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ON THE SMOOTHNESS OF SPHERE EXTENSIONS

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Let $\mathscr{H}$ be an infinite dimensional complex separable Hilbert space. By $\mathscr{L}(\mathscr{H})$ and $\mathscr{K}(\mathscr{H})$ we shall denote the C^*-algebras of bounded operators and compact operators on $\mathscr{H}$, respectively, and $Q(\mathscr{H})$ will denote the quotient $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ with canonical surjection $\pi\colon \mathscr{L}(\mathscr{H}) \to Q(\mathscr{H})$. For X a compact metrizable space an extension of the algebra $C(X)$ of complex continuous functions on X by $\mathscr{K}(\mathscr{H})$ is defined by a unital $*$-monomorphism $\rho\colon C(X) \to Q(\mathscr{H})$ [1], [2]. If X is embedded in $\mathbf{C}^n$, then the co-ordinate functions $\{z_i\}_{i=1}^n$ determine canonical elements $\{\rho(z_i)\}_{i=1}^n$ of $Q(\mathscr{H})$ and we can consider n-tuples of operators $\{T_i\}_{i=1}^n \subset \mathscr{L}(\mathscr{H})$ such that $\rho(z_i) = \pi(T_i)$ for $i = 1,2, \ldots, n$. For $\mathscr{I}$ an ideal in $\mathscr{K}(\mathscr{H})$ containing the finite rank operators, the extension ρ is said to be $\mathscr{I}$-smooth if the $\{T_i\}_{i=1}^n$ can be chosen such that the commutators $[T_i, T_j^*]$, $[T_i, T_j]$ for $i, j = 1,2, \ldots, n$ all lie in $\mathscr{I}$.

In [5] this notion was introduced and various results on $\mathscr{C}_1$-smooth elements were obtained where $\mathscr{C}_p$ denotes the Schatten-von Neumann p-class. It was conjectured for finite complexes X (and proved for $\dim X \leqslant 3$) that a $\mathscr{C}_1$-smooth element of $\mathrm{Ext}(X)$ comes from the one-skeleton. It is reasonable to believe that analogous results hold for $\mathscr{C}_p$-smooth elements and q-skeletons. This paper arose in an attempt at understanding this higher dimensional phenomena.

The basic technique used in [5] in studying $\mathscr{C}_1$-smooth elements depends on the work of Helton-Howe [6], [7] and in particular on the fact that the index of operators in the "smooth" matrix algebras generated by the $\{T_i\}$ can be expressed in terms of traces of commutators. Our principal result in this paper is a partial generalization to the case of spheres. This will depend on a fundamental combinatorial identity relating formal traces of powers in the Grassmann algebra to antisymmetrizations.

If M is a C^∞-manifold and ρ is an element of $\mathrm{Ext}(M)$, then using the functional calculus given in [7] one can show that the smoothness of ρ is the same for any C^∞-embedding of M in $\mathbf{C}^n$. Thus smoothness depends on the differentiable structure for M a C^∞-manifold. A basic ingredient in the study of the smoothness of extensions for C^∞-manifolds is the determination of just how smooth are the extensions for spheres. We show that a $\mathscr{C}_{n-1}$-smooth extension of $S^{2n-1} (n > 1)$, embedded as the unit sphere in $\mathbf{C}^n$, is necessarily trivial, but that nontrivial extensions exist which are $\mathscr{C}_p$-smooth for $p > n$. That all $\mathscr{C}_1$-smooth extensions of S^{2n-1} $(n > 1)$ are trivial was proved by Helton-Howe [6], [7]. Curto gave a proof of this [4] based on an index formula of Markus-Feldman [9] which was our starting point.

We begin with the algebraic identity on which our considerations will be based after introducing the necessary notation.

Let $e_1, \ldots, e_n$ be the canonical orthonormal basis for $\mathbf{C}^n$ as a Hilbert space. Let $\wedge\mathbf{C}^n = \wedge^0\mathbf{C}^n \oplus \wedge^1\mathbf{C}^n \oplus \ldots \oplus \wedge^n\mathbf{C}^n$ be the Grassmann algebra over $\mathbf{C}^n$ with the Hilbert space structure corresponding to the orthonormal basis $(e_J)_{J\subset\{1,\ldots,n\}}$, where $e_\emptyset = 1$ and $e_J = e_{j_1}\wedge\ldots\wedge e_{j_k}$ if $J = \{j_1, \ldots, j_k\}$, $j_1 < \ldots < j_k$. On $\wedge\mathbf{C}^n$ we define as usual the operators a_i by $a_i h = e_i \wedge h$ satisfying the anticommutation relations

$$a_i a_j + a_j a_i = 0 \qquad \text{for all } i, j,$$

$$a_i^* a_j + a_j a_i^* = 0 \qquad \text{for } i \neq j,$$

and

$$a_i^* a_i + a_i a_i^* = I \qquad \text{for all } i.$$

Now let $\mathscr{A}$ be an algebra with unit over $\mathbf{C}$ and consider the algebraic tensor product $\mathscr{A} \otimes \mathscr{L}(\wedge\mathbf{C}^n)$, which can be identified with the $2^n \times 2^n$ matrices $(x_{J,K})_{J,K\subset\{1,\ldots,n\}}$ over $\mathscr{A}$. We shall denote by

$$\tau\colon \mathscr{A} \otimes \mathscr{L}(\wedge\mathbf{C}^n) \to \mathscr{A}$$

the map given by the trace; that is, $\tau(x \otimes a) = x\cdot\mathrm{Tr}(a)$ for x in $\mathscr{A}$ and a in $\mathscr{L}(\wedge\mathbf{C}^n)$, or equivalently in the matricial setting

$$\tau((x_{J,K})_{J,K\subset\{1,\ldots,n\}}) = \sum_{J\subset\{1,\ldots,n\}} x_{J,J}\,.$$

Further, we shall denote by P_o, $P_e \in \mathscr{L}(\wedge\mathbf{C}^n)$ the orthogonal projections onto $\wedge^o(\mathbf{C}^n) = \wedge^1\mathbf{C}^n \oplus \wedge^3\mathbf{C}^n \oplus \ldots$ and respectively $\wedge^e(\mathbf{C}^n) = \wedge^0\mathbf{C}^n + \wedge^2\mathbf{C}^n \oplus \ldots$.

Finally, let $[x_1, \ldots, x_m]$ denote the complete antysimmetric sum of $x_1, \ldots, x_m \in \mathscr{A}$:

$$[x_1, \ldots, x_m] = \sum_\sigma \varepsilon(\sigma) x_{\sigma(1)} \ldots x_{\sigma(m)}\,,$$

where σ in the right-hand sum runs over the symmetric group on $\{1, \ldots, m\}$ and $\varepsilon(\sigma)$ is the sign of the permutation σ.

PROPOSITION 1. *Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathscr{A}$ and consider*

$$d' = x_1 \otimes a_1 + \ldots. + x_n \otimes a_n$$

$$d'' = y_1 \otimes a_1^* + \ldots + y_n \otimes a_n^*.$$

Then we have:

$$\tau((d'+d'')^k(1\otimes P_e - 1\otimes P_o)) =$$

$$= \begin{cases} 0 & \text{for } 1 \leqslant k < 2n \\ (-1)^n [x_1, y_1, x_2, y_2, \ldots, x_n, y_n] & \text{for } k = 2n. \end{cases}$$

Proof. Since $a_i \wedge^o \subset \wedge^e$, $a_i \wedge^e \subset \wedge^o$, $a_i^* \wedge^o \subset \wedge^e$, $a_i^* \wedge^e \subset \wedge^o$ it is easily seen that for $k = 2p+1$ the diagonal entries of the matrix which corresponds to $(d'+d'')^k(1\otimes P_e - 1\otimes P_o)$ are zero and hence $\tau((d'+d'')^k(1\otimes P_e - 1\otimes P_o)) = 0$. Thus it will be sufficient to concentrate on the case when k is even so let $k = 2p$.

To prove the proposition for $k = 2p$, $1 \leqslant p < n$ it will be sufficient to prove that

$$\tau(b(1\otimes P_e - 1\otimes P_o)) = 0$$

for b any of the $(2n)^{2p}$ terms in the expansion of

$$(x_1\otimes a_1 + \ldots + x_n\otimes a_n + y_1\otimes a_1^* + \ldots + y_n\otimes a_n^*)^{2p}.$$

Such a term b is of the form

$$b = X\otimes E$$

where X is a monomial of degree $2p$ in $x_1, \ldots, x_n, \ldots, y_n$ and E is the corresponding monomial in $a_1, \ldots, a_n, a_1^*, \ldots, a_n^*$ obtained by replacing x_i with a_i and y_i with a_i^* in the expression of X. We have:

$$\tau(b(1\otimes P_e - 1\otimes P_o)) = \mathrm{Tr}(E(P_e - P_o))X.$$

Thus, it will be sufficient to prove that for $p < n$ we have $\mathrm{Tr}(E(P_e - P_o)) = 0$. The matrix of $P_e - P_o$ with respect to the basis $(e_J)_{J\subset\{i,\ldots,n\}}$ being diagonal, it will be clearly sufficient to consider only those E, the matrices of which have non-zero diagonal entries. Taking into account the anticommutation relations it is easily seen that this implies that the number of times a certain a_i and respectively, the corresponding a_i^*, appear in the expression of E must be equal. Moreover, in order that $E \neq 0$ it is necessary that between two consecutive occurrences of a_i (respectively a_i^*) in the expression of E, there also be an a_i^* (respectively a_i). Using again the anticommutation relations we get that

$$E = \pm f_{i_1}^+ f_{i_2}^+ \ldots f_{i_s}^+ f_{j_1}^- \ldots f_{j_t}^-$$

where $i_1 < i_2 < \ldots < i_s$, $j_1 < \ldots < j_t$,

$$\{i_1, i_2, \ldots, i_s\} \cap \{j_1, j_2, \ldots, j_t\} = \emptyset, \quad 0 \leqslant s,\ 0 \leqslant t,\ s+t \leqslant p$$

and $f_i^{\pm}$ are the idempotents:

$$f_i^+ = a_i^* a_i, \quad f_i^- = a_i a_i^*.$$

Now, $f_{i_1}^+ \dots f_{i_s}^+ f_{j_1}^- \dots f_{j_t}^-$ is the projection onto the subspace of $\wedge \mathbf{C}^n$ spanned by e_K, where K runs over all subsets of $\{1, \dots, n\}$ such that

$$\{j_1, \dots, j_t\} \subset K \subset \{1, \dots, n\} \setminus \{i_1, \dots, i_s\}.$$

Equivalently, $K = \{j_1, \dots, j_t\} \cup K'$, where

$$K' \subset \{1, \dots, n\} \setminus \{i_1, \dots, i_s, j_1, \dots, j_t\}.$$

Since $n - s - t > 0$ the number of subsets K' with an even number of elements and the number of subsets K' with an odd number of elements are equal, implying

$$\mathrm{Tr}(E(P_e - P_o)) = 0.$$

Turning now to the case $p = n$ $(k = 2n)$ we begin again by looking at $\tau(b(1 \otimes P_e - 1 \otimes P_o))$, where $b = X \otimes E$ is one of the monomials from the expansion of

$$(x_1 \otimes a_1 + \dots + x_n \otimes a_n + y_1 \otimes a_1^* + \dots + y_n \otimes a_n^*)^{2n}.$$

By the preceding discussion, we see that $\tau(b(1 \otimes P_e - 1 \otimes P_o))$ is zero unless each a_i and a_i^* occurs exactly once in the expression of E. This means that

$$E = \pm f_{i_1}^+ \dots f_{i_s}^+ f_{j_1}^- \dots f_{j_t}^-$$

where $s + t = n$ and $\{i_1, \dots, i_s, j_1, \dots, j_t\} = \{1, 2, \dots, n\}$. Then $f_{i_1}^+ \dots f_{i_s}^+ f_{j_1}^- \dots f_{j_t}^-$ is the projection of $\wedge \mathbf{C}^n$ onto the subspace $\mathbf{C}e_J$, where $J = \{j_1, \dots, j_t\}$.

Using these facts it is easily seen that the diagonal entry of $(d' + d'')^{2n}$ corresponding to e_J where $J = \{j_1, \dots, j_t\}$, $\{1, 2, \dots, n\} \setminus J = \{i_1, \dots, i_s\}$, $j_1 < \dots < j_t$, $i_1 < \dots < i_s$, can be written as

$$\varepsilon(\sigma_J) \sum_{\sigma \in S_J} \varepsilon(\sigma) X_\sigma.$$

Here, S_J is the set of permutations σ of $\{1, 2, \dots, 2n\}$ such that $\sigma(2j_k - 1) < \sigma(2j_k)$, $\sigma(2i_l - 1) > \sigma(2i_l)$, $X_\sigma = X_{\sigma(1)} \dots X_{\sigma(2n)}$ where $X_{2m-1} = x_m$, $X_{2m} = y_m$ and σ_J is the permutation which is the product of the transpositions $(2i_l - 1, 2i_l)$, $l = 1, \dots, s$. We have used here the fact that the idempotents $f_1^+, \dots, f_n^+, f_1^-, \dots, f_n^-$ commute.

Thus we obtain:

$$\tau((d' + d'')^{2n}(1 \otimes P_e - 1 \otimes P_o)) =$$

$$= \sum_{J \subset \{1, \dots, n\}} ((-1)^{|J|} \varepsilon(\sigma_J) \sum_{\sigma \in S_J} \varepsilon(\sigma) X_\sigma).$$

Since $\varepsilon(\sigma_J) = (-1)^{n-|J|}$ and the S_J's form a partition of the symmetric group on $\{1, \ldots, 2n\}$, we have

$$\tau((d' + d'')^{2n}(1 \otimes P_e - 1 \otimes P_o)) =$$

$$= (-1)^n \sum_{\sigma} \varepsilon(\sigma) X_\sigma =$$

$$= (-1)^n [x_1, y_1, x_2, y_2, \ldots, x_n, y_n].$$

Q.E.D.

We can now pass to the extensions of $C(S^{2n-1})$. An extension of $C(S^{2n-1})$ is determined by $\rho\colon C(S^{2n-1}) \to Q(\mathscr{H})$. Identifying S^{2n-1} with the unit sphere of $\mathbf{C}^n$, $\rho\colon C(S^{2n-1}) \to Q(\mathscr{H})$ is equivalent to specifying an n-tuple $(x_1, \ldots, x_n) \subset Q(\mathscr{H})$ such that $[x_i, x_j] = 0$, $[x_i, x_j^*] = 0$ for all $1 \leqslant i, j \leqslant n$ and $\sum_{1 \leqslant i \leqslant n} x_i^* x_i = 1$. The corresponding ρ is determined by $\rho(z_i) = x_i$. Passing to preimages in $\mathscr{L}(\mathscr{H})$ of $x_1, \ldots, x_n$ we see that ρ will be determined by an n-tuple $(T_1, \ldots, T_n) \subset \mathscr{L}(\mathscr{H})$ such that $[T_i, T_j]$, $[T_i, T_j^*]$, and $\sum_{1 \leqslant i \leqslant n} T_i^* T_i - I$ are all in $\mathscr{K}(\mathscr{H})$.

Recall from [2] that $\mathrm{Ext}(S^{2n-1}) \cong \mathbf{Z}$ with the isomorphism being given by the homomorphism $\mathrm{Ext}(S^{2n-1}) \to \mathrm{Hom}_{\mathbf{Z}}(K^1(S^{2n-1}), \mathbf{Z})$ associated with index. More precisely, there is a certain unitary matrix α with entries in $C(S^{2n-1})$, which is a generator of $K^1(S^{2n-1})$, such that $\mathrm{Ext}(S^{2n-1}) \to \mathbf{Z}$ is given by $[\tau] \to \mathrm{index}\,\rho(\alpha)$ where $\rho(\alpha)$ is viewed as a unitary in $Q(\mathscr{H} \oplus \ldots \oplus \mathscr{H})$.

With the notation preceding Proposition 1 for an extension ρ of $C(S^{2n-1})$ determined by $T_1, \ldots, T_n$, a preimage of $\rho(\alpha)$ can be described as an element of $\mathscr{L}(\mathscr{H} \otimes \wedge^e(\mathbf{C}^n))$ in the following way. Let $\eta\colon \wedge^o(\mathbf{C}^n) \to \wedge^e(\mathbf{C}^n)$ be unitary, let

$$d' = T_1 \otimes a_1 + \ldots + T_n \otimes a_n$$

and

$$d'' = T_1^* \otimes a_1^* + \ldots + T_n^* \otimes a_n^*$$

be elements of $\mathscr{L}(\mathscr{H} \otimes \wedge(\mathbf{C}^n))$, and define

$$A = (I \otimes \eta)(d' + d'') \mid (\mathscr{H} \otimes \wedge^e(\mathbf{C}^n)) \in \mathscr{L}(\mathscr{H} \otimes \wedge^e(\mathbf{C}^n)),$$

which is the preimage of the $\tau(\alpha)$ we shall use. Such matrices A appear in the work of Vasilescu [10] and Curto [4] on Fredholm n-tuples of operators. Note also that if we identify $\wedge(\mathbf{C}^n) = \wedge^e(\mathbf{C}^n) \oplus \wedge^o(\mathbf{C}^n)$ with $\wedge^e(\mathbf{C}^n) \oplus \wedge^e(\mathbf{C}^n)$, then $d' + d''$ can be written as

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Moreover, A is essentially unitary.

Defining

$$\tau_e\colon \mathscr{L}(\mathscr{H} \otimes \wedge^e(\mathbf{C}^n)) \simeq \mathscr{L}(\mathscr{H}) \otimes \mathscr{L}(\wedge^e(\mathbf{C}^n)) \to \mathscr{L}(\mathscr{H})$$

by

$$\tau_e(S \otimes X) = (\mathrm{Tr}X)S,$$

we see that

$$\tau_e((A^*A)^p - (AA^*)^p) =$$
$$= \tau((d' + d'')^{2p}(I \otimes P_e - I \otimes P_o))$$

where τ is the map used in Proposition 1 in the case $\mathscr{A} = \mathscr{L}(\mathscr{H})$.

We have used here the fact that

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}^{2p} = \begin{pmatrix} (A^*A)^p & 0 \\ 0 & (AA^*)^p \end{pmatrix}.$$

PROPOSITION 2. *Let $T_1, \ldots, T_n \in \mathscr{L}(\mathscr{H})$ be such that*

$$[T_i, T_j] \in \mathscr{C}_n,\ [T_i, T_j^*] \in \mathscr{C}_n \quad \textit{for all } 1 \leqslant i, j \leqslant n$$

and

$$I - \sum_{i=1}^{n} T_i^* T_i \in \mathscr{C}_n.$$

Then for A defined as above, we have

$$\mathrm{index}A = \mathrm{Tr}[T_1, T_1^*, T_2, T_2^*, \ldots, T_n, T_n^*].$$

Proof. First we shall prove that $A^*A - I \in \mathscr{C}_n$, $AA^* - I \in \mathscr{C}_n$. Since

$$(d' + d'')^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}$$

it will be sufficient to show that

$$(d' + d'')^2 - I \in \mathscr{C}_n.$$

We have:

$$(d' + d'')^2 =$$
$$= \sum_{i \neq j} (T_i^* T_j \otimes a_i^* a_j + T_i T_j^* \otimes a_i a_j^*) +$$
$$+ \sum_i (T_i^* T_i \otimes a_i^* a_i + T_i T_i^* \otimes a_i a_i^*) =$$
$$= \sum_{i \neq j} [T_i^*, T_j] \otimes a_i^* a_j + \sum_i [T_i, T_i^*] \otimes a_i a_i^* +$$
$$+ (\sum_i T_i^* T_i - I) \otimes 1 + I \otimes 1$$

which shows that indeed $(d' + d'')^2 - I \in \mathscr{C}_n$.

Now, since $A^*A - I$, $AA^* - I \in \mathscr{C}_n$, we can use Lemma 7.1 of [8] which gives:

$$\operatorname{index} A = \operatorname{Tr}((I - A^*A)^n - (I - AA^*)^n).$$

Since the trace of an operator-valued matrix is equal to the trace of the sum of its diagonal entries, using Proposition 1, we have:

$$\operatorname{index} A = \operatorname{Tr}(\tau_e((I - A^*A)^n - (I - AA^*)^n)) =$$

$$= \operatorname{Tr}\left(\tau_e\left(\sum_{p=0}^{n} \frac{n!}{p!(n-p)!}(-1)^p((A^*A)^p - (AA^*)^p)\right)\right) =$$

$$= \operatorname{Tr}\left(\sum_{p=0}^{n} \frac{n!}{p!\,(n-p)!}(-1)^p \tau((d' + d'')^{2p}(I \otimes P_e - I \otimes P_o))\right) =$$

$$= \operatorname{Tr}[T_1, T_1^*, T_2, T_2^*, \ldots, T_n, T_n^*]. \qquad \text{Q.E.D.}$$

There is unfortunately one short-coming of Proposition 2, which we must mention. We do not know whether there exist operators $T_1, \ldots, T_n$ satisfying the assumptions of Proposition 2, and such that $\operatorname{index} A \neq 0$. The obvious candidate for such an n-tuple is the Toeplitz operators on $H^2(\partial B_n)$ (where B_n is the unit ball of $\mathbf{C}^n$) with symbols $z_1, \ldots, z_n$ (cf. [3]), but this n-tuple satisfies $[T_i, T_j^*] \in \mathscr{C}_p$ only for $p > n$.

However, Proposition 2 can be used to give a triviality result for certain extensions. Although the conditions in Proposition 2 are apparently more restrictive than $\mathscr{C}_n$-smoothness, we shall show that the additional assumption concerning $I - \sum T_i^* T_i$ can be satisfied by perturbing $T_1, \ldots, T_n$.

PROPOSITION 3. *The $\mathscr{C}_{n-1}$-smooth extensions of $\mathscr{C}(S^{2n-1})$ ($n \geqslant 2$) are trivial.*

Proof. First we shall prove that $T_1, \ldots, T_n$ defining a $\mathscr{C}_{n-1}$-smooth extension can be chosen so as to satisfy $I - \sum T_i^* T_i \in \mathscr{C}_{n-1}$ besides $[T_i, T_j] \in \mathscr{C}_{n-1}$, $[T_i, T_j^*] \in \mathscr{C}_{n-1}$. Indeed, since $I - \sum T_i^* T_i \in \mathscr{K}(\mathscr{H})$ we can find $X \in \mathscr{L}(\mathscr{H})$, $X \geqslant 1/2 I$ such that $X - \sum T_i^* T_i$ is finite rank. Since $[X, T_i] \in \mathscr{C}_{n-1}$ and the function $t \to t^{-\frac{1}{2}}$ is C^∞ in a neighborhood of $[1/2, \|X\|]$ it follows by the Fourier-transform method of [6] that $[X^{-\frac{1}{2}}, T_i] \in \mathscr{C}_{n-1}$ for $1 \leqslant i \wedge n$. Clearly $X^{-\frac{1}{2}} - I \in \mathscr{K}(\mathscr{H})$ so that replacing $T_1, \ldots, T_n$ by $T_1 X^{-\frac{1}{2}}, \ldots, T_n X^{-\frac{1}{2}}$ will leave the extension unchanged and since $[X^{-\frac{1}{2}}, T_i] \in \mathscr{C}_{n-1}$, we shall also have

$$[T_i X^{-\frac{1}{2}}, T_j X^{-\frac{1}{2}}] \in \mathscr{C}_{n-1}, \quad [T_i X^{-\frac{1}{2}}, (T_j X^{-\frac{1}{2}})] \in \mathscr{C}_{n-1}.$$

Moreover

$$I - \sum_{1 \leqslant i \leqslant n} (T_i X^{-\frac{1}{2}})^* (T_i X^{-\frac{1}{2}}) =$$

$$= X^{-\frac{1}{2}}(X - \sum_{1 \leqslant i \leqslant n} T_i^* T_i) X^{-\frac{1}{2}} \in \mathscr{C}_{n-1}.$$

To see that the extension defined by $T_1, \ldots, T_n$ is trivial we have to prove that index$A = 0$. This can be seen either by using Proposition 1.1 of [7] which will give

$$\mathrm{Tr}[T_1, T_1^*, T_2, T_2^*, \ldots, T_n, T_n^*] = 0$$

or by going through the proof of Proposition 2, and noting that since

$$I - A^*A \in \mathscr{C}_{n-1}, \quad I - AA^* \in \mathscr{C}_{n-1}$$

we have

$$\mathrm{index}A = \mathrm{Tr}((I - A^*A)^{n-1} - (I - AA^*)^{n-1})$$

which is zero by the same kind of computation involved in Proposition 1.

Q.E.D.

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