from $B_1(0, r) \subset W_1(T)$, take $z_1 = re^{-i\theta_1}, \ldots, z_n = re^{-i\theta_n}$. With these choices, (3.18) becomes

$$H_n(\lambda) \geqslant r \left(\sum_{j=1}^{N} \langle \rho v_j, v_j \rangle \right) = r(\mathrm{tr}\rho) = r|\lambda|_1, \tag{3.19}$$

where $|\cdot|_1$ is the trace norm of a matrix. From standard matrix theory,

$$\mathrm{Re}(\mathrm{tr}(\lambda^* p)) \leqslant |\mathrm{tr}(\lambda^* p)| \leqslant \|p\| \, |\lambda|_1. \tag{3.20}$$

Combining (3.17), (3.19) and (3.20) then gives the inclusion $W_n(T) \supseteq B_n(0, r)$, and the proof is complete.

REMARK 3.1. If $W_1(T)$ does not contain a disk, then it is a point or a line segment. In either case, T has the form $T = z_1 I + z_2 S$ where S has numerical range $[-1,1]$. Matrix ranges for self-adjoint operators are known [2, p. 302]. In particular, $W_n(S) = \{p \in M_n : p = p^*, \ \|p\| \leqslant 1\}$ and hence,

$$W_n(T) = \{p \in M_n : p = z_1 I_n + z_2 q, \ q = q^*, \ \|q\| \leqslant 1\}. \tag{3.21}$$

Direct calculation or a modified version of the proof of Theorem 3.2 then implies that if the interval joining $z_0 + re^{i\theta}$ and $z_0 - re^{i\theta}$ is contained in $W_1(T)$, then $W_n(T)$ includes the set of all matrices of the form $z_0 I_n + re^{i\theta} q$, where $q = q^*$, $\|q\| \leqslant 1$

Two corollaries are immediate:

COROLLARY 3.3. *If z_0 is in the relative interior of $W_1(T)$, then $z_0 I_n$ is in the relative interior of $W_n(T)$.*

Proof. Apply Theorem 3.2 and Remark 3.1.

The next corollary characterizes the affine hulls of matrix ranges.

COROLLARY 3.4. *If the affine hull of $W_1(T)$ is a point z_1, or a line through the points $z_1 \pm z_2$ ($z_2 \neq 0$), or all of $\mathbf{C}$, then the affine hull of $W_n(T)$ is $\{z_1 I_n\}$, or $\{p \in M_n : p = z_1 I_n + z_2 q, \ q = q^*\}$, or M_n, respectively.*

Proof. Again apply Theorem 3.2 and Remark 3.1.

There is an interesting interplay among the matrix range, vector-matrix range, and essential matrix range which is analogous to that among the numerical range, spatial numerical range, and essential numerical range. The results which follow resemble those obtained by Lancaster [10].

For $T \in \mathscr{B}(\mathscr{H})$, the n^{th} *essential matrix range* is

$$W_n^e(T) = \{p \in M_n : p = \varphi(T), \ \varphi \in \mathrm{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n], \ \mathscr{C}(\mathscr{H}) \subset \ker\varphi\}, \tag{3.22}$$

where $\mathscr{C}(\mathscr{H})$ consists of all compact operators on $\mathscr{H}$. The following result is a restatement of a result of Bunce and Salinas in terms of vector matrix ranges.

THEOREM 3.3. *If T is in $\mathscr{B}(\mathscr{H})$, then*

$$W_n(T) = \text{mconv}[w_n(T) \cup W_n^e(T)]. \tag{3.23}$$

Proof. From [5, Theorem 3.7], $W_n(T)$ is the matricial convex hull of $\mathscr{W}_n(T) \cup W_n^e(T)$. Clearly, $\mathscr{W}_n(T)$ may be replaced by its matricial convex hull, $w_n(T)$ (see Theorem 3.1); hence, (3.23) holds.

REMARKS. The result is analogous to a theorem of Lancaster [10, Theorem 1], but not completely. The presence of the matricial convex hull, rather than the convex hull, is due to the fact that the extreme points of the state space of norm-one positive linear functionals are either of the form $\langle (\cdot)x, x\rangle$, $x \in \mathscr{H}$, $\|x\| = 1$, or annihilate the compact operators. A similar dichotomy does not occur for the generalized state spaces, $\text{CP}[\mathscr{B}(\mathscr{H}), M_n; I_n]$.

The following two corollaries to Theorem 3.3 illustrate nicely the analogy between vector-matrix ranges and the spatial numerical range. (For the spatial numerical range case, see [10, Corollary 1 and Corollary 7].)

COROLLARY 3.5. *If $T \in \mathscr{B}(\mathscr{H})$, then $w_n(T) = W_n(T)$ if and only if $W_n^e(T) \subseteq w_n(T)$.*

Proof. If $W_n^e(T) \subseteq w_n(T)$, then

$$\text{mconv}(w_n(T) \cup W_n^e(T)) = \text{mconv}(w_n(T)). \tag{3.24}$$

From (3.23), the left-side above is $W_n(T)$; from Corollary 3.1, $\text{mconv}(w_n(T)) = w_n(T)$; hence, $W_n(T) = w_n(T)$. The converse is a trivial consequence of the inclusion, $W_n^e(T) \subseteq W_n(T)$.

COROLLARY 3.6. *If $T \in \mathscr{B}(\mathscr{H})$ is compact, and if 0 is in the relative interior of $W_1(T)$, then $w_n(T) = W_n(T)$ for all $n \geqslant 1$.*

Proof. Direct application of the definition gives that $W_n^e(T) = \{0_n\}$ for every $T \in \mathscr{C}(\mathscr{H})$. Making the additional assumption that $0 \in \text{ri}(W_1(T))$ and using Corollary 3.3 yields

$$0_n \in \text{ri}(W_n(T)). \tag{3.25}$$

From (3.16) and (3.25) it follows that $W_n^e(T) \subseteq w_n(T)$; Corollary 3.5 then implies that $w_n(T) = W_n(T)$.

4. CLOSING REMARKS

The authors wish to note that Vern Paulsen has looked at problems similar to those studied here. [Private communication].

There are several interesting questions which arise in the course of the work:

QUESTION 1. In defining the vector-matrix range $w_n(T)$, the unital completely positive maps were restricted to having length less than $\sqrt{3}\, n$. Is it possible to use a smaller bound? Will n do?

QUESTION 2. Let $\mathscr{A}$ be a C^*-algebra having identity I and let $T \in \mathscr{A}$; let S denote the set of norm-one positive linear functionals on $\mathscr{A}$ and $\text{ext}(S)$ the extreme points of S. Let $\mathscr{E}(T) = \{\varphi(T): \varphi \in \text{ext}(S)\}$. When is $\mathscr{E}(T) = W(T)$, the numerical range? When is $\mathscr{E}(T)$ convex?

QUESTION 3. Let $\mathscr{A}$, T, and I be as in Question 2. If $S_n = \text{CP}[\mathscr{A}, M_n;\ I_n]$, then what subsets of S_n sweep out the matrix range $W_n(T)$?

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