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EXT OF CERTAIN FREE PRODUCT C^*-ALGEBRAS

LAWRENCE G. BROWN

§ 1.

In this note we prove an exact sequence which makes it possible to compute $\mathrm{Ext}(A \underset{C}{*} B)$ up to a group extension, where $A \underset{C}{*} B$ is the amalgamated free product, C is finite dimensional, and $\mathrm{Ext}(A)$ and $\mathrm{Ext}(B)$ are groups. Since $A \underset{C}{*} B$ is not generally nuclear, it is not possible to deduce this result from the general theory; and our proof is quite elementary.

The Calkin algebra Q is $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, where $\mathscr{L}(\mathscr{H})$ is the C^*-algebra of bounded operators on the separable infinite dimensional Hilbert space $\mathscr{H}$, and $\mathscr{K}(\mathscr{H})$ is the ideal of compact operators. The quotient map from $\mathscr{L}(\mathscr{K})$ to Q is denoted by π. Since all separable infinite dimensional Hilbert spaces are isomorphic, we are free to change $\mathscr{H}$ when convenient. For A a separable C^*-algebra, $\mathrm{Ext}(A)$ is defined as a set of equivalence classes $[\tau]$. Here $\tau\colon A \to Q$ is a $*$-monomorphism and $\tau_1 \sim \tau_2$ if and only if there is a partial isometry $u \in Q$ such that $u^*u\tau_1(a) = \tau_1(a)$ and $\tau_2(a) = u\tau_1(a)u^*$ for all $a \in A$. An operation, $+$, is defined on Ext (A) by means of the direct sum operation for Hilbert spaces, and $\mathrm{Ext}(A)$ becomes an abelian semigroup τ or $[\tau]$ is called trivial if there is a $*$-homomorphism $\sigma\colon A \to \mathscr{L}(\mathscr{H})$ such that $\tau = \pi\sigma$. Voiculescu's theorem [11] shows that there is only one trivial element of $\mathrm{Ext}(A)$ and that it is a unit for $\mathrm{Ext}(A)$. Ext is a contravariant functor. If A has a unit an object $\mathrm{Ext}_s(A)$ containing equivalence classes $[\tau]_s$ can be defined similarly. The only differences are that τ (and σ) are required to be unital and $\tau_1 \underset{s}{\sim} \tau_2$ if and only if there is a unitary $U \in \mathscr{L}(\mathscr{H})$ such that $\tau_2 = \pi(U)\tau_1\pi(U)^{-1}$. Further details on Ext can be found in [2], [3], [5], [8], [9], and other works. Voiculescu's theorem is the only major result we will use.

If P_1 and P_2 are projections in $\mathscr{L}(\mathscr{H})$ such that $\pi(P_1) = \pi(P_2)$, then an integer $[P_1\colon P_2]$, called the essential codimension, can be defined [4, § 4.9]. $[\ :\ ]$ has the following properties:

1. If $V_i^* V_i = 1$, $V_i V_i^* = P_i$, $i = 1, 2$, then $[P_1\colon P_2] = \mathrm{index}\, V_2^* V_1$;

2. If $P_2 \leqslant P_1$, $[P_1: P_2] = [P_1\mathscr{H}: P_2\mathscr{H}]$ (usual codimension);

3. $[P_2: P_1] = -[P_1:P_2]$;

4. $[P_1 + P_1': P_2 + P_2'] = [P_1: P_2] + [P_1': P_2']$, when sensible;

5. $[P_1: P_2] = 0$ if and only if there is a unitary U such that $\pi(U) = 1$ and $P_2 = UP_1U^{-1}$.

For A sufficiently nice a group $\mathrm{Ext}_0(A)$ is usually defined as $\mathrm{Ext}(SA)$, where S denotes suspension. For present purposes we need a different approach (cf. [1]). For A an arbitrary separable C^*-algebra we consider pairs (σ_1, σ_2) of $*$-homomorphisms from A to $\mathscr{L}(\mathscr{H})$ such that $\pi\sigma_1 = \pi\sigma_2$, $\pi\sigma_1$ is one-to-one, and σ_1 and σ_2 have infinite dimensional degeneracy subspaces. We define „Ext_0"(A) as the set of equivalence classes $[(\sigma_1, \sigma_2)]$, where $(\sigma_1, \sigma_2) \sim (\sigma_1', \sigma_2')$ if and only if there are unitaires U_1, U_2 such that $\sigma_i' = U_i\sigma_iU_i^{-1}$ and $\pi(U_1)=\pi(U_2)$. Just as for Ext, "Ext_0" (A) is an abelian semigroup. If A has a unit, a similar object "$\widetilde{\mathrm{Ext}}_0$"$(A)$ is defined by replacing the last condition on (σ_1, σ_2) with the condition that σ_1 and σ_2 be unital. The equivalence relation just defined is rather fine. For A nice a coarser equivalence relation could be defined (cf. [5, pp. 317—318]) to yield a group isomorphic to $\mathrm{Ext}_0(A)$, but no knowledge of the relationship between "Ext_0" and Ext_0 is needed to read this note.

If A is finite dimensional, say $A \cong M_{n_1} \oplus \ldots \oplus M_{n_r}$, an isomorphism of "$\mathrm{Ext}_0$"$(A)$ with $\mathbf{Z}^r$ is defined as follows: Let e_i be a minimal projection in M_{n_i}, $i = 1, \ldots, r$. Map $[(\sigma_1, \sigma_2)]$ to $[\sigma_1: \sigma_2] = (\ldots, [\sigma_1(e_i): \sigma_2(e_i)], \ldots)$. Similarly "$\widetilde{\mathrm{Ext}}_0(A)$" $\cong \left\{(m_1, \ldots, m_r) \in \mathbf{Z}^r: \sum_1^r m_in_i = 0\right\}$. For convenience, we will drop the quotation marks when A is finite dimensional. (Again, it is not necessary for the reader to know that this dropping of quotation marks is correct.)

Finally we define $A \underset{C}{*} B$, when A, B, C are C^*-algebras and C is a C^*-subalgebra of each, as the Hausdorff completion of $A \underset{\mathrm{alg}}{*} B$, the algebraic free product. $A \underset{\mathrm{alg}}{*} B$ is the $*$-algebra of all formal finite sums of mononials ($a_1b_1a_2b_2 \ldots$, or $b_1a_1b_2 \ldots$, $a_i \in A$, $b_i \in B$). No monomials of length 0 are permitted and hence $A \underset{\mathrm{alg}}{*} B$ has no unit. If ρ_1 and ρ_2 are $*$-representations of A and B on the same Hilbert space $\mathscr{H}'$, there is a corresponding $*$-homomorphism ρ: $A \underset{\mathrm{alg}}{*} B \to \mathscr{L}(\mathscr{H}')$. The topology on $A \underset{\mathrm{alg}}{*} B$ is defined by a C^*-seminorm:

$$\|P\| = \sup\{\|\rho(P)\|: \rho_1|_C = \rho_2|_C\}.$$

It is routine to verify that $A \underset{C}{*} B$ is a C^*-algebra containing (isomorphic copies of) A and B and containing only one copy of C. (The definition of the seminorm is such that the two copies of C in $A \underset{\mathrm{alg}}{*} B$—one in A and one in B— are identified in the Hausdorff completion.) $A \underset{C}{*} B$ frequently has no unit, but it does have a unit if each of A and B has a unit *and* C contains either the unit of A or the unit of B.

§ 2.

THEOREM. *Let A and B be separable C^*-algebras such that* Ext(A) *and* Ext(B) *are groups and C a finite dimensional C^*-algebra contained in A and B. Then* $\mathrm{Ext}(A \underset{C}{*} B)$ *is a group and exact sequences exist as follows:*

1.
$$\text{“}\mathrm{Ext}_0\text{”}(A) \oplus \text{“}\mathrm{Ext}_0\text{”}(B) \xrightarrow{\alpha} \mathrm{Ext}_0(C) \xrightarrow{\beta} \mathrm{Ext}(A \underset{C}{*} B) \xrightarrow{\gamma} \to \mathrm{Ext}(A) \oplus \mathrm{Ext}(B) \xrightarrow{\delta} \mathrm{Ext}(C)$$

(in all cases);

2.
$$\text{“}\widetilde{\mathrm{Ext}}_0\text{”}(A) \oplus \text{“}\widetilde{\mathrm{Ext}}_0\text{”}(B) \xrightarrow{\alpha} \widetilde{\mathrm{Ext}}_0(C) \xrightarrow{\beta} \mathrm{Ext}_s(A \underset{C}{*} B) \xrightarrow{\gamma} \to \mathrm{Ext}_s(A) \oplus \mathrm{Ext}_s(B) \xrightarrow{\delta} \mathrm{Ext}_s(C)$$

(if A and B are unital and C contains both units);

3.
$$\text{“}\mathrm{Ext}_0\text{”}(A) \oplus \text{“}\widetilde{\mathrm{Ext}}_0\text{”}(B) \xrightarrow{\alpha} \mathrm{Ext}_0(C) \xrightarrow{\beta} \mathrm{Ext}_s(A \underset{C}{*} B) \xrightarrow{\gamma} \to \mathrm{Ext}(A) \oplus \mathrm{Ext}_s(B) \xrightarrow{\delta} \mathrm{Ext}(C)$$

(if A and B are unital and C contains the unit of A but not the unit of B).

Proof. The three cases are similar and will be treated simultaneously. Let i_1 and i_2 be the inclusions of C into A and B and j_1 and j_2 the inclusions of A and B into $A \underset{C}{*} B$. In Case 2 $A \underset{C}{*} B$ is unital and i_1, i_2, j_1 and j_2 are unital; in Case 3 $A \underset{C}{*} B$ is unital and j_2 (also i_1) is unital.

γ is defined as (j_1^*, j_2^*), and α and δ are defined as $i_1^* - i_2^*$. It is necessary to show that the range of α is a subgroup, since “Ext_0” is only a semigroup. This follows since in $\mathrm{Ext}_0(C)$ the inverse of $[(\sigma_1, \sigma_2)]$ is $[(\sigma_2, \sigma_1)]$ and the unit is $[(\sigma_1, \sigma_1)]$. “Ext_0” is closed under these operations, even when it is not a group.

The definition of β is the main part of the proof and the place where finite dimensionality of C is used. Let (σ_1, σ_2) define an element of $\mathrm{Ext}_0(C)$ ($\widetilde{\mathrm{Ext}}_0(C)$ in Case 2). Extend σ_1 to a representation ρ_1 of A and σ_2 to a representation ρ_2 of B. In Case 2 ρ_1 and ρ_2 are unital since σ_1 and σ_2 are; in Case 3 ρ_2 is required to be unital. This is possible since any two representations of C which are faithful modulo $\mathcal{K}$ and have degeneracy subspaces of equal dimension are unitarily equivalent, since A and B possess representations faithful modulo $\mathcal{K}$, and since the unitality and non-unitality conditions fit together properly. Since $\pi\sigma_1 = \pi\sigma_2$, $\pi\rho_1$ and $\pi\rho_2$ define a $*$-homomorphism $\tau\colon A \underset{C}{*} B \to Q$. We define $\beta([\sigma_1, \sigma_2]) = [\tau]$ ($[\tau]_s$ in Cases 2 and 3). It must be proved that β is well defined. Thus let (σ_1', σ_2') be such that $[(\sigma_1', \sigma_2')] = [(\sigma_1, \sigma_2)]$ and let ρ_1', ρ_2' be appropriate extensions, giving rise to $\tau'\colon A \underset{C}{*} B \to Q$. Let $[(\sigma_1'', \sigma_2'')] = -[(\sigma_1, \sigma_2)]$ and ρ_1'', ρ_2'' appropriate extensions giving rise to τ''. We will show that $\tau \oplus \tau''$ and $\tau' \oplus \tau''$ are trivial.

Then:

$$[\tau] + ([\tau''] + [\tau']) = ([\tau] + [\tau'']) + [\tau']$$

$$[\tau] + 0 = 0 + [\tau']$$

$$[\tau] = [\tau'].$$

After a change of notation it suffices to prove τ is trivial whenever $[(\sigma_1, \sigma_2)] = 0$. To do this we note that (from the theory of essential codimension) there is a unitary $U \in 1 + \mathcal{K}$ such that $\sigma_2 = U\sigma_1 U^{-1}$. Then $U\sigma_1 U^{-1}$ and ρ_2 define a representation of $A \underset{C}{*} B$ which lifts τ.

Now it is obvious that $\delta\gamma = 0$ and $\gamma\beta = 0$. We show that $\beta\alpha = 0$. We have to show that if (λ_1, λ_2) defines an element of "Ext_0"(A) (the case of "Ext_0"(B) is similar), then $\beta([\lambda_1|_C, \lambda_2|_C]) = 0$. To do this we take ρ_2 an appropriate extension of $\lambda_2|_C$ and $\rho_1 = \lambda_1$, thus getting $\tau\colon A \underset{C}{*} B \to Q$. Then λ_2 and ρ_2 define a lifting of τ and hence τ is trivial.

Exactness at $\mathrm{Ext}(A) \oplus \mathrm{Ext}(B)$: Let $\tau_1\colon A \to Q$ and $\tau_2\colon B \to Q$ be such that $\tau_1|_C \sim \tau_2|_C$. We may assume that $\tau_1(A)$ (in Cases 1 and 3) and $\tau_2(B)$ (in Case 1) do not contain the unit of Q. In all cases it is then true that there is a *unitary* $u \in Q$ such $\tau_2|_C = u(\tau_1|_C)u^{-1}$. In Case 2 u is even $\pi(U)$ for U unitary. Then $u\tau_1 u^{-1}$ and τ_2 give rise to $\tau\colon A \underset{C}{*} B \to Q$ and $\gamma([\tau]) = [\tau_1] \oplus [\tau_2]$.

Exactness at $\mathrm{Ext}(A \underset{C}{*} B)$: Let $\tau\colon A \underset{C}{*} B \to Q$ be such that $\tau|_A$ and $\tau|_B$ are trivial. Let ρ_1 be a lifting of $\tau|_A$ (unital in Case 2) and ρ_2 a lifting of $\tau|_B$ (unital in Cases 2 and 3). Let $\sigma_i = \rho_i|_C$, $i = 1,2$. Then (σ_1, σ_2) defines an element of $\mathrm{Ext}_0(C)$ ($\widetilde{\mathrm{Ext}}_0(C)$ in Case 2), ρ_i is an appropriate extension of σ_i, and hence $\beta([\sigma_1, \sigma_2]) = [\tau]$.

Exactness at $\mathrm{Ext}_0(C)$: Let $[(\sigma_1, \sigma_2)]$ be in the kernel of β. Let ρ_1 and ρ_2 be appropriate extensions of σ_1 and σ_2 giving rise to $\tau\colon A \underset{C}{*} B \to Q$. We may assume ρ_1, in Cases 1 and 3, and ρ_2, in Case 1, have infinite dimensional degeneracy subspaces. Let λ_1 and λ_2 be representations of A and B giving rise to a lifting of τ. We may assume λ_1 and λ_2 satisfy the same unitality or non-unitality conditions.

$$[(\sigma_1, \sigma_2)] = [(\sigma_1, \lambda_1|_C)] + [(\lambda_1|_C, \lambda_2|_C)] + [(\lambda_2|_C, \sigma_2)].$$

Note that $[(\sigma_1, \lambda_1|_C)]$ is the image of $[(\sigma_1, \lambda_1)] \in$ "Ext_0"(A) ("$\widetilde{\mathrm{Ext}}_0$"$(A)$ in Case 2), $[(\lambda_1|_C, \lambda_2|_C)] = 0$ since $\lambda_1|_C = \lambda_2|_C$, and $[(\lambda_2|_C, \sigma_2)]$ is the image of $[(\lambda_2, \rho_2)] \in$ $\in$ "Ext_0"(B) ("$\widetilde{\mathrm{Ext}}_0$"$(B)$ in Cases 2 and 3).

Finally we prove $\mathrm{Ext}(A \underset{C}{*} B)$ is a group. Let $x \in \mathrm{Ext}(A \underset{C}{*} B)$. By exactness the range of γ is a group and hence there is y such that $\gamma(x + y) = 0$. Then $x + y = \beta z$ for some z. Thus $y + \beta(-z)$ is an inverse of x.

REMARK. For any unital separable C^*-algebra, Ext_s is a group if and only if Ext is a group.

§ 3. EXAMPLES AND REMARKS

1. For a countable discrete group G, let $C^*(G)$ denote the enveloping C^*-algebra. Then $C^*(G*H) \cong C^*(G) \underset{\mathbf{C}1}{*} C^*(H)$. More generally if K is a subgroup of G and H, $C^*(G \underset{K}{*} H) \cong C^*(G) \underset{C^*(K)}{*} C^*(H)$. If K is finite and $\mathrm{Ext}(C^*(G))$ and $\mathrm{Ext}(C^*(H))$ are groups, the theorem applies. The case where G and H are finite was done (with different terminology) by Karoubi and de la Harpe [7, Théorème 2].

2. Consider $A = M_n \underset{\mathbf{C}1}{*} C(X)$, where X is a compact metric space. It is easy to see that the map from "Ext_0"$(C(X))$ to $\mathrm{Ext}_0(\mathbf{C})$ is onto. Hence the theorem implies that $\mathrm{Ext}(A) \cong \mathrm{Ext}(C(X))$. Two special cases:

a. $X = S^\circ$ ($C(X) = \mathbf{C} \oplus \mathbf{C}$). Then $\mathrm{Ext}\,(A) = 0$. It is easy to see that $A \cong M_n \otimes \otimes G_n^{\mathrm{nc}}$, where G_n^{nc}, the *non-commutative Grassmanian*, is defined by generators and relations. G_n^{nc} is generated by an identity and elements P_{ij}, $i, j = 1, \ldots, n$, subject to the relations that cause the $n \times n$ matrix $[P_{ij}]$ to be a projection. ($P_{ij}^* = P_{ji}$ and $P_{ij} = \sum_k P_{ik} P_{kj}$). The abelianization of G_n^{nc} is $C\left(\bigcup_{k=0}^{n} G_{k,n}\right)$, where $G_{k,n}$ is the usual Grassmanian of k dimensional subspaces of $\mathbf{C}^n$.

b. $X = S^1$. Then $\mathrm{Ext}\,(A) \cong \mathbf{Z}$. Now $A \cong M_n \otimes U_n^{\mathrm{nc}}$, where U_n^{nc} is the non-commutative version of $C(U_n)$, U_n the group of $n \times n$ unitary matrices. U_n^{nc} is generated by an identity and elements $U_{i,j}$, $i, j = 1, \ldots, n$, subject to the relations that cause $[U_{i,j}]$ to be unitary. The abelianization of U_n^{nc} is $C(U_n)$.

The algebras of G_n^{nc} and U_n^{nc} were originally intended to be useful in generalizing Atiyah's proof of the Künneth theorem of K-theory to non-commutative C^*-algebras. What we wanted was to obtain a C^*-algebra $\tilde{A}$ with $K_0(\tilde{A})$ and $K_1(\tilde{A})$ free and a map $\theta\colon \tilde{A} \to A$, A given in advance, which induces surjective maps from $K_i(\tilde{A})$ to $K_i(A)$. $\tilde{A}$ was to be a free product of G_n^{nc}'s and U_n^{nc}'s with one G_n^{nc} for each class of projections in $A \otimes M_n$ and one U_n^{nc} for each class of unitaries in $A \otimes M_n$. Unfortunately we do not know the K-theory of such $\tilde{A}$ despite the fact that it is easy to calculate $\mathrm{Ext}(\tilde{A})$ at least for finite free products. Now this is all unnecessary, since Schochet has observed that it is sufficient to have a map $\theta\colon \tilde{A} \to A \otimes \mathscr{K}$. It is then possible to choose a more tractable algebra for $\tilde{A}$. ([10]).

3. It was observed by W. Paschke that the algebras $\mathcal{O}_n$ of Cuntz [6] are related to free products. Precisely $\mathcal{O}_n \otimes M_{n+1} \cong M_{n+1} \underset{\mathbf{C}^2}{*} M_2$. Here $\mathbf{C}^2$ is imbedded in M_2 as the diagonal matrices and in M_{n+1} as the diagonal matrices whose first n entries are equal.

4. It was pointed out to us by V. Paulsen that when $\mathrm{Ext}(A \underset{C}{*} B)$ is known a priori to have homotopy invariance, another method can be used to prove that β is well-defined, which does not require C to be finite dimensional. By recent results

of Kasparov [8], this is available whenever $A \underset{C}{*} B$ is nuclear. Since $\mathrm{Ext}_0(A \underset{C}{*} B) = \mathrm{Ext}(S(A \underset{C}{*} B)) = \mathrm{Ext}(SA \underset{SC}{*} SB)$ (if $C \neq 0$), whenever the above applies, it applies also to Ext_0; and we get a six-term cyclic exact sequence. Of course $A \underset{C}{*} B$ is usually not nuclear, but there are some cases when it is, such as Examples 3 above and 5 below[1)].

5. It was pointed out to us by P. de la Harpe that $\mathbf{Z} \underset{2\mathbf{Z}}{*} \mathbf{Z}$ is amenable. Hence its C^*-algebra is nuclear. Thus we can compute

$$\mathrm{Ext}(C^*(\mathbf{Z} \underset{2\mathbf{Z}}{*} \mathbf{Z})) \cong \mathbf{Z}$$

and

$$\mathrm{Ext}_0(C^*(\mathbf{Z} \underset{2\mathbf{Z}}{*} \mathbf{Z})) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

6. The properties of finite dimensional algebras used in this proof are the same ones that make possible a direct proof of the $\lim^{(1)}$ sequence for AF-algebras.

7. [8] and [9] consider more general Ext groups in which the ideal $\mathcal{K}(\mathcal{H})$ is replaced by another ideal J. The same methods used above should work in this case. The ideal J would have to be such that the theory of essential codimension could still be carried out. The range of the essential codimension would be $K_0(J)$. We mention that the definition of essential codimension (point 1 in the list of properties given) fits very well with our favorite treatment of the relevant K-theory.

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1) Note that when C is general there are additional cases when $A \underset{C}{*} B$ can be unital. It is necessary and sufficient that C generate A as hereditary C^*-sub-algebra and B be unital, or vice-versa.

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LAWRENCE G. BROWN
Department of Mathematics,
Purdue University,
West Lafayette, IN 47907,
U.S.A.

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