J. OPERATOR THEORY
6(1981), 113–123

AN ABSTRACT KATO INEQUALITY FOR GENERATORS OF POSITIVE OPERATORS SEMIGROUPS ON BANACH LATTICES

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1. THE CONJECTURE

The main theme in the theory of strongly continuous semigroups of linear operators on Banach spaces is the interplay between three objects:

(i) the *semigroup* $\{T(t)\}_{t\geqslant 0}$, which is a subset of $L(E)$, E a Banach space, such that $t \mapsto T(t)$ is a continuous semigroup homomorphism for the strong operator topology on $L(E)$.

(ii) the *generator* $(A, D(A))$, which is a (generally unbounded) linear operator with dense domain $D(A)$ in E defined by

$$Ax := \lim_{t\to 0} \frac{1}{t}(T(t)x - x) \qquad \text{for } x \in D(A).$$

(iii) the *resolvent* $\mathrm{R}(\lambda, A)$ which is a bounded linear operator defined as the inverse of $(\lambda - A)$ for large positive $\lambda \in \mathbf{R}$.

The famous Hille-Yosida theorem characterizes those (unbounded) linear operators $(A, D(A))$ which are generators of such semigroups $\{T(t)\}_{t\geqslant 0}$, but its conditions rely heavily on properties of the resolvent $\mathrm{R}(\lambda, A)$ (see [18], Ch. IX). On the other hand, the Lumer-Phillips theorem succeeds in characterizing the generator more directly (again see [18]) but unfortunately this theory works only for contraction semigroups.

Due to the fact that many semigroups appearing in the applications (e.g., partial differential equations, probability theory) are semigroups of *positive* operators on function spaces bearing a natural order structure (making those spaces into *Banach lattices)*, analogous problems arise. Using the Hille-Yosida theorem and the relations

$$T(t) = \lim_{n\to\infty} \left[\frac{n}{t}\,\mathrm{R}\left(\frac{n}{t}, A\right)\right]^n$$

and

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s}\, T(s)\, ds$$

(in the strong sense), one can easily characterize the generator of positive semigroups in terms (and by the positivity) of the resolvent. A direct characterization of such generators is desirable but more difficult to obtain.

One of the first results in this direction, used in the theory of Markov processes and analogous to the Lumer-Phillips theorem, was found for positive contraction semigroups on spaces $C_{\mathbf{C}}(X)$, X compact. It may be stated as follows.

1.1. THEOREM (Feller, 1952; see [8]) *For a linear operator A with domain $D(A)$ in $C_{\mathbf{C}}(X)$, X compact, the following assertions are equivalent:*

(a) *$(A, D(A))$ is the generator of a strongly continuous semigroup of positive contractions on $C_{\mathbf{C}}(X)$.*

(b) *$(A, D(A))$ is closed, densely defined, $(\lambda\,\mathrm{Id} - A)$ is surjective for some $\lambda > 0$ and A is dispersive in the sense that $f \in D(A)$ and $x \in X$ satisfying $f(x) = \sup\{f(y) : y \in X\} \geqslant 0$, implies that $\mathrm{Re}(Af(x)) \leqslant 0$.*

In the spirit of Lumer-Phillips and motivated by the above theorem, Phillips [9], Hasegawa [5], Kunita [7] and Sato [12] developed a theory which enabled them to characterize the generators of positive contraction semigroups on arbitrary Banach lattices.

For positive semigroups which are not necessarily contractive Sato [12] solved the problem if the generator is bounded or if the Banach lattice is $C(X)$.

A quite different approach was initiated in 1973, when Kato [6] observed that the Laplacian $\triangle$ on $L^2(\mathbf{R}^n)$, interpreted in terms of distributions, satisfies the following inequality:

1.2. THEOREM (Kato, 1973; see [10])

$$\triangle |f| \geqslant \mathrm{Re}(\mathrm{sgn} f \cdot \triangle f)$$

if f and $\triangle f$ are in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

This inequality plays a central role in the investigation of Schrödinger operators. B. Simon realized that it is closely connected to the positivity of the semigroup generated by $\triangle$. He even showed that an inequality of the above type is equivalent to the positivity of the semigroup. But his theory is essentially tied to the Hilbert space structure of L^2 and to the self-adjointness of the operators. In particular, the domain of the generator is enlarged to the form domain of the corresponding quadratic form. We refer to [11] for proofs and interesting applications. A similar and even more systematic treatment of the L^2-case is presented by Fukushima [3].

Thus it still seems to be very tempting to characterize the generators of arbitrary semigroups on arbitrary Banach lattices by an inequality of the above type. In particular, besides its potential practical use, there are some heuristic arguments supporting such a view. For instances, the positivity of an operator $A \in L(E)$, E a Banach lattice, is characterized by the inequality

$$A|x| \geqslant |Ax|$$

and implies the positivity of the semigroup $\{T(t)\} = \{e^{tA}\}$. The Kato Inequality

$$A|x| \geqslant \mathrm{Re}((\mathrm{sgn}\, x)\, Ax)$$

is weaker and might still imply the positivity of $\{T(t)\}$. Therefore, we formulate the following conjecture.

1.3. CONJECTURE. *Let* $\{T(t)\}_{t \geqslant 0}$ *be a strongly continuous semigroup of bounded linear operators on a (complex) Banach lattice* E *with generator* $(A, D(A))$. *The following are equivalent*:

(a) $T(t)$ *is positive for all* $t \geqslant 0$.

(b) A *satisfies an abstract Kato Inequality, i.e.*

$$A|x| \geqslant \mathrm{Re}((\mathrm{sgn}\, x) Ax) \quad \textit{for all } x \in D(A).$$

Even from the most elementary examples $\left(A = \dfrac{\mathrm{d}}{\mathrm{d}x},\ A = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\right)$ it is immediately clear that the main problem consists in a reasonable interpretation of $A|x|$ for $x \in D(A)$, while it is easy (see 2.1) to define the expression sgn. In full generality the problem is still unsolved, but we will present a positive answer in two particular cases.

Moreover we refer to [15] for a survey of recent developments in the theory of positive semigroups and their generators.

2. THE DERIVATIVE OF $x \mapsto |x|$

In this preparatory section we develop a method which will clarify considerably the nature of the Kato Inequality.

We always denote by E a complex Banach lattice (see [13] for the terminology employed and basic results). In particular, we denote by $E_{\mathbf{R}}$ the real part of E, by $\bar{x} = x_1 - \mathrm{i}x_2$ the *conjugate* and by $\mathrm{Re}\, x = x_1$ the *real part* of $x = x_1 + \mathrm{i}x_2 \in E = E_{\mathbf{R}} \oplus \mathrm{i}E_{\mathbf{R}}$. Now consider the mapping V which associates to every $x \in E$ its absolute value

$$V(x) := |x|.$$

Even if we have learned in calculus that this mapping is not differentiable, we will insist in differentiating it. To this aim recall that V is (right) partially differentiable at x in the direction y if there exists $z \in E$ such that

$$\lim_{t \to 0} t^{-1} \| V(x + ty) - V(x) - tz \| = 0.$$

z is called the *partial derivative* (at x in the direction y) and will be denoted by $\partial^+ V|_x(y)$. If V is differentiable at x in every direction $y \in E$, we call it *Gâteaux-differentiable* at x and obtain a map

$$\partial^+ V|_x : y \mapsto \partial^+ V|_x(y)$$

from E into E, called the *Gâteaux derivative* of V at x. This concept has to be distinguished from the (stronger concept of) Fréchet derivative. To avoid confusion we point out that all of these derivatives are only *right* derivatives!

Next we explain the so-called signum operator $\operatorname{sgn} x$ appearing in the Kato Inequality. Its intuitive meaning is quite obvious, but we prefer the following abstract approach, which was communicated to us by W. Arendt.

2.1. LEMMA. *Let E be a complex Banach lattice and choose $x \in E$. If $|x|$ is a quasi-interior point of E_+ or if E is order complete, then there exists a unique bounded linear operator $S \in L(E)$ such that*

(i) $Sx = |x|$,

(ii) $|Sy| \leqslant |y|$ *for every* $y \in E$,

(iii) $Sy = 0$ *for every* $y \in E$ *which is orthogonal to* x.

Proof. 1. $E = C(X)$. By assumption, the function $|x|$ is strictly positive and, consequently, the multiplication operator

$$Sy := \frac{\bar{x}}{|x|} \cdot y, \qquad y \in C(X)$$

satisfies (i)—(iii). Its uniqueness follows from [14], 1.3.a.

2. $|x|$ quasi-interior point. From 1. and the Kakutani representation theorem (see [13]) follows that there exists a unique operator $S_0 \in L(E_{|x|})$ satisfying (i)—(iii). But (ii) implies the continuity of S_0 for the norm of E and S may be obtained as the continuous extension of S_0 to E.

3. E order complete. Again denote by S_0 the unique operator on $E_{|x|}$ satisfying (i)—(iii). S_0 is a regular operator on $E_{|x|}$ (see [13], IV.1.5) and can be decomposed into $S_1 = (\operatorname{Re} S_0)^+$, $S_2 = (\operatorname{Re} S_0)^-$, $S_3 = (\operatorname{Im} S_0)^+$, $S_4 = (\operatorname{Im} S_0)^-$, such that each S_j is positive and satisfies (ii) on $E_{|x|}$. Consider the projection band $B := \{|x|\}^{\perp\perp}$ and define

$$\tilde{S}_j z := \sup\{S_j y : \ y \in E_{|x|} \cap [0, z]\}$$

for $0 \leqslant z \in B$. By linear extension we obtain operators $\tilde{S}_j \in L(B)$. The operator $S \in L(E)$ defined by

$$Sz := \begin{cases} \tilde{S}_1 z - \tilde{S}_2 z + i(\tilde{S}_3 z - \tilde{S}_4 z) & \text{for } z \in B \\ 0 & \text{for } z \in B^\perp \end{cases}$$

satisfies (i)—(iii). Since (ii) implies order continuity, the uniqueness of S follows from the uniqueness of S_0.

The operator S defined in the above lemma will be called the *signum operator* corresponding to x and it will be denoted by $\operatorname{sgn} x$. Its importance for us lies in the fact that — in some sense — it is the derivative of the absolute value mapping V at the point x. More precisely:

2.2. THEOREM. *Assume that $|x|$ is a quasi-interior point in E_+. Then the absolute value mapping V is Gâteaux-differentiable in x with derivative $\partial^+ V|_x(y) =$ $\operatorname{Re}((\operatorname{sgn} x)y)$ for all $y \in E$.*

Proof. Identify the principal ideal $E_{|x|}$ with $C_C(X)$ for some compact space X. Then $|x|$ corresponds to the unit function $\mathbf{1}$ and we are able to multiply elements of $E_{|x|}$. First, choose $y \in E$ with $|y| \leqslant \frac{1}{2}|x|$ and decompose $x = x_1 + ix_2$ and $y = y_1 + iy_2$ into real and imaginary parts. Then $x_1^2 + x_2^2 = \mathbf{1}$ and $\operatorname{Re}((\operatorname{sgn} x)y) =$ $= x_1y_1 + x_2y_2$. Therefore we obtain the following estimate:

$$|V(x+y) - V(x) - \operatorname{Re}((\operatorname{sgn} x)y)| = ||x+y| - |x| - (x_1y_1 + x_2y_2)| \leqslant$$

$$\leqslant ||x+y| - |x| - (x_1y_1 + x_2y_2)| \cdot ||x+y| + |x| + (x_1y_1 + x_2y_2)| =$$

$$= |(x_1+y_1)^2 + (x_2+y_2)^2 - (|x|^2 + 2(x_1y_1 + x_2y_2) + (x_1y_1 + x_2y_2)^2)| =$$

$$= |y_1^2 + y_2^2 - (x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2)| =$$

$$= |y_1^2 + y_2^2 - ((x_1^2 + x_2^2)y_1^2 + (x_1^2 + x_2^2)y_2^2 - x_2^2y_1^2 + 2x_1y_1x_2y_2 - x_1^2y_2^2)| =$$

$$= (x_2y_1 - x_1y_2)^2 \leqslant |y|^2.$$

This shows that the partial derivative of V at x exists for every direction y with $|y| \leqslant \frac{1}{2}|x|$.

Now choose $0 < \varepsilon < 1$ and $y \in E$. Since $E_{|x|}$ is dense in E, we find $\tilde{y} \in E_{|x|}$ such that $\|y - \tilde{y}\| < \varepsilon/4$. For $n \in \mathbf{N}$ with

$$\frac{1}{n}|\tilde{y}| \leqslant \frac{\varepsilon}{2} \cdot \min(1, \|x\|^{-1})\, |x|$$

and for $0 < t < \frac{1}{n^2}$ we obtain

$$|t\tilde{y}| \leqslant \frac{1}{2}|x|$$

and

$$t|\tilde{y}|^2 \leqslant \left(\frac{1}{n}|\tilde{y}|\right)^2 \leqslant \frac{1}{n}|\tilde{y}| \leqslant \frac{\varepsilon}{2\|x\|}|x|.$$

Therefore

$$t^{-1}\|V(x+ty)+V(x)-t\mathrm{Re}((\mathrm{sgn}x)y)\| \leqslant$$

$$\leqslant t^{-1}(\|\,|x+t\tilde{y}|-|x|-\mathrm{Re}((\mathrm{sgn}x)t\tilde{y})\|+\|\,|x+ty|-|x+t\tilde{y}|\,\|+$$

$$+t\|\mathrm{Re}((\mathrm{sgn}x)(y-\tilde{y}))\|) \leqslant t^{-1}\|t^2|\tilde{y}|^2\|+\varepsilon/2<\varepsilon.$$

For arbitrary elements $x \in E$ we obtain that the absolute value mapping V is partially differentiable at x in every direction $y \in \bar{E}_{|x|}$, where $\bar{E}_{|x|}$ is the closed ideal generated by $|x|$. Since in Banach lattices with order continuous norm every closed ideal is already a projection band ([13], II.5.14), the result becomes very elegant under additional assumptions on E:

2.3. COROLLARY. *Let E be a Banach lattice with order continuous norm. Then the absolute value mapping $V: x \mapsto |x|$ is (right) Gâteaux differentiable at every $x \in E$.*

Proof. It remains to show that V is differentiable at x in every direction $y \in \{x\}^{\perp}$. But for such y we have

$$|x+ty|-|x|-t|y|=0\ ,$$

and therefore $\partial^+ V|_x(y)=|y|$.

2.4. EXAMPLES. 1. As can be seen from the proof of (2.3), the absolute value mapping in $C(X)$ is Fréchet differentiable at every nowhere vanishing function.

2. In general, V is not Fréchet differentiable: Take $E=L^1[0,1]$, $x=\mathbf{1}$ and $y_n=-2\chi_{[0,n^{-1}]}$. The Fréchet derivative, if it exists, has to be equal to $\partial^+V|_x(y)$ for every y. But

$$\|y_n\|^{-1}\,\|V(x+y_n)-x-\partial^+V|_x(y_n)\|=1 \quad \text{for all } n \in \mathbf{N},$$

while $\lim\limits_{n\to\infty} y_n=0$.

REMARK. A complete analysis of the derivative of V can be found in [16].

3. THE GENERATORS OF LATTICE SEMIGROUPS

In this section we attack the problem of characterizing the generators of positive semigroups $\{T(t)\}$ on Banach lattices E. But we will make additional assumptions and study only lattice semigroups, i.e., strongly continuous semigroups of lattice homomorphisms. The basic idea consists in differentiating the mapping

$$t \mapsto V(T(t)x)=|T(t)x|=T(t)|x|.$$

To this aim we need the following well known *chain rule.*

3.1. LEMMA. *Assume that* g: $[0,1] \to E$ *and* $f: E \to E$ *are mappings such that* g *is (right) differentiable in* 0, f *is partially differentiable at* $g(0)$ *in the direction* $g'(0)$ *and is Lipschitz continuous. Then*

$$(f \circ g)'(0) = \partial^+ f\,|_{g(0)}(g'(0)).$$

This chain rule will be applied to the mappings $t \overset{g}{\mapsto} T(t)x$ and $x \overset{f}{\mapsto} V(x) = |x|$. But before doing so we introduce a property of (unbounded) operators on Banach lattices which plays a surprisingly great role in this theory.

3.2. DEFINITION. Let $(B, D(B))$ be a closed linear operator with dense domain $D(B)$ in a Banach lattice E. The operator $(B, D(B))$ is called *local* if $Bx \in \{x\}^{\perp\perp}$ for every $x \in D(B)$.

If B is bounded and E is represented as a function lattice, the local operators are multiplication operators defined by a function in the representation.

In the unbounded case the class of local operators is considerably larger (and much more important) and includes all differential operators (in appropriate function spaces). In an abstract context, local operators may be obtained through the following result.

3.3. PROPOSITION. *The generator of a strongly continuous semigroup of lattice homomorphisms on a Banach lattice is local.*

Proof. Denote by $(A, D(A))$ the generator of the lattice semigroup $\{T(t)\}$ on the Banach lattice E and choose $x \in D(A)$ and $x \perp y \in E$. Using the computation rules for the absolute value in vector lattices (see [13], II.1) we obtain

$$\left|\frac{1}{t}(T(t)x - x)\right| \wedge |y| \leqslant \left|\frac{1}{t}T(t)x\right| \wedge |y| + |x| \wedge |y| \leqslant$$

$$\leqslant \frac{1}{t}T(t)|x| \wedge |T(t)y - y| + \left(T(t)\frac{1}{t}|x|\right) \wedge T(t)|y| \leqslant$$

$$\leqslant |T(t)y - y| + T(t)\left(\frac{1}{t}|x| \wedge |y|\right) = |T(t)y - y|.$$

The continuity of the lattice operations implies $Ax \perp y$.

REMARK. It is obvious that lattice homomorphisms and consequently generators of lattice semigroups are *real* operators, i.e., operators leaving invariant the real part E_{R} of E. In particular they commute with the complex conjugation in E.

After these preparations we are able to characterize the generator of a lattice semigroup, on Banach lattices with order continuous norm, by an abstract Kato *equality*.

3.4. THEOREM. *Let $\{T(t)\}$ be a strongly continuous semigroup with generator $(A, D(A))$ in a complex Banach lattice E. If E has order continuous norm, the following conditions are equivalent:*

(a) *The operators $T(t)$, $t \geqslant 0$, are lattice homomorphisms.*

(b) *The generator A is a real, local operator and its domain $D(A)$ is a vector sublattice of E.*

(c) *$x \in D(A)$ implies $|x| \in D(A)$ and $A|x| = \mathrm{Re}((\mathrm{sgn}\, x)\, Ax)$.*

Proof. (a) $\Rightarrow$ (c): Recall first that $\eta_x(t) := T(t)x$ is differentiable if and only if $x \in D(A)$. Moreover, since E has order continuous norm, the absolute value mapping $V : x \mapsto |x|$ is Gâteaux differentiable at every $x \in E$ by (2.3). In particular, its derivative in the direction Ax is $\mathrm{Re}((\mathrm{sgn}\, x)Ax)$ (use 3.3). Now we apply (3.1) to the composition $V \circ \eta_x$, $x \in D(A)$, and obtain $(V \circ \eta_x)'(0) = \mathrm{Re}((\mathrm{sgn}\, x)Ax)$. By hypothesis

$$V \circ \eta_x(t) = T(t)\,|x| \text{ for all } t \geqslant 0.$$

Therefore $\lim_{t \to 0} t^{-1}(T(t)|x| - |x|)$ exists, $|x| \in D(A)$, and $A|x| = \mathrm{Re}((\mathrm{sgn}\, x)Ax)$.

(c) $\Rightarrow$ (b) is obvious, since the signum operator is local.

(b) $\Rightarrow$ (a): Choose $x \in D(A) \cap E_{\mathbf{R}}$ and $s > 0$. First we show that $T(s)$ is real, i.e. $\overline{T(s)x} = T(s)\bar{x} = T(s)x$.

Consider the differentiable mapping

$$t \mapsto T(s-t)\overline{T(t)x} \qquad \text{for } t \in [0, s],$$

which has the derivative

$$-AT(s-t)\,\overline{T(t)x} + T(s-t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{T(t)x} =$$

$$= -\,AT(s-t)\,\overline{T(t)x} + T(s-t)\,\overline{AT(t)x} = 0$$

since A is real. From the mean value theorem we conclude that $T(s)x = T(s)\bar{x}$.

Next denote by P_+ (resp. P_-) the band projection onto $\{x^+\}^{\perp\perp}$ (resp. $\{x^-\}^{\perp\perp}$). Since A is local and $D(A)$ is a vector sublattice we have

$$P_+Ax = P_+Ax^+ = Ax^+$$

and

$$-\,P_-Ax = P_-Ax^- = Ax^-.$$

Consequently we obtain

$$A|x| = Ax^+ + Ax^- = P_+Ax - P_-\,Ax = \mathrm{sgn}\, x \cdot Ax \quad \text{for all } x \in D(A) \cap E_{\mathbf{R}}.$$

Observe that this real part of $D(A)$ is invariant under V (by hypothesis) and $T(s)$.

In the final step we define the mapping

$$\xi_x : [0,s] \to E$$

by

$$\xi_x(t) := T(s-t)\,|T(t)x| \quad (t \in \mathbf{R}_+).$$

By hypothesis and by the previous considerations, ξ_x is differentiable and

$$\xi_x'(t) = -AT(s-t)\,|\,T(t)x\,| + T(s-t)\frac{\mathrm{d}}{\mathrm{d}t}(V(T(t)x)) =$$

$$= -AT(s-t)\,|T(t)x\,| + T(s-t)\,(\mathrm{sgn}T(t)x \cdot AT(t)x) = 0.$$

Again by the mean value theorem, ξ_x is constant and

$$T(s)|x| = \xi(0) = \xi(s) = |T(s)x|.$$

3.5. REMARK. We have deliberately chosen a very restrictive hypothesis for the above theorem *(order continuous norm)*. Even so it can still be applied to interesting examples (e.g., Sobolev spaces) and statement and proof become extremely simple, while at the same time the essential ideas carry over to much more general situations. But it might be necessary to point out that in general (e.g., $E = C(X)$) the domain of the generator of a lattice semigroup is not a vector sublattice. Such generators and their domain in Banach lattices $C(X)$ are characterized in [2], while a complete solution for the general situation is contained in [16]. For the origin of the above implication (a) ⇒ (b), see [17]. Finally we refer to [4] where the above "Kato equality" proves its usefulness for the spectral theory of strongly continuous positive groups.

4. THE GENERATORS OF UNIFORMLY CONTINUOUS POSITIVE SEMIGROUPS

Explaining $A|x|$ for $x \in D(A)$ is the main difficulty in verifying the Conjecture 1.3. In Theorem 3.5 we simply assumed that $D(A)$ is a vector sublattice and obtained, if A is local, a rather restricted class of positive semigroups. Another situation in which no such difficulties occur is when $D(A) = E$, i.e. when A is bounded or equivalently, when the semigroup $\{T(t)\}$ is uniformly continuous.

4.1. THEOREM. *Let $\{T(t)\}$ be a uniformly continuous semigroup with generator $A \in L(E)$, on an order complete Banach lattice. The following conditions are equivalent:*

(a) *The operators $T(t)$, $t \geqslant 0$, are positive;*

(b) *The generator V satisfies the abstract Kato Inequality $A|x| \geqslant \geqslant \mathrm{Re}((\mathrm{sgn}x)Ax)$ for all $x \in E$;*

(c) *$A + \|A\|\,\mathrm{Id}$ is a positive operator.*

From the proof we isolate the following lemma, which contains a consequence of the Kato Inequality that is very useful for further investigations.

4.2. LEMMA. *If, under the above assumptions, A satisfies the Kato Inequality, then $(Ax)^- \in \{x\}^{\perp\perp}$ for every $0 \leqslant x \in E$.*

Proof. For $0 \leqslant x \in E$ the Kato Inequality becomes

$$Ax \geqslant \operatorname{sgn} x \cdot Ax = PAx,$$

where P is the band projection onto $\{x\}^{\perp\perp}$. In particular, $(\mathrm{Id} - P)Ax \geqslant 0$ and $(\mathrm{Id} - P)\,(Ax)^- = 0$.

Proof of 4.1. (a) $\Rightarrow$ (b): If the operators $T(t)$ are positive, we have the following inequality for every $x \in E$:

$$\xi_1(t) := \mathrm{Re}((\operatorname{sgn} x) T(t)x) \leqslant |T(t)x| \leqslant T(t)|x| =: \xi_2(t).$$

Since both maps are differentiable, and since

$$\xi_1(0) = \xi_2(0) = |x|,$$

we obtain the Kato Inequality.

(b) $\Rightarrow$ (c): Assume that $A + \lambda\mathrm{Id}$ is not positive for some $\lambda > \|A\|$. There exists $0 < x \in E$ and a band projection P such that

$$0 > \lambda Px + PAx = \lambda Px + PAPx + PA\,(\mathrm{Id} - P)x.$$

From the Lemma 4.2 it follows that $(A(\mathrm{Id}-P)x)^- \in \{(\mathrm{Id} - P)x\}^{\perp\perp}$. Therefore we have

$$0 > \lambda Px + PAPx + P(A(\mathrm{Id} - P)x)^+ \geqslant \lambda Px + PAPx$$

or

$$\lambda Px < -PAPx,$$

which implies the contradiction $\lambda \leqslant \|A\|$.

(c) $\Rightarrow$ (a): If $B := A + \lambda$ is positive, so are the semigroups

$$S(t) := \mathrm{e}^{tB} \quad \text{and} \quad T(t) = \mathrm{e}^{tA} = \mathrm{e}^{-\lambda t}\mathrm{e}^{tB}.$$

4.3. REMARKS. 1. The theorem is valid without the assumption of order completeness (use the representation theory of [1] to define the signum operator).

2. For matrices $A = (a_{ij})$ we obtain that $T(t) = \mathrm{e}^{tA}$ is positive for all $t \geqslant 0$ if and only if $a_{ij} \geqslant 0$ for $i \neq j$.

3. The condition (c) was also found by Ando [1]. He showed also that Kato's Inequality characterizes generators of positive semigroups on $C(X)$ thereby retrieving Sato's results ([12], Sect. 6).

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Received August 23, 1980.