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FREDHOLM OPERATORS AND THE CONTINUITY OF THE LEFSCHETZ NUMBER

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1.

Let X be a complex Banach space and let $\mathscr{L}(X)$ be the algebra of all linear bounded operators on X. If $A \in \mathscr{L}(X)$ and M, N are closed subspaces of X that are invariant under A, such that $N \subset M$ and $\dim M/N < \infty$, then we denote by $\mathrm{Tr}_{M/N}(A)$ the *trace* of the operator induced by A in M/N. In particular, if $N = 0$, then $\dim M < \infty$, and we write simply $\mathrm{Tr}_M(A)$ instead of $\mathrm{Tr}_{M/0}(A)$.

Now consider another Banach space Y, and let us denote by $\mathscr{L}(X, Y)$ the space of all bounded linear operators from X into Y. If $S \in \mathscr{L}(X, Y)$ is a Fredholm operator and $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(Y)$ have the property $SA = BS$, then we define

$$\mathrm{L}_S(A, B) = \mathrm{Tr}_{\mathrm{N}(S)}(A) - \mathrm{Tr}_{Y/\mathrm{R}(S)}(B), \tag{1.1}$$

where $\mathrm{N}(S)$ and $\mathrm{R}(S)$ stand for the *null-space* and the *range* of S, respectively. The number (1.1) will be called the *Lefschetz number* of the pair (A, B) with respect to S (see [1], for a similar context).

The aim of this paper is to prove the norm-continuity of the function

$$(S, A, B) \to \mathrm{L}_S(A, B),$$

when S runs over all Fredholm operators in $\mathscr{L}(X, Y)$, and $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(Y)$ satisfy $SA = BS$. Note that for $A = 1_X$ and $B = 1_Y$, we have the equality

$$\mathrm{L}_S(1_X, 1_Y) = \mathrm{ind}\, S,$$

therefore the property of continuity of the function $\mathrm{L}_S(A, B)$ is more comprehensive than that of continuity (i.e., stability) of the index for Fredholm operators.

The last section of the present work contains some applications, which are derived from the above mentioned continuity. In particular, the existence of eigenvalues for some types of operators is proved.

Now we shall state, without proof, two results concerning the trace of an operator, that are more or less known.

1.1. LEMMA. *Let $A \in \mathscr{L}(X)$, and let $M \subset X$ be a closed subspace that is invariant under A, with $\dim X/M < \infty$. Let N be a complement of M in X, and let P denote the projection of X onto N along M. If $A_N = PA \mid N$, then*

$$\mathrm{Tr}_N(A_N) = \mathrm{Tr}_{X/M}(A).$$

1.2. LEMMA. *Let $A \in \mathscr{L}(X)$, and let M, N be closed subspaces of X that are invariant under A, such that $N \subset M$, $\dim X/M < \infty$ and $\dim M/N < \infty$. Then we have the equality*

$$\mathrm{Tr}_{X/N}(A) = \mathrm{Tr}_{X/M}(A) + \mathrm{Tr}_{M/N}(A).$$

2.

This section contains some auxiliary results, that will be used in the next section. For each pair of closed subspaces M, N in the Banach space X, $M \neq 0$, $N \neq 0$, we denote by $\hat{\mathrm{d}}(M, N)$ the Hausdorff distance between the unit spheres of those subspaces [2, IV.2.1].

2.1. LEMMA. *Let X be a Banach space, let $M \subset X$ be a proper complemented subspace of X and let P denote a projection of X onto M. If N is a closed subspace of X that satisfies*

$$\hat{\mathrm{d}}(M, N) < \min\{\|P\|^{-1}, \|1_X - P\|^{-1}\}, \tag{2.1}$$

then N is also complemented by $(1_X - P)X$ and the projection Q of X onto N along $(1_X - P)X$ satisfies

$$\|Q - P\| \leqslant 2\,\|P\|\,\|1_X - P\|\hat{\mathrm{d}}(M, N)(1 - \|1_X - P\|\hat{\mathrm{d}}(M, N))^{-1}. \tag{2.2}$$

The proof of this result, with some minor changes, can be found in [3,13 H].

We denote by δ_{jk} the usual Kronecker symbol, where j, k are natural numbers.

2.2. LEMMA. *Let X be a Banach space, let $M \subset X$ be a subspace of X with $\dim M = m < \infty$ and let P_M denote a projection of X onto M. If N is a closed subspace of X such that*

$$\hat{\mathrm{d}}(M, N) < \min\{\|P_M\|^{-1}, \|1_X - P_M\|^{-1}, \|1_X - P_M\|^{-1}(1 + 2\,\|P_M\|)^{-1}\}, \tag{2.3}$$

then $\dim N = \dim M$ and for every bases $\{x_1, \ldots, x_m\}$ in M and $\{x_1^*, \ldots, x_m^*\}$ in M^* with $x_j^*(x_k) = \delta_{jk}$ $(j, k = 1, \ldots, m)$ we can find the bases $\{y_1, \ldots, y_m\}$, in N and $\{y_1^*, \ldots, y_m^*\}$ in N^*, such that $y_j^*(y_k) = \delta_{jk}$ $(j, k = 1, \ldots, m)$ and

$$\|x_j - y_j\| \leqslant r(d), \quad \|x_j^* - y_j^*\| \leqslant r(d), \quad j = 1, \ldots, m,$$

where $r(d) \to 0$ as $d = \hat{\mathrm{d}}\,(M, N) \to 0$. (Here the bases $\{x_1^*, \ldots, x_m^*\}$ and $\{y_1^*, \ldots, y_m^*\}$ are viewed as systems of vectors in X^*, by a certain extension.)

Proof. We are indebted to Prof. B. Gramsch (Mainz) for some suggestions concerning this proof.

The condition (2.3) implies (2.1); therefore, by Lemma 2.1, the subspace N is complemented. Let P_N be the projection of X onto N given by Lemma 2.1. From (2.3), we also infer that $\|P_M - P_N\| < 1$. Consider the operator

$$G = P_N P_M + (1_X + P_N)(1_X - P_M) = 1_X - (P_M - P_N),$$

which is invertible since $\|1_X - G\| < 1$. We note that G maps the space M onto the space N. Indeed, by Lemma 2.1, both M and N have the same complement in X, namely $(1_X - P_M)X$, therefore they are isomorphic. Since G maps M into N, and G is invertible, G should map M onto N.

The basis $\{x_1^*, \ldots, x_m^*\}$ of M^* will be regarded as a system of vectors in X^*, by an arbitrary norm-preserving extension. Let us define

$$y_j = Gx_j,\ y_j^* = G^{*-1}x_j^* \qquad (j = 1, \ldots, m).$$

Then we have $\|y_j - x_j\| \leqslant \|1_X - G\|\,\|x_j\|$ and $\|y_j^* - x_j^*\| \leqslant \|1_X - G^{-1}\|\,\|x_j^*\|$. Let us also note that

$$\|1_X - G\| = \|P_M - P_N\|,\ \|1_X - G^{-1}\| \leqslant \|P_M - P_N\|(1 - \|P_M - P_N\|)^{-1},$$

and $\|P_M - P_N\| \to 0$ as $d = \hat{d}(M,N) \to 0$, by (2.2).

Finally, the system $\{y_1^*, \ldots, y_m^*\}$, restricted to N, satisfies

$$y_j^*(y_k) = (G^{*-1}x_j^*)(Gx_k) = x_j^*(x_k) = \delta_{jk}$$

for all $j,k = 1, \ldots, m$, which finishes the proof.

3.

In this section we prove the main result of the paper. We start with a particular case of it.

3.1. LEMMA. *Let $S \in \mathscr{L}(X,Y)$ be a surjective Fredholm operator and let $A \in \mathscr{L}(X)$ have the property $A\mathrm{N}(S) \subset \mathrm{N}(S)$. Then for every $\varepsilon > 0$ one can find $\eta(\varepsilon) > 0$ such that if $\tilde{S} \in \mathscr{L}(X,Y)$ and $\tilde{A} \in \mathscr{L}(X)$ satisfy $\|\tilde{S} - S\| < \eta(\varepsilon)$, $\|\tilde{A} - A\| < \eta(\varepsilon)$ and $\tilde{A}\mathrm{N}(\tilde{S}) \subset \mathrm{N}(\tilde{S})$, then*

$$|\mathrm{Tr}_{\mathrm{N}(\tilde{S})}(\tilde{A}) - \mathrm{Tr}_{\mathrm{N}(S)}(A)| < \varepsilon.$$

Moreover, for $\varepsilon > 0$ sufficiently small, $\mathrm{N}(\tilde{S})$ and $\mathrm{N}(S)$ have the same complement in X.

Proof. It is known that if $\|\tilde{S} - S\| < \gamma(S)$, then $\tilde{S}$ is also surjective, where $\gamma(S)$ is the *reduced minimum modulus* of S [2]. Moreover,

$$\gamma(\tilde{S})^{-1} \leqslant \gamma(S)^{-1}(1 - \|\tilde{S} - S\|\,\gamma(S)^{-1})^{-1} \tag{3.1}$$

(see, for instance, Lemma 2.1 and Corollary 2.2 in [4]). Then by [4, Lemma 2.6], [2, IV.2.1] and (3.1), we have that

$$\hat{\mathrm{d}}(\mathrm{N}(S), \mathrm{N}(\tilde{S})) \leqslant \max\{2\|\tilde{S} - S\|\gamma(S)^{-1}, 2\|\tilde{S} - S\|\gamma(S)^{-1}(1 - \|\tilde{S} - S\|\gamma(S)^{-1})^{-1}\},$$

therefore we can apply Lemma 2.2, if $\|\tilde{S} - S\|$ is sufficiently small. Let $\{x_1, \ldots, x_m\} \subset \mathrm{N}(S)$, $\{x_1^*, \ldots, x_m^*\} \subset \mathrm{N}(S)^*$, $\{y_1, \ldots, y_m\} \subset \mathrm{N}(\tilde{S})$ and $\{y_1^*, \ldots, y_m^*\} \subset \mathrm{N}(\tilde{S})^*$ be the bases given by Lemma 2.2. Then we can write

$$|\mathrm{Tr}_{\mathrm{N}(\tilde{S})}(\tilde{A}) - \mathrm{Tr}_{\mathrm{N}(S)}(A)| =$$

$$= \left| \sum_{j=1}^{m} (y_j^*(\tilde{A}y_j) - x_j^*(Ax_j)) \right| \leqslant$$

$$\leqslant \sum_{j=1}^{m} (|y_j^*(\tilde{A}y_j) - x_j^*(\tilde{A}y_j)| + |x_j^*(\tilde{A}y_j) - x_j^*(Ay_j)| + |x_j^*(Ay_j) - x_j^*(Ax_j)|) \leqslant$$

$$\leqslant m\|\tilde{A}\| \max_{1 \leqslant j \leqslant m} \|y_j^*\| \max_{1 \leqslant j \leqslant m} \|y_j^* - x_j^*\| +$$

$$+ m \max_{1 \leqslant j \leqslant m} \|x_j^*\| \max_{1 \leqslant j \leqslant m} \|y_j\| \|\tilde{A} - A\| +$$

$$+ m\|A\| \max_{1 \leqslant j \leqslant m} \|x_j^*\| \max_{1 \leqslant j \leqslant m} \|y_j - x_j\|.$$

Note that if G is defined as in Lemma 2.2 for $M = \mathrm{N}(S)$ and $N = \mathrm{N}(\tilde{S})$, and if $\|1_X - G\| \leqslant \rho < 1$ then $\|G\| \leqslant 1 + \rho$ and $\|G^{-1}\| \leqslant 1 + \rho(1 - \rho)^{-1}$; hence the families $\{\|y_j\|;\ 1 \leqslant j \leqslant m\}$ and $\{\|y_j^*\|;\ 1 \leqslant j \leqslant m\}$ are uniformly bounded when $\|\tilde{S} - S\| \to 0$. If $r(d)$ is the function given by Lemma 2.2, where $d = \hat{\mathrm{d}}(\mathrm{N}(S), \mathrm{N}(\tilde{S}))$, then we have $r(d) \to 0$ as $\|\tilde{S} - S\| \to 0$, therefore

$$|\mathrm{Tr}_{\mathrm{N}(\tilde{S})}(\tilde{A}) - \mathrm{Tr}_{\mathrm{N}(S)}(A)| \to 0$$

when both $\|\tilde{S} - S\| \to 0$ and $\|\tilde{A} - A\| \to 0$.

The last assertion follows directly from Lemma 2.2.

3.2. COROLLARY. *Let $S \in \mathscr{L}(X, Y)$ be an injective Fredholm operator and let $B \in \mathscr{L}(Y)$ have the property $B\mathrm{R}(S) \subset \mathrm{R}(S)$. Then for every $\varepsilon > 0$ one can find $\eta(\varepsilon) > 0$ such that if $\tilde{S} \in \mathscr{L}(X, Y)$ and $\tilde{B} \in \mathscr{L}(Y)$ satisfy $\|\tilde{S} - S\| < \eta(\varepsilon)$, $\|\tilde{B} - B\| < \eta(\varepsilon)$ and $\tilde{B}\mathrm{R}(\tilde{S}) \subset \mathrm{R}(\tilde{S})$, then*

$$|\mathrm{Tr}_{Y/\mathrm{R}(\tilde{S})}(\tilde{B}) - \mathrm{Tr}_{Y/\mathrm{R}(S)}(B)| < \varepsilon.$$

Moreover, for $\varepsilon > 0$ sufficiently small, $\mathrm{R}(S)$ and $\mathrm{R}(\tilde{S})$ have the same complement in Y.

Proof. We apply a duality argument. The operator $S^* \in \mathscr{L}(Y^*, X^*)$ is Fredholm and surjective; therefore, by the previous lemma,

$$\varepsilon > |\mathrm{Tr}_{\mathrm{N}(\tilde{S}^*)}(\tilde{B}^*) - \mathrm{Tr}_{\mathrm{N}(S^*)}(B^*)| =$$

$$= |\mathrm{Tr}_{Y/\mathrm{R}(\tilde{S})}(\tilde{B}) - \mathrm{Tr}_{Y/\mathrm{R}(S)}(B)|,$$

when $\|\tilde{S}^* - S^*\| = \|\tilde{S} - S\| < \eta(\varepsilon)$ and $\|\tilde{B}^* - B^*\| = \|\tilde{B} - B\| < \eta(\varepsilon)$.

Since $\mathrm{N}(\tilde{S}^*)$ and $\mathrm{N}(S^*)$ have the same complement in Y^*, we obtain readily that $\mathrm{R}(\tilde{S})$ and $\mathrm{R}(S)$ have the same complement in Y.

3.3. LEMMA. *Let $S \in \mathscr{L}(X, Y)$, and consider the finite dimensional subspaces $N \subset X$ and $M \subset Y$. Let X_0 be a complement of N in X and let P_0 be the projection of X onto N along X_0. We extend S to $X \oplus M$ by the formula $S_1(x \oplus y) = Sx + y$ for all $x \oplus y \in X \oplus M$, and assume that $\mathrm{R}(S_1) = Y$. Then we take the restriction $S_0 = S \mid X_0$, and assume that S_0 is injective and that $\mathrm{R}(S_0)$ is a complement of M in Y. Let us denote by Q_0 the projection of Y onto M along $\mathrm{R}(S_0)$.*

Let us also consider $A \in \mathscr{L}(X)$ and $B \in \mathscr{L}(Y)$ such that $SA = BS$. We define $A_1 \in \mathscr{L}(X \oplus M)$ by the relation $A_1(x \oplus y) = Ax \oplus 0 + S_2^{-1}By$ for all $x \oplus y \in X \oplus M$, where $S_2 = S_1 \mid X_0 \oplus M$. We also define $B_0 = B_0' \oplus B_0'' \in \mathscr{L}(Y)$, where $B_0' \in \mathscr{L}(\mathrm{R}(S_0))$ is equal to $S_0A_0S_0^{-1}$, with $A_0 = (1_X - P_0)A \mid X_0$, and $B_0'' = Q_0B \mid M$. Then we have the relations $S_1A_1 = BS_1$, $S_0A_0 = B_0S_0$, $B_0\mathrm{R}(S) \subset \mathrm{R}(S)$ and

$$\mathrm{Tr}_{\mathrm{N}(S_1)/\mathrm{N}(S)\oplus 0}(A_1) = \mathrm{Tr}_{\mathrm{R}(S)/\mathrm{R}(S_0)}(B_0). \tag{3.2}$$

Moreover, the operators B and B_0 induce the same action in $Y/\mathrm{R}(S)$.

Proof. Let us note that we have

$$S_1A_1(x \oplus y) = S_1(Ax \oplus 0 + S_2^{-1}By) = SAx + By =$$

$$= BSx + By = BS_1(x \oplus y),$$

for all $x \oplus y \in X \oplus M$.

The relation $B_0S_0 = S_0A_0$ is obvious.

Let us show that $B_0\mathrm{R}(S) \subset \mathrm{R}(S)$. Indeed, since M is a complement of $\mathrm{R}(S_0)$ in Y, we have $\mathrm{R}(S) = \mathrm{R}(S_0) + \mathrm{R}(S) \cap M$, where the sum is direct. It is enough to show that $B_0(\mathrm{R}(S) \cap M) \subset \mathrm{R}(S)$. If $y \in \mathrm{R}(S) \cap M$ then $y = Sx$ and $B_0Sx = Q_0BSx = Q_0SAx$. But we can write $SAx = S_0u_0 + y_0$, with $y_0 \in M \cap \mathrm{R}(S)$, therefore $B_0Sx = y_0$ is an element of $\mathrm{R}(S)$. We have obtained, in fact, that the space $\mathrm{R}(S) \cap M$ is invariant under B_0.

Now we have to prove the equality (3.2). Let us define the operators J and T_0 from $X \oplus M$ into Y as follows: $J(x \oplus y) = y$ and $T_0(x \oplus y) = Sx$ for all $x \oplus y \in X \oplus M$. Notice that we have

$$T_0A_1 = BT_0 + BJ - JA_1, \tag{3.3}$$

and

$$Q_0BJ = JA_1. \tag{3.4}$$

Indeed, $T_0 = S_1 - J$ and $S_1A_1 = BS_1$, from which we derive (3.3).

If $x \oplus y \in X \oplus M$ and $By = S_0x_0 + y_0$, with $y_0 \in M$, then we have $Q_0By = y_0$. On the other hand, since $S_2^{-1}(S_0x_0 + y_0) = x_0 \oplus y_0$, we have $JA_1(x \oplus y) = JS_2^{-1}By = y_0$, which implies (3.4). From (3.3), (3.4) and the equality $Q_0J = J$, we infer that

$$T_0A_1 \mid \mathrm{N}(S_1) = Q_0BT_0 \mid \mathrm{N}(S_1), \tag{3.5}$$

since $A_1\mathrm{N}(S_1) \subset \mathrm{N}(S_1)$ and $T_0N(S_1) = \mathrm{R}(S) \cap M$. If $\hat{T}_0$ is the operator induced by T_0 in $\mathrm{N}(S_1)/\mathrm{N}(S) \oplus 0$ and A_1 is the operator induced by $\hat{A}_1$ in the same space, then $\mathrm{R}(\hat{T}_0) = \mathrm{R}(S) \cap M$ and

$$\hat{T}_0\hat{A}_1 = (Q_0B \mid \mathrm{R}(S) \cap M)\hat{T}_0 = (B_0 \mid \mathrm{R}(S) \cap M)\hat{T}_0, \tag{3.6}$$

which is deduced from (3.5). Since $\hat{T}_0$ is invertible on $\mathrm{R}(S) \cap M$, from (3.6) we infer that

$$\mathrm{Tr}_{\mathrm{N}(S_1)/\mathrm{N}(S)\oplus 0}(A_1) = \mathrm{Tr}_{\mathrm{R}(S) \cap M}(B_0).$$

By Lemma 1.1, we have the equality

$$\mathrm{Tr}_{\mathrm{R}(S) \cap M}(B_0) = \mathrm{Tr}_{\mathrm{R}(S)/\mathrm{R}(S_0)}(B_0),$$

showing that (3.2) holds.

Finally, if $y \in Y$ is arbitrary, then $y = S_0x_0 + y_0$, with $y_0 \in M$, and we have

$$By - B_0y = BS_0x_0 - B_0S_0x_0 + By_0 - B_0y_0 =$$
$$= S(A - A_0)x_0 + (1_Y - Q_0)\, By_0 \in \mathrm{R}(S),$$

which completes the proof.

Now we can prove the main result of the paper. Let us mention that every direct sum of Banach spaces $X \oplus M$ (as used in the previous lemma) will be endowed with the norm $\|x \oplus y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \oplus y \in X \oplus M$.

3.4. THEOREM. *Let $S \in \mathscr{L}(X, Y)$ be a Fredholm operator and consider $A \in \mathscr{L}(X)$ and $B \in \mathscr{L}(Y)$ with $SA = BS$. Then for every $\varepsilon > 0$ one can find $\eta(\varepsilon) > 0$ such that if $\tilde{S} \in \mathscr{L}(X, Y)$, $\tilde{A} \in \mathscr{L}(X)$, $\tilde{B} \in \mathscr{L}(Y)$ satisfy $\tilde{S}\tilde{A} = \tilde{B}\tilde{S}$, $\|\tilde{S} - S\| < \eta(\varepsilon)$, $\|\tilde{A} - A\| < \eta(\varepsilon)$ and $\|\tilde{B} - B\| < \eta(\varepsilon)$ then $\mathrm{L}_{\tilde{S}}(\tilde{A}, \tilde{B})$ is well-defined and*

$$(3.7) \qquad |\mathrm{L}_{\tilde{S}}(\tilde{A}, \tilde{B}) - \mathrm{L}_S(A, B)| < \varepsilon.$$

Proof. Let us make the following notations: $N = \mathrm{N}(S)$, X_0 stands for a complement of N in X and M stands for a complement of $\mathrm{R}(S)$ in Y. Let S_1 and $\tilde{S}_1$ be the extensions of S and $\tilde{S}$ (respectively) to $X \oplus M$, in the sense of Lemma 3.3. Then S_1 is a surjective Fredholm operator. When $\|\tilde{S}_1 - S_1\| (= \|\tilde{S} - S\|)$ in sufficiently small, then $\tilde{S}_1$ is surjective as well, and $\mathrm{N}(\tilde{S}_1)$, $\mathrm{N}(S_1)$ have the same complement in $X \oplus M$, namely $X_0 \oplus M$, by Lemma 3.1. In particular, $S_0 = S \mid X_0$ and $\tilde{S}_0 = \tilde{S} \mid X_0$ are injective. At the same time, for $\|\tilde{S}_0 - S_0\| (\leqslant \|\tilde{S} - S\|)$ sufficiently small, the spaces $\mathrm{R}(S_0)(= \mathrm{R}(S))$ and $\mathrm{R}(\tilde{S}_0)$ have the same complement in Y, namely M, by Corollary 3.2. Under these assumptions, we can realize the conditions of Lemma 3.3, for both operators S and $\tilde{S}$.

Let A_1 and $\tilde{A}_1$ be the extensions of A and $\tilde{A}$ to $X \oplus M$, in the sense of Lemma 3.3. Also consider the operators B_0 and $\tilde{B}_0$, obtained from the same lemma, corresponding to B and $\tilde{B}$, respectively. By Lemma 3.1, if $\varepsilon > 0$ is given, we can find $\eta_1(\varepsilon) > 0$ such that if $\|\tilde{S}_1 - S_1\| < \eta_1(\varepsilon)$ and $\|\tilde{A}_1 - A_1\| < \eta_1(\varepsilon)$, then

$$(3.8) \qquad |\mathrm{Tr}_{\mathrm{N}(\tilde{S}_1)}(\tilde{A}_1) - \mathrm{Tr}_{\mathrm{N}(S_1)}(A_1)| < \frac{\varepsilon}{2}.$$

Since $\mathrm{N}(S_1) = \mathrm{N}(S) \oplus 0$, we have $\mathrm{Tr}_{\mathrm{N}(S_1)}(A_1) = \mathrm{Tr}_{\mathrm{N}(S)}(A)$. By Lemma 1.2, we also have

$$\mathrm{Tr}_{\mathrm{N}(\tilde{S}_1)}(\tilde{A}_1) = \mathrm{Tr}_{\mathrm{N}(\tilde{S})}(\tilde{A}) + \mathrm{Tr}_{\mathrm{N}(\tilde{S}_1)/\mathrm{N}(\tilde{S})\oplus 0}(\tilde{A}_1).$$

From Corollary 3.2, we obtain a number $\eta_2(\varepsilon) > 0$ such that if $\|\tilde{S}_0 - S_0\| < \eta_2(\varepsilon)$ and $\|\tilde{B}_0 - B_0\| < \eta_2(\varepsilon)$ then

$$(3.9) \qquad |\mathrm{Tr}_{Y/\mathrm{R}(\tilde{S}_0)}(\tilde{B}_0) - \mathrm{Tr}_{Y/\mathrm{R}(S_0)}(B_0)| < \frac{\varepsilon}{2}.$$

Since $\mathrm{R}(S_0) = \mathrm{R}(S)$, we have $\mathrm{Tr}_{Y/\mathrm{R}(S_0)}(B_0) = \mathrm{Tr}_{Y/\mathrm{R}(S)}(B)$. According to Lemma 1.2, we can write

$$\mathrm{Tr}_{Y/\mathrm{R}(\tilde{S}_0)}(\tilde{B}_0) = \mathrm{Tr}_{Y/\mathrm{R}(\tilde{S})}(\tilde{B}) + \mathrm{Tr}_{\mathrm{R}(\tilde{S})/\mathrm{R}(\tilde{S}_0)}(\tilde{B}_0),$$

since $\tilde{B}_0$ and $\tilde{B}$ induce the same action in $Y/\mathrm{R}(\tilde{S})$, by Lemma 3.3. From (3.2), (3.8), (3.9) and the previous remarks we obtain

$$|\mathrm{L}_{\tilde{S}}(\tilde{A}, \tilde{B}) - \mathrm{L}_S(A, B)| \leqslant$$

$$\leqslant |\mathrm{Tr}_{\mathrm{N}(\tilde{S}_1)}(\tilde{A}_1) - \mathrm{Tr}_{\mathrm{N}(S_1)}(A_1)| + |\mathrm{Tr}_{Y/\mathrm{R}(\tilde{S}_0)}(\tilde{B}_0) - \mathrm{Tr}_{Y/\mathrm{R}(S_0)}(B_0)| < \varepsilon,$$

if $\|\tilde{S} - S\| < \eta_3(\varepsilon)$, $\|\tilde{A}_1 - A_1\| < \eta_3(\varepsilon)$, $\|\tilde{B}_0 - B_0\| < \eta_3(\varepsilon)$, where

$$\eta_3(\varepsilon) = \min\{\eta_1(\varepsilon), \eta_2(\varepsilon)\}.$$

The only fact to be proved is that $\|\tilde{A}_1 - A_1\|$ and $\|\tilde{B}_0 - B_0\|$ are as small as we desire, when $\|\tilde{S} - S\|$, $\|\tilde{A} - A\|$ and $\|\tilde{B} - B\|$ are sufficiently small. Note that we can write

$$\|\tilde{A}_1 - A_1\| \leqslant \|\tilde{A} - A\| + \|\tilde{S}_2^{-1}\tilde{B} - S_2^{-1}B\| \leqslant$$

$$\leqslant \|\tilde{A} - A\| + \|\tilde{B}\|\,\|\tilde{S}_2^{-1} - S_2^{-1}\| + \|S_2^{-1}\|\,\|\tilde{B} - B\|,$$

where $S_2 = S_1 \mid X_0 \oplus M$ and $\tilde{S}_2 = \tilde{S}_1 \mid X_0 \oplus M$ (see Lemma 3.3). Since $\|\tilde{S}_2 - S_2\| \leqslant \|\tilde{S}_1 - S_1\| = \|\tilde{S} - S\|$, the norm $\|\tilde{S}_2^{-1} - S_2^{-1}\|$ can be made as small as we want when $\|\tilde{S} - S\|$ is small enough; hence the norm $\|\tilde{A}_1 - A_1\|$ tends to zero as the norms $\|\tilde{A} - A\|$, $\|\tilde{B} - B\|$ and $\|\tilde{S} - S\|$ tend to zero.

Now, let Q_0 be the projection of Y onto M along $\mathrm{R}(S_0)$, and let $\tilde{Q}_0$ be the projection of Y onto M along $\mathrm{R}(\tilde{S}_0)$. Then the norm $\|\tilde{Q}_0 - Q_0\|$ can be majorized in terms of $\|\tilde{S}_0 - S_0\|$, by (2.2) (see also the proof of Lemma 3.1 and Corollary 3.2). Then we have

$$\tilde{B}_0 - B_0 = (\tilde{B}_0(1_Y - \tilde{Q}_0) - B_0(1_Y - Q_0)) + (\tilde{B}_0\tilde{Q}_0 - B_0Q_0) =$$

$$= (\tilde{S}_0^{-1}\tilde{A}_0\tilde{S}_0(1_Y - \tilde{Q}_0) - S_0^{-1}A_0S_0(1_Y - Q_0)) + (\tilde{Q}_0\tilde{B}\tilde{Q}_0 - Q_0BQ_0),$$

where $A_0 = (1_X - P_0)A \mid X_0$ and $\tilde{A}_0 = (1_X - P_0)\tilde{A} \mid X_0$, with P_0 the projection of X onto N along X_0 (see Lemma 3.3). Since, as above, the norm $\|\tilde{S}_0^{-1} - S_0^{-1}\|$ is controlled by the norm $\|\tilde{S} - S\|$, and the norm $\|\tilde{A}_0 - A_0\|$ is controlled by the norm $\|\tilde{A} - A\|$, by the usual trick of Banach algebras when proving the continuity of the product, we obtain that the norm $\|\tilde{B}_0 - B_0\|$ tends to zero as the norms $\|\tilde{S} - S\|$, $\|\tilde{A} - A\|$ and $\|\tilde{B} - B\|$ tend to zero. This completes the proof of our theorem.

Let us rephrase the statement of Theorem 3.4, giving it a more geometric aspect. For every $S \in \mathscr{L}(X, Y)$, we define the set

$$\mathrm{I}(S) = \{(A, B) \in \mathscr{L}(X) \times \mathscr{L}(Y); \quad SA = BS\}.$$

The set $\mathrm{I}(S)$ is, in fact, a Banach subalgebra with unit of the Banach algebra $\mathscr{L}(X) \times \mathscr{L}(Y)$ (endowed with the multiplication $(A_1, B_1)\cdot(A_2, B_2) = (A_1A_2, B_1B_2)$ and with the norm $\|(A, B)\| = \max\{\|A\|, \|B\|\}$). We denote by $\Phi(X, Y)$ the set of all Fredholm operators in $\mathscr{L}(X, Y)$. Let us define the space

$$\mathscr{F}(X, Y) = \{(S, A, B); \ S \in \Phi(X, Y), \ (A, B) \in \mathrm{I}(S)\}.$$

We give $\mathscr{F}(X, Y)$ the topology induced by the (norm) topology of $\mathscr{L}(X, Y) \times \times \mathscr{L}(X) \times \mathscr{L}(Y)$. Then the triple

$$(\mathscr{F}(X, Y), \Phi(X, Y), \pi), \tag{3.10}$$

where $\pi: \mathscr{F}(X, Y) \to \Phi(X, Y)$ is the canonical projection, forms a *fiber space* ("espace découpé", in the terminology of Godement), whose fibers are Banach algebras with unit. Theorem 3.4 asserts that the function

$$\mathscr{F}(X, Y) \ni (S, A, B) \to \mathrm{L}_S(A, B) \in \mathbf{C}$$

is continuous.

We end this section with the following question: Is the triple (3.10) a vector bundle ?

4.

In this section we give some applications of the previous results and make some final comments.

4.1. PROPOSITION. *Let $S_0 \in \mathscr{L}(X, Y)$ be a semi-Fredholm operator with* $\dim \mathrm{N}(S_0) < \infty$, *and let D_1 be an open disc in the complex plane, with center at zero. Also consider $S_1 \in \mathscr{L}(X, Y)$ and let us define $S(\lambda) = S_0 + \lambda S_1$ for $\lambda \in D_1$. If $\lambda \to A(\lambda)$ is a continuous function from D_1 into $\mathscr{L}(X)$, satisfying $A(\lambda)\mathrm{N}(S(\lambda)) \subset \mathrm{N}(S(\lambda))$ for all $\lambda \in D_1$, then there exists an open disc $D_0 \subset D_1$ with center at zero, such that the function*

$$D_0 \setminus \{0\} \ni \lambda \to \mathrm{Tr}_{\mathrm{N}(S(\lambda))}(A(\lambda)) \in \mathbf{C}$$

is continuous.

Proof. It follows from [2, Theorem IV.5.31] that there exist two closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $S(\lambda)X_0 \subset Y_0$ for all $\lambda \in D_1$, $\mathrm{R}(S_0 \mid X_0) = Y_0$

and $N(S(\lambda) \mid X_0) = N(S(\lambda))$ for all $\lambda \in D_1 \setminus \{0\}$. Then $R(S(\lambda) \mid X_0) = Y_0$ for λ in an open disc $D_0 (\subset D_1)$ with center at zero, and we can apply Lemma 3.1, which provides the conclusion. We only note that the proof of Lemma 3.1 still works, although the operator $A(\lambda) \mid X_0$ does not necessarily have values in X_0.

We assume that $X = Y$, and let us define $\Phi(X) = \Phi(X, X)$. Let $C(S)$ be the commutant of the operator $S \in \mathscr{L}(X)$. Consider the space

$$\mathscr{F}(X) = \{(S,\ A);\ S \in \Phi(X),\ A \in C(S)\}.$$

For every $(S, A) \in \mathscr{F}(X)$, we set $L_S(A) = L_S(A, A)$. The number $L_S(A)$ will be called the *Lefschetz number* of A with respect to S. By Theorem 3.4, the function $L_S(A)$ is continuous on $\mathscr{F}(X)$.

4.2. PROPOSITION. *Let $(S,A) \in \mathscr{F}(X)$ be with the property $L_S(A) \neq 0$. Then there exists a number $\eta > 0$ such that if $(T, B) \in \mathscr{F}(X)$ satisfies $\|T - S\| < \eta$, $\|B - A\| < \eta$, then either B or B^* has an eigenvalue.*

Proof. We apply Theorem 3.4 with $\varepsilon = |L_S(A)| > 0$. Let $\eta = \eta(\varepsilon)$. Then we have

$$|L_T(B)| \geqslant \varepsilon - |L_T(B) - L_S(A)| > 0;$$

therefore, either $N(T) \neq 0$ or $R(T) \neq X$. Accordingly, we obtain that either B or B^* has an eigenvalue.

Proposition 4.2 applies even in the case when $\operatorname{ind} S = 0$.

4.3. EXAMPLE. There are pairs $(S, A) \in \mathscr{F}(X)$ such that $\operatorname{ind} S = 0$ but $L_S(A) \neq 0$. Indeed, if H is a separable Hilbert space and T is the unilateral shift on H, then the operator $S = T \oplus T^*$, defined on $X = H \oplus H$, has the property $\operatorname{ind} S = 0$. However, if $A = \alpha \cdot 1_H \oplus \beta \cdot 1_H$, with α, β complex numbers, $\alpha \neq \beta$, then one can easily see that $L_S(A) = \beta - \alpha \neq 0$.

4.4. REMARKS. 1) The proof of Theorem 3.4 is independent of the stability theorem under small perturbations for Fredholm operators.

2) The Lefschetz number can be defined for finite systems of operators which are homomorphisms of Fredholm complexes (see [1] and [4] for the background). It is plausible that Theorem 3.4 can be extended to such a context.

3) M. Putinar has noticed that the proof of Theorem 3.4 can be slightly modified in order to obtain the continuity of the function

$$\mathscr{F}(X,\ Y) \ni (S,\ A,\ B) \to \frac{\mathrm{Det}_{N(S)}(A)}{\mathrm{Det}_{Y/R(S)}(B)} \in \mathbf{C},$$

at those points $(S, A, B) \in \mathscr{F}(X, Y)$ in which $\mathrm{Det}_{Y/R(S)}(B) \neq 0$, where "Det" stands for the *determinant*.

REFERENCES

1. ATIYAH, M. F.; BOTT, R., A Lefschetz fixed point formula for elliptic complexes. I, *Ann. of Math.*, **86** (1967), 347–407; II. Applications, ibid., **88** (1968), 451–491.
2. KATO, T., *Perturbation theory for linear operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1966.
3. MASSERA, J. L.; SCHÄFFER, J. J., *Linear differential equations and functions spaces*, Academic Press, New York and London, 1966.
4. VASILESCU, F.-H., Stability of the index of a complex of Banach spaces, *J. Operator Theory*, **2**(1979), 247–275.

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