J. OPERATOR THEORY
6(1981), 125–133

K-THEORY FOR ACTIONS OF THE CIRCLE GROUP ON C^*-ALGEBRAS

WILLIAM L. PASCHKE

0. INTRODUCTION

M. Pimsner and D. Voiculescu [12] (see also [5], [3]) have recently obtained the following cyclic exact sequence for the K-groups of a C^*-algebra A and its crossed product B by a single automorphism θ:

$$(1)\qquad \ldots \to K_j(A) \xrightarrow{\theta_*-\mathrm{id}} K_j(A) \xrightarrow{i_*} K_j(B) \to K_{1-j}(A) \xrightarrow{\theta_*-\mathrm{id}} \ldots \qquad (j=0,1).$$

Our project here is to extend this important result to the situation in which B is a C^*-algebra equipped with an action ρ of the circle group whose spectral subspaces are "large" (see §2), and A is the fixed-point algebra for ρ. (The prototypical case is when $B = C^*(A, \theta)$ and ρ is the action dual to θ as in [15].) In this more general setting, with the additional technical assumption that B has a strictly positive element, we show that there is an exact sequence of the form (1), where θ is a certain automorphism of $A \otimes K$. (Here, K is, as usual, the C^*-algebra of compact operators on separable infinite-dimensional Hilbert space.) The automorphism θ is not, of course, unique. One way to concoct a suitable θ is via the spectral subspace $E_1 = \{x \in B \colon \rho_\lambda(x) = \lambda x \ \ \forall \lambda\}$, which we assume to be large in the sense that $E_1^* E_1$ and $E_1 E_1^*$ are dense in A. We may use [2, Thm. 3.4] to associate to E_1^* (viewed as an $A - A$ equivalence bimodule with the obvious left and right A-valued inner products) an automorphism of $A \otimes K$ which turns out to have the right effect on $K_*(A)$. Another approach invokes Theorem 2 of [10] to see that $A \otimes K$ is isomorphic to the crossed product of $B \otimes K$ by $\rho \otimes \mathrm{id}_K$, whereupon Takai's duality theorem [15] gives $B \otimes K$ as the crossed product of $A \otimes K$ by the automorphism corresponding to the generator of the action dual to $\rho \otimes \mathrm{id}_K$. These two constructions, which we show induce the same map on $K_*(A)$, are non-formulaic and involve several arbitrary choices. Computation of $\theta_* \colon K_*(A) \to K_*(A)$ thus becomes a problem in specific examples. We determine θ_* when B is unital in as explicit a fashion as may be hoped for, and as a consequence obtain the sequence

$$\ldots \to K_j(A) \xrightarrow{\sigma_* - \mathrm{id}} K_j(A) \xrightarrow{i_*} K_j(C^*(A, S)) \to K_{1-j}(A) \xrightarrow{\sigma_* - \mathrm{id}} \ldots \qquad (j = 0,1)$$

for the algebras treated in [11].

The paper is organized as follows. In § 1 we put on record a proof of the fact that the inclusion map of C^*-subalgebra C into bigger C^*-algebra D induces an isomorphism on K-theory when C is full hereditary in D and both have strictly positive elements. This is crucial for some of the subsequent manipulations. Section 2 deals with consequences of the assumption that ρ has large spectral subspaces, including the two methods mentioned above for obtaining automorphisms of $A \otimes K$. The K-theoretic results discussed in the previous paragraph are proved in § 3, and applied in § 4 to crossed products by endomorphisms. In particular, we show in this last section that for any countable subgroup H of $\mathbf{R}$, there is a unital, infinite, simple C^*-algebra A such that $K_0(A) \approx H$.

The K-theory used here is that developed in [16, §§ 5—10]. The arguments given there are valid for non-commutative as well as for commutative Banach algebras. We follow the standard practice of using projections (resp. unitaries) in place of arbitrary idempotents (resp. invertibles) when working with C^*-algebras; this changes nothing ([9, Thms. 2, 6, 27], [4]) and is sometimes useful in small ways. Two important features of K-theory for C^*-algebras are stability ($K_\#(A) \approx K_\#(A \otimes K)$ via the map on $K_\#$ induced by $a \mapsto a \otimes e_{11}$) and periodicity ($K_j(A) \approx K_{1-j}(A \otimes C_0(\mathbf{R}))$, $j = 0,1$). We will use square brackets to denote equivalence classes in $K_\#$.

1. FULL HEREDITARY SUBALGEBRAS

Let D be a C^*-algebra. A C^*-subalgebra C of D is said to be *full* if it is contained in no proper closed ideal of D, and *hereditary* if $0 \leqslant h \leqslant k$, $k \in C$ implies $h \in C$. When both these conditions are met (and C and D both have strictly positive elements), L. G. Brown [1] has shown that $C \otimes K$ and $D \otimes K$ are isomorphic, and furthermore that the inclusion $i\colon C \to D$ induces an isomorphism on Ext (for separable D). We will need the K-theoretic analog of this latter result, a special case of which is proved in [13]. The following lemma will be useful in this connection and later on.

1.1. LEMMA. *Let B be a Banach algebra. If x and y in B are such that $1 - xy$ is invertible in B^+ (the algebra obtained by adjoining a unit to B, if necessary), then $[1 - xy] = [1 - yx]$ in $K_1(B)$.*

Proof. We have

$$\begin{pmatrix} 1-xy & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix},$$

Shrinking the off-diagonal entries to zero in the two outer factors connects them through invertibles to the identity, and conjugating the middle factor by a path of invertibles from the identity to $e_{21} + e_{12}$ joins this factor to the matrix obtained from it by interchanging x and y, which in turn can be joined to $(1 - yx) \oplus 1$.

1.2. PROPOSITION. *Let C be a full hereditary C^*-subalgebra of a C^*-algebra D and assume that each of C and D has a strictly positive element. The map $i_*: K_\#(C) \to K_\#(D)$ induced by the inclusion of C into D is an isomorphism.*

Proof. As in [1], we first consider the case in which C is a full corner of D, so there is a projection q in the multiplier algebra $M(D)$ such that $C = qDq$. By [1, Lemma 2.5], there is a multiplier v in $M(D \otimes K)$ such that $v^*v = 1$ and $vv^* = q\otimes 1$. The map $x \to v^*xv$ is an isomorphism of $C \otimes K$ with $D\otimes K$. It will suffice to show that this map has the same effect on $K_\#(C)$ as the inclusion map. To see this for K_1, let u be a unitary in $(C \otimes K)^+ \otimes M_n$, so $u = w + W$, where $w \in C \otimes K \otimes M_n$ and W is a unitary in M_n. Since $K \otimes M_n \approx K$ and the unitary group of M_n is connected, there is no loss of generality in assuming that $n = 1$ and $u = w + 1$. Since w is normal, we can factor w as $w = ab$, where $a, b \in C^*(w)$. With $x = bv$ and $y = v^*a$, we have $x, y \in D \otimes K$ (since v is a multiplier of $D \otimes K$), $xy = w$ (since a and b are commuting elements of $C \otimes K$ and $vv^* = q \otimes 1$), and $yx = v^*wv$. Lemma 1.1 now shows that $[u] = [v^*wv + 1]$, so $i_*: K_1(C) \to K_1(D)$ is an isomorphism. Tensoring C and D with $C_0(\mathbf{R})$ and using periodicity [16, § 9] gives the same result for K_0. The general case now follows by the argument used to prove [1, Thm. 2.10].

2. FIXED-POINT ALGEBRAS AND CROSSED PRODUCTS

In this section B will be a C^*-algebra equipped with a (point-norm) continuous action ρ of a compact abelian group G. For a character χ in the dual group $\hat{G}$, we set $E_\chi = \{b \in B\colon \rho_s(b) = \chi(s)b \;\; \forall s \in G\}$. Letting A denote the fixed-point algebra for ρ, we say that ρ has *large spectral subspaces* if $\overline{E_\chi^* E_\chi} = A$ for each χ in $\hat{G}$. (By $E_\chi^* E_\chi$ we mean the linear span of $\{x^*y\colon x, y \in E_\chi\}$.) Recall that the crossed product $C^*(B, \rho)$ of B by ρ is the completion in the greatest C^*-norm of $L^1(G, B)$, the space of norm-integrable functions from G to B, considered as a $*$-algebra with multiplication and involution defined by

$$(fg)(s) = \int_G f(t)\rho_t(g(s-t))\,\mathrm{d}t \tag{2}$$

$$f^*(s) = \rho_s(f(-s)^*), \tag{2)'}$$

$\mathrm{d}t$ being Haar measure on G. Define $j\colon A \to L^1(G, B) \subseteq C^*(B, \rho)$ by $j(a)(s) = a$ (s in G). It is immediate that j is a $*$-monomorphism.

The next proposition probably comes as no surprise to crossed product specialists. It will be used in the proof of Theorem 3.2.

2.1. PROPOSITION. *If ρ has large spectral subspaces, then $j(A)$ is a full corner of $C^*(B, \rho)$.*

Proof. That $j(A)$ is a corner of $C^*(B, \rho)$ is shown in [14]. It remains to show that $j(A)$ is contained in no proper closed ideal of $C^*(B, \rho)$. Let L be a closed ideal containing $j(A)$. Given χ, ψ in $\hat{G}$, let $x \in E_{\chi\bar{\psi}}$ and $y \in E_\psi$, and define f and g in $L^1(G, B)$ by $f(s) = x$, $g(s) = \psi(s)y$ (s in G). An easy computation using (2) shows that $(fj(a)g)(s) = \psi(s)xay$ for any a in A. Because ρ has large spectral subspaces, the linear span of the products xay is dense in E_χ. (Using an approximate unit for A, we have $E_\chi = \overline{AE_\chi} = \overline{E^*_{\chi\bar{\psi}}E_{\chi\bar{\psi}}E_\chi} \subseteq \overline{E_{\chi\bar{\psi}}E_\psi} \subseteq E_\chi$.) Hence every h in $L^1(G, B)$ of the form $h(s) = H(s)w$, where $H \in C(G)$ and $w \in E_\chi$, belongs to L. The E_χ's taken together span a dense subspace of B (since for any φ in B^* annihilating all the E_χ's, the Fourier coefficients of $s \to \varphi(\rho_s(b))$ vanish for every b in B), so we may now argue as in the proof of [11, Thm. 2] to conclude that $L = C^*(B, \rho)$.

2.2. REMARK. If G is separable, ρ has large spectral subspaces, and B has a strictly positive element, then [10, Thm. 2] shows that $C^*(B, \rho) \otimes K$ and $A \otimes K$ are isomorphic. (This also follows from [1, Cor. 2.6], 2.1 above, and the easy fact that $C^*(B, \rho)$ has a strictly positive element under the given circumstances.) Let $\hat{\rho}$ be the action of $\hat{G}$ dual to ρ on $C^*(B, \rho)$ (so $(\hat{\rho}_\chi f)(s) = \chi(s)f(s)$ for f in $L^1(G, B)$), and let σ be the action of $\hat{G}$ on $A \otimes K$ corresponding to $\hat{\rho} \otimes \mathrm{id}_K$ on $C^*(B, \rho) \otimes K$. The duality theorem in [15] now shows that $B \otimes K$ is (isomorphic to) the crossed product of $A \otimes K$ by σ.

Suppose that $G = \mathbf{T}$, the circle group. In this case there is a second, ostensibly different way to exhibit $B \otimes K$ as the crossed product of $A \otimes K$ by an action of $\hat{G}$, based on the treatment of Picard groups in [2, § 3]. If ρ has large spectral subspaces, then in particular we have $\overline{E_1^* E_1} = A = \overline{E_1 E_1^*}$. (It is easy to check that this condition is in fact equivalent to ρ having large spectral subspaces.) The two-sided A-module E_1^*, with left and right A-valued inner products $\langle x, y\rangle_L = xy^*$ and $\langle x, y\rangle_R = x^*y$, is thus what [2] calls an $A - A$ equivalence bimodule. The same holds for $E_1^* \otimes K$ with respect to $A \otimes K$, so if A has a strictly positive element, [2, Thm. 3.4] yields an automorphism θ of $A \otimes K$ and a linear bijection $F\colon A \otimes K \to E_1^* \otimes K$ satisfying

(3) $$F(a_1)^*F(a_2) = \theta(a_1^*a_2),$$

(4) $$F(a_1)\theta(a_2) = F(a_1a_2),$$

(5) $$F(a_1)F(a_2)^* = a_1a_2^*,$$

(6) $$a_1F(a_2) = F(a_1a_2),$$

for a_1, a_2 in $A \otimes K$.

2.3. THEOREM. *Let ρ be an action of* $\mathbf{T}$ *on B with large spectral subspaces, and suppose that the fixed point algebra A has a strictly positive element. If θ is an automorphism of $A \otimes K$ arising from $E_1^* \otimes K$ as in* [2, Thm. 3.4], *then $B \otimes K$ is isomorphic to $C^*(A \otimes K, \theta)$ in such a way that $\rho \otimes \mathrm{id}_k$ corresponds to the action $\hat{\theta}$ of* $\mathbf{T}$ *on $C^*(A \otimes K, \theta)$ dual to θ.*

Proof. Replacing A and B by their tensor products with K and ρ by $\rho \otimes \mathrm{id}_K$, we may assume that A and B are stable, that θ is an automorphism of A, and that there is a linear bijection $F: A \to E_1^*$ satisfying (3), (4), (5), (6). We define linear bijections $F_{-k}: A \to E_k^*$ $(= E_{-k})$ for $k = 1,2, \ldots$ satisfying

$$(3)_{-k} \qquad F_{-k}(a_1)^* F_{-k}(a_2) = \theta^k(a_1^* a_2),$$

$$(4)_{-k} \qquad F_{-k}(a_1)\theta^k(a_2) = F_{-k}(a_1 a_2),$$

(5), (6) by induction as follows. Set $F_{-1} = F$ and suppose that F_{-k} has been constructed and satisfies $(3)_{-k}$, $(4)_{-k}$, (5), (6). For $a_1, \ldots, a_n, b_1, \ldots, b_n$ in A we have

$$\|\sum_i F_{-k}(a_i)F(\theta^k(b_i))\|^2 = \|\sum_{i,j} F_{-k}(a_i)\theta^k(b_i b_j^*)F_{-k}(a_j)^*\| = \qquad \text{(by (5))}$$

$$= \|\sum_{i,j} F_{-k}(a_i b_i b_j^*)F_{-k}(a_j)^*\| = \qquad \text{(by } (4)_{-k})$$

$$= \|\sum_{i,j} a_i b_i b_j^* a_j^*\| = \qquad \text{(by (5))}$$

$$= \|\sum_i a_i b_i\|^2,$$

so there is a well-defined linear isometry $F_{-(k+1)}: A \to E_{k+1}^*$ satisfying

$$(7) \qquad F_{-(k+1)}(ab) = F_{-k}(a)F(\theta^k(b))$$

for a, b in A. This map is surjective because $\overline{E_k^* E_1^*} = E_{k+1}^*$, and routine computations establish that $F_{-(k+1)}$ satisfies $(3)_{-k-1}$, $(4)_{-k-1}$, (5), (6). For $j = 1,2, \ldots$, define $F_j: A \to E_j$ by

$$(8) \qquad F_j(a) = F_{-j}(\theta^{-j}(a^*))^*,$$

and let $F_0: A \to E_0(=A)$ be the identity map. We claim that

$$(9) \qquad F_{m+n}(ab) = F_m(a)F_n(\theta^{-m}(b))$$

for a,b in A and all integers m, n. [For j, $k \geqslant 0$, we obtain $F_{-k-j}(a_1 a_2 \ldots a_j b) = F_{-k}(a_1)F_{-j}(\theta^k(a_2 \ldots a_j b))$ by iterating (7) j times, so (9) holds for m, $n \leqslant 0$.

For $k \geqslant j \geqslant 0$, we have

$$F_{-k}(a_1a_2)F_j(\theta^k(b)) = F_{j-k}(a_1)F_{-j}(\theta^{k-j}(a_2))F_{-j}(\theta^{k-j}(b^*))^* = F_{j-k}(a_1a_2b)$$

using (8), (5), $(4)_{j-k}$. Hence (9) holds for $m \leqslant m+n \leqslant 0$. The remaining cases can be checked using similar computations.] We are now in a position to define a $*$-homomorphism $\Delta\colon C^*(A, \theta) \to B$. Let $f\colon \mathbf{Z} \to A$ be a finitely supported function. (Multiplication and involution for such functions are defined by (2) and (2)′, with $\theta_n = \theta^n$ in place of ρ_λ, to give a $*$-algebra whose enveloping C^*-algebra is $C^*(A, \theta)$). Set $\Delta(f) = \sum_n F_n(f(n))$. By (8) (which holds for all integers j) and (9), Δ is a $*$-homomorphism on the $*$-algebra of finitely supported functions and hence extends to $C^*(A, \theta)$. Since the E_n's span a dense subspace of B, Δ is surjective. We have $\Delta(\hat{\theta}_\lambda(f)) = \rho_\lambda(\Delta(f))$ for finitely supported f and λ in $\mathbf{T}$, so $\Delta\hat{\theta}_\lambda = \rho_\lambda\Delta$. Integrating the action $\hat{\theta}$ over $\mathbf{T}$ gives a faithful conditional expectation of $C^*(A, \theta)$ on the canonical copy of A that it contains. Notice that the restriction of Δ to this copy of A is injective, merely identifying it with the copy of A in B, whence it follows that Δ is injective

2.4. Remark. The only positive functional on $C^*(A, \theta)$ annihilating the canonical copy of A is the zero functional, so the hypotheses of 2.3 imply that $B \otimes K$, and hence B, has a strictly positive element.

3. K-THEORETIC RESULTS

3.1. Theorem. *Let A be the fixed-point algebra for an action with large spectral subspaces of* $\mathbf{T}$ *on a C^*-algebra B, and suppose that A has a strictly positive element. There is an automorphism θ of $A \otimes K$ such that, upon identifying $K_\#(A \otimes K)$ with $K_\#(A)$, we have a cyclic exact sequence*

$$\ldots \to K_j(A) \xrightarrow{\theta_*-\mathrm{id}} K_j(A) \xrightarrow{i_*} K_j(B) \to K_{1-j}(A) \xrightarrow{\theta_*-\mathrm{id}} \ldots \qquad (j = 0, 1).$$

Proof. Take θ as in 2.3 or set $\theta = \sigma_1$ as in 2.2. After also identifying $K_\#(B \otimes K)$ with $K_\#(B)$, [12, Thm. 2.4] applies directly.

The next result shows that the automorphisms given by 2.2 and 2.3 have the same effect on $K_\#(A)$.

3.2. Theorem. *Let B, ρ, and A be as in* 3.1 *and let $\theta = \sigma_1$ as in* 2.2. *Let α, β be $*$-homomorphisms of $A \otimes K$ into itself for which there exists a map $g\colon A \otimes K \to E_1^* \otimes K$ satisfying $g(a_1)^*g(a_2) = \alpha(a_1^*a_2)$ and $g(a_1)g(a_2)^* = \beta(a_1a_2^*)$. Then $\alpha_* = \theta_*\beta_*$.*

Proof. Passing to $A \otimes K$, $B \otimes K$, and $\rho \otimes \mathrm{id}_K$, and identifying $C^*(B \otimes K, \rho \otimes \mathrm{id}_K)$ with $C^*(B, \rho) \otimes K$, we may assume that A and B are stable, so that the maps α, β, and g are defined on A, with g mapping into E_1^*. Let $\hat{\rho} = \hat{\rho}_1$, the generating automorphism for the dual action on $C^*(B, \rho)$ (so $(\hat{\rho}F)(\lambda) = \lambda F(\lambda)$ for F in $L^1(\mathbf{T}, B)$ and λ in $\mathbf{T}$). With j as in 2.1, we have $j_*\theta_* = \hat{\rho}_* j_*$ and hence $\theta_* = j_*^{-1}\hat{\rho}_* j_*$, since j_* is an isomorphism (by 1.2 and 2.1). We deal first with K_1. As in the proof of 1.2, it will suffice to consider unitaries in A^+ of the form $1 + w$, where $w \in A$. We factor w as $w = ab^*$, where a and b belong to the commutative C^*-subalgebra of A generated by w, and define X, Y in $L^1(\mathbf{T}, B)$ by $X(\lambda) = g(a)$, $Y(\lambda) = \lambda g(b)^*$. Formula (2) gives $(XY)(\lambda) = \lambda g(a)g(b)^* = \lambda\beta(w)$ and $(YX)(\lambda) = g(b)^*g(a) = \alpha(w)$. That is, $XY = (\hat{\rho}j\beta)(w)$ and $YX = (j\alpha)(w)$, so $\rho_* j_* \beta_*([1 + w]) = j_*\alpha_*([1 + w])$. We compose with j_*^{-1} on the left to obtain $\theta_*\beta_* = \alpha_*$ as maps on $K_1(A)$. Replacing A, B, and E_1^* by their tensor products with $C_0(\mathbf{R})$, α, β, g, and ρ by their tensor products with $\mathrm{id}_{C_0(\mathbf{R})}$, and identifying $C^*(B \otimes C_0(\mathbf{R}), \rho \otimes \mathrm{id})$ with $C^*(B, \rho) \otimes C_0(\mathbf{R})$ causes j and $\hat{\rho}$ to be replaced by their tensor products with $\mathrm{id}_{C_0(\mathbf{R})}$, so we have

$$(\hat{\rho} \otimes \mathrm{id})_*(j \otimes \mathrm{id})_*(\beta \otimes \mathrm{id})_* = (j \otimes \mathrm{id})_*(\alpha \otimes \mathrm{id})_*$$

as maps from $K_1(A \otimes C_0(\mathbf{R}))$ to $K_1(C^*(B, \rho) \otimes C_0(\mathbf{R}))$. By periodicity, this means that $\hat{\rho}_* j_* \beta_* = j_*\alpha_*$ on $K_0(A)$, so $\alpha_* = \theta_*\beta_*$ on $K_0(A)$.

We can use 3.2 to identify θ_* more explicitly when B is unital. In this case, under the assumptions of 3.1, the dense ideal $E_1^*E_1$ is all of A and as in the proof of [1, Prop 2.1] we can find $x_1, \ldots, x_n$ in E_1 such that $x_1^*x_1 + \ldots + x_n^*x_n = 1$. Consider the map $\gamma: A \to A \otimes M_n$ defined by

$$\gamma(a) = \sum_{i,j} x_i a x_j^* \otimes e_{ij},$$

which is easily seen to be a $*$-monomorphism. (This map is used in the proof of [13, Prop. 2.4].)

3.3. PROPOSITION. *The maps γ and θ induce the same map on $K_*(A)$.*

Proof. Define $h: A \to E_1^* \otimes M_n$ and $\delta: A \to A \otimes M_n$ by $h(a) = \sum_j a x_j^* \otimes e_{1j}$ and $\delta(a) = a \otimes e_{11}$. We have $h(a_1)h(a_2)^* = \delta(a_1a_2^*)$ and $h(a_1)^*h(a_2) = \gamma(a_1^*a_2)$ for a_1, a_2 in A. Taking tensor products with K and identifying $M_n \otimes K$ with K, we obtain from γ and δ, respectively, $*$-endomorphisms α and β of $A \otimes K$, and from h a map $g: A \otimes K \to E_1^* \otimes K$, such that $\gamma_* = \alpha_*$, $\beta_* = \mathrm{id}$, $g(a_1)^*g(a_2) = \alpha(a_1^*a_2)$ and $g(a_1)g(a_2)^* = \beta(a_1a_2^*)$. The proof is completed by invoking 3.2.

4. CROSSED PRODUCTS BY ENDOMORPHISMS

We consider the case in which B is the crossed product of A by a single endomorphism, more or less as in [11]. Specifically, let A be unital and suppose we have a $*$-isomorphism $\sigma: A \to pAp$, where p is a proper projection of A. By considering the completion in the greatest C^*-norm of an appropriate $*$-algebra, we can find a faithful $*$-representation of A on a Hilbert space H and an isometry S of H satisfying $SaS^* = \sigma(a)$ (a in A) with the following universal property: given a $*$-representation π of A and an isometry R of the representation space satisfying $R\pi(a)R^* = \pi(\sigma(a))$, there is a $*$-homomorphism from $C^*(A, S)$ to $C^*(\pi(A), R)$ taking a to $\pi(a)$ and S to R. We write $C^*(A, \sigma)$ in place of $C^*(A, S)$, and call it the crossed product of A by σ. (It was shown in [11] that if A is strongly amenable and has no nontrivial σ-invariant ideals, then $C^*(A, \sigma)$ is simple.) By the universal property, there in an action ρ of $\mathbf{T}$ on $C^*(A, \sigma)$ such that $\rho_\lambda(a) = a$, $\rho_\lambda(S) = \lambda S$ (a in A, λ in $\mathbf{T}$) whose fixed-point algebra is precisely A.

4.1. THEOREM. *Let A and σ be as above and assume that $\sigma(A)$ is a full corner of A. Then there is a cyclic exact sequence*

$$\dots \to K_j(A) \xrightarrow{\sigma_*-\mathrm{id}} K_j(A) \xrightarrow{i_*} K_j(C^*(A,\sigma)) \to K_{1-j}(A) \xrightarrow{\sigma_*-\mathrm{id}} \dots \qquad (j = 0,1),$$

where σ is regarded as a map of A into itself.

Proof. For the action ρ of $\mathbf{T}$ on $C^*(A, \sigma)$, we have S in E_1, so $E_1E_1^*$ contains $SAS^* = \sigma(A)$ and hence $E_1E_1^* = A$, since no proper ideal of A contains $\sigma(A)$. We have $E_1^*E_1 = A$ because $S^*S = 1$. Now apply 3.3, with $\gamma = \sigma$, and 3.1.

4.2. COROLLARY. *Let A and σ be as in* 4.1 *and suppose in addition that A is an* AF-*algebra. Then* $K_0(C^*(A,\sigma)) \approx K_0(A)/\mathrm{Im}(\sigma_* - \mathrm{id})$ *and* $K_1(C^*(A, \sigma)) \approx \ker(\sigma_* - \mathrm{id})$.

Proof. Since $K_1(A) = (0)$, this follows immediately from 4.1.

We can use 4.2 to construct "0_n-like" C^*-algebras (separable, unital, nuclear, infinite, simple) with tailor-made K_0 groups. Let G be a countable Riesz group (in the sense of [6]) with order unit u, and let σ_* be an order-automorphism of G such that $\sigma_*(u) < u$ and G^+ has no non-trivial σ_*-invariant faces. By [6, Thm. 2.2], there is a separable unital AF-algebra A with $K_0(A)$ order-isomorphic to G in such a way that u corresponds to [1]. Moreover, application of [8, Thm. 4.3] (see [7] for the statement of this in terms of K_0) yields a proper projection p of A and a $*$-isomorphism $\sigma\colon A \to pAp$ which, when regarded as a map from A into itself, induces the given order-automorphism σ_* on G. Our requirement that G^+ have no non-trivial σ_*-invariant faces means precisely that A has no non-trivial σ-invariant ideals. Under these circumstances, $C^*(A, \sigma)$ has the "0_n-like" properties indicated above [11]. We have $K_0(C^*(A, \sigma)) \approx G/\mathrm{Im}\,(\sigma_* - \mathrm{id})$.

One can, for instance, choose G and σ_* such that $G/\mathrm{Im}\,(\sigma_* - \mathrm{id})$ is isomorphic to any prescribed countable subgroup H of $(\mathbf{R}, +)$. The construction is as follows. Choose λ in (0,1) such that λ is not a root of any non-zero polynomial with coefficients in H; such a λ exists for the same reason that transcedental numbers exist.

Let $G = \left\{ \sum_{-N}^{N} x_j \lambda^j : x_j \in H \text{ for } -N \leqslant j \leqslant N \right\}$, so G is countable and totally ordered by the order inherited from $\mathbf{R}$. Let σ_* be multiplication by λ. Note that every non-zero element t in G^+ is an order unit (so G^+ has no non-trivial faces) and satisfies $\sigma_*(t) < t$ (so we may pick u arbitrarily in $G^+ \setminus \{0\}$). Deprived of its order, G is the (weak) direct sum of countably many copies of H, indexed by $\mathbf{Z}$, and σ_* is the shift. It is a routine matter to check that $G/\mathrm{Im}\,(\sigma_* - \mathrm{id})$ is isomorphic to H.

Research for this paper partially supported by NSF Grant MCS 77–01850.

REFERENCES

1. Brown, L. G., Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pacific J. Math.*, **71**(1977), 335–348.
2. Brown, L. G. ; Green, P. ; Rieffel, M. A., Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pacific J. Math.*, **71**(1977), 349–363.
3. Connes, A., An analogue of the Thom isomorphism for cross products of a C^*-algebra by an action of $\mathbf{R}$, *Advances in Math.*, to appear.
4. Cuntz, J., K-theory for certain C^*-algebras, preprint SFB 123, *Ann. of Math.*, to appear.
5. Cuntz, J., K-theory for certain C^*-algebras. II, *J. Operator Theory*, **5**(1981), 101 – 108.
6. Effros, E. G. ; Handelman, D. ; Shen, C.-L., Dimension groups and their affine representations, *Amer. J. Math.*, **102**(1980), 385–407.
7. Effros, E. G. ; Rosenberg, J. ; C^*-algebras with approximately inner flip, *Pacific J. Math.*, **77**(1978), 417–443.
8. Elliott, G., On the classification of inductive limits of sequences of finite-dimensional algebras, *J. Algebra*, **38** (1976), 29–44.
9. Kaplansky, I., *Rings of operators*, W. A. Benjamin, New York, 1968.
10. Kishimoto, A. ; Takai, H., Some remarks on C^*-dynamical systems with a compact abelian group, *Publ. RIMS Kyoto Univ.*, **14**(1978), 383–397.
11. Paschke, W. L., The crossed product of a C^*-algebra by an endomorphism, *Proc. Amer. Math. Soc.*, **80**(1980), 113–118.
12. Pimsner, M. ; Voiculescu, D., Exact sequences for K-groups and Ext-groups of certain cross-products C^*-algebras, *J. Operator Theory*, **4**(1980), 93–118.
13. Rieffel, M. A., C^*-algebras associated with irrational rotations, *Pacific J. Math.*, to appear.
14. Rosenberg, J., Appendix to O. Bratteli's paper on "Crossed products of UHF algebras", *Duke Math. J.*, **46** (1979), 25–26.
15. Takai, H., On a duality for crossed products of C^*-algebras, *J. Functional Analysis*, **19**(1975), 25–39.
16. Taylor, J., Banach algebras and topology, in *Algebras in Analysis* (ed. J. Williamson), Academic Press, 1975.

WILLIAM L. PASCHKE
Department of Mathematics,
University of Kansas,
Lawrence, KS 66045,
U.S.A.

Received September 9, 1980.