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GAUGE ACTIONS ON $\mathcal{O}_A$

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1. INTRODUCTION

Suppose A is a zero-one $n \times n$ matrix, satisfying certain standard mild conditions as in [12, 9, 10]. Then $\mathcal{O}_A$, the C^*-algebra generated by n non-zero partial isometries $S_1, \ldots, S_n$ satisfying

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j)\, S_j S_j^* \quad i = 1, 2, \ldots, n$$

was introduced in [12] and shown to be uniquely determined by these relations. Here we consider the natural gauge action of the circle group $\mathbf{T}$ on $\mathcal{O}_A$, where $t \in \mathbf{T}$ sends a generator S_i of $\mathcal{O}_A$ to tS_i. If $G = G(p)$ is the torsion subgroup of $\mathbf{T}$ determined by a generalized integer p, we consider $C^*(\mathcal{O}_A, G(p))$ the crossed product of $\mathcal{O}_A$ under this action of $G(p)$, and compute the K-groups of this crossed product. in particular taking $\mathcal{O}_n$, the C^*-algebra of the full n-shift, this enables us to show that for fixed n, p is a complete invariant for $C^*(\mathcal{O}_n, G(p))$.

In [18] we showed how to construct $\mathcal{O}_n$ from shift operators on Fock space $\bigoplus_{m=0}^{\infty} (\otimes^m \mathbf{C}^n)$. In §2 this construction is modified to get $\mathcal{O}_A$ from shift operators on a subspace F_A of Fock space. This enables one to give a canonical definition of $\mathcal{O}_A$ for any matrix A without restriction. Thus if A is a permutation matrix, $\mathcal{O}_A$ is canonically isomorphic to $B \otimes C(\mathbf{T})$ where B is a finite dimensional C^*-algebra. In the other extreme we consider the full n-shift, and give an elementary proof of simplicity of $\mathcal{O}_n$.

In §2 we consider the C^*-algebra produced by considering weighted shifts on F_A of period p, for some positive integer p. This algebra can be identified as $\mathcal{O}_{A(p)}$ for the matrix $A(p) = A \otimes J_p$ where J_p is an irreducible $p \times p$ permutation matrix. $\mathcal{O}_{A(p)}$ can also be identified as $C^*(\mathcal{O}_A, G(p))$, [11, 17]. This particularly concrete representation of $\mathcal{O}_{A(p)}$ naturally leads us to embedd $\mathcal{O}_{A(p_1)}$ in $\mathcal{O}_{A(p_2)}$ if $p_1 | p_2$ and define

$\mathcal{O}_{A(p)} = \left(\bigcup_{i=1}^{\infty} \mathcal{O}_{A(p_i)}\right)^-$ if p is a generalized integer determined by a sequence of positive integers (p_i) where $p_i | p_{i+1}$. Then $\mathcal{O}_{A(p)}$ can be identified with $C^*(\mathcal{O}_A, G(p))$. If $n = 1$, $A = 1$, these algebras are precisely the weighted shift algebras of Bunce and Deddens [4].

In §4 we compute the K-groups of $\mathcal{O}_{A(p)}$, and deduce that p is a complete invariant for $\mathcal{O}_{n(p)} = C^*(\mathcal{O}_n, G(p))$ for fixed n. In particular we recover the result of Bunce and Deddens [4] when $n = 1$. In §5 we study the stable algebra of $\mathcal{O}_{A(p)}$ and show that it can be expressed as the crossed product of an AF algebra by an automorphism, (cf. [12]). Moreover if an integer p_1 divides a generalized integer p, we show that $\mathcal{O}_{A(p)}$ and $\mathcal{O}_{A^{p_1}(p/p_1)}$ are stably isomorphic, but not isomorphic in general. In particular if p is finite, $\mathcal{O}_{A(p)}$ and $\mathcal{O}_{A^p}$ are stably isomorphic. In this case, this can be regarded as a consequence of the flow equivalence of $A(p)$ and A^p, and the fact that $\mathcal{O}_A$ is a stable isomorphism invariant of flow equivalence [12, 9].

We let $\mathcal{B}(\mathcal{H})$, respectively $\mathcal{K}(\mathcal{H})$ denote the bounded, respectively compact operators on a Hilbert space $\mathcal{H}$, and $Q(\mathcal{H})$ will denote the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. If $\mathcal{A}$ is a C^*-algebra, $\mathcal{M}(\mathcal{A})$ will denote its multiplier algebra.

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2. SHIFT ALGEBRAS

Let $\Sigma = \{1, 2, \ldots, n\}$, and H be a n-dimensional Hilbert space, with orthonormal basis $e_1, \ldots, e_n$. Define Fock space $F(H)$ to be $\bigoplus_{m=0}^{\infty} (\otimes^m H)$, where $\otimes^0 H$ denotes a one-dimensional Hilbert space spanned by a unit vector Ω, the vacuum. If u is a unitary on H, define $F(u)$ to be the unitary $\bigoplus_{m=0}^{\infty} (\otimes^m u)$ on $F(H)$ where $\otimes^0 u = 1$. If $h \in H$, let $O^F(h)$ [18] denote the operator on $F(H)$ such that $O^F(h)f = h \otimes f$, $f \in \otimes^m H$, $m \geqslant 1$, and $O^F(h)\Omega = h$. Then

$$F(u)O^F(h)F(u)^* = O^F(uh),$$

$$O^F(h)^* O^F(h') = \langle h', h\rangle 1,$$

$$\sum_{i=1}^{n} O^F(e_i)O^F(e_i)^* = 1, \quad \text{for } h, h' \in H,$$

u unitary on H.

Let $A = [A(i,j)]$ be a $n \times n$ matrix with $A(i,j) \in \{0, 1\}$, and let $\mathcal{M}_A^m$ for $m \geqslant 2$ denote the set of multi-indices $\mu = (\mu_1, \ldots, \mu_m) \in \Sigma^m$, for which $A(\mu_i, \mu_{i+1}) = 1$, $i = 1, \ldots, m-1$, and put $\mathcal{M}_A^1 = \Sigma$, $\mathcal{M}_A = \bigcup_{m=1}^{\infty} \mathcal{M}_A^m$. If $\mu = (\mu_1, \ldots, \mu_m) \in \Sigma^m$, let $e_\mu = e_{\mu_1} \otimes \ldots \otimes e_{\mu_m} \in \otimes^m H$. Let F_A^m for $m \geqslant 1$, denote the subspace of $\otimes^m H$ spanned by $\{e_\mu : \mu \in \mathcal{M}_A^m\}$, and $F_A = \bigoplus_{m=0}^{\infty} F_A^m$ where $F_A^0 = \mathbf{C}\Omega$, and let p be the projection of $F(H)$ on F_A. For $i \in \Sigma$, define $\bar{S}_i = pO^F(e_i)p \in \mathscr{B}(F_A)$. Then $\bar{S}_i$ is a partial isometry such that $\bar{S}_i\Omega = e_i$

$$\bar{S}_i e_{\mu_1} \otimes \ldots \otimes e_{\mu_m} = \begin{cases} e_i \otimes e_{\mu_1} \otimes \ldots \otimes e_{\mu_m} & \text{if } A(i, \mu_1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $(\mu_1, \ldots, \mu_m) \in \mathcal{M}_A^m$, $m \geqslant 1$. Then

$$\bar{S}_i^* \bar{S}_i = \sum_j A(i, j) \bar{S}_j \bar{S}_j^* + \Omega \otimes \bar{\Omega}, \quad i \in \Sigma$$

where $\Omega \otimes \bar{\Omega}$ denotes the projection on the vacuum. If $\mu = (\mu_1, \ldots, \mu_m) \in \Sigma^m$, let $\bar{S}_\mu = \bar{S}_{\mu_1} \ldots \bar{S}_{\mu_m}$, so that $\mu \in \mathcal{M}_A^m$ if and only if $\bar{S}_\mu \neq 0$, in which case $e_\mu = \bar{S}_\mu \Omega$. If μ, $\nu \in \mathcal{M}_A$, then

$$\bar{S}_\mu [\bar{S}_i^* \bar{S}_i - \sum_j A(i,j) \bar{S}_j \; \bar{S}_j^*] \bar{S}_\nu^*$$

is the rank one operator, $e_\mu \otimes \bar{e}_\nu$. Thus $\mathcal{O}_A^F$, the C^*-algebra generated by the operators $\{\bar{S}_i : i \in \Sigma\}$ contains the compact operators $\mathscr{K}(F_A)$ on F_A. We can thus form the quotient $\mathcal{O}_A^F/\mathscr{K}(F_A)$ which we call $\mathcal{O}_A$ (c.f. [12]), and denote the image of $\bar{S}_\mu$, $\mu \in \Sigma^m$ in this quotient algebra by S_μ. Then $\mathcal{O}_A$ is generated by partial isometries S_i which satisfy

$$S_i^* S_i = \sum_j A(i, j) S_j S_j^*, \quad i \in \Sigma \tag{2.1}$$

(cf. [12]). This modification of [17] to get operators from Fock space satisfying (2.1) has been observed by a number of people including K. Schmidt and [15]. Note that S_j is non-zero if and only if

$$\{(\mu_k)_{k=1}^{\infty} : \mu_k \in \Sigma, \; A(j, \mu_1) = 1 = A(\mu_k, \mu_{k+1}), \; k = 1, 2, \ldots\} \neq \emptyset.$$

In particular, $S_j \neq 0$ for every j if and only if every column of A is non-zero.

If $t \in \mathbf{T}$, $F(t)$ leaves F_A invariant and if we let $G(t) = F(t)|_{F_A}$, then $G(t)\bar{S}_i G(t)^* = t\bar{S}_i$, $t \in \mathbf{T}$, $i \in \Sigma$. Let $\{\alpha(t) : t \in \mathbf{T}\}$ denote the induced strongly continuous automorphism group of $\mathcal{O}_A$ such that $\alpha(t)S_i = tS_i$, $t \in \mathbf{T}$, $i \in \Sigma$. In particular, if $\mu \in \Sigma^m$ and $S_\mu \neq 0$, then $\sigma(S_\mu) \supset \mathbf{T}$.

If $n = 1$, $A = 1$, then S_1 is the unilateral shift on $F(\mathbf{C}) \cong \ell_2$, $\mathcal{O}_1^F = C^*(S_1)$, and $\mathcal{O}_1 = C^*(S_1) \cong C(\mathbf{T})$, as S_1 is unitary with $\sigma(S_1) = \mathbf{T}$, (cf. [5]). Thus $\mathcal{O}_1 \cong C(\mathbf{T})$ as in [8].

Let $\mathcal{F}_A$ denote the fixed point algebra of α. Then $\mathcal{F}_A$ is an AF algebra and can be described as follows. Let P_i denote the range projection $S_iS_i^*$ of S_i, $i \in \Sigma$. Then $\mathcal{F}_A$ is the inductive limit of

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

where

$$\mathcal{F}_0 = \mathbf{C}P_1 \oplus \ldots \oplus \mathbf{C}P_n, \quad \mathcal{F}_k = \mathcal{F}_k^1 \oplus \ldots \oplus \mathcal{F}_k^n,$$

where

$$\mathcal{F}_k^i = C^*(S_\mu P_i S_\nu^* : \mu, \nu \in \mathcal{M}_A^k)$$

are simple matrix algebras and the multiplicities of the embedding of $\mathcal{F}_k$ in $\mathcal{F}_{k+1}$ is given by A^t; see [12, 9] for details.

A property of A called (I) was isolated in [12, 9] and shown to be equivalent to the uniqueness of the C^*-algebra generated by n non-zero partial isometries satisfying (2.1). Moreover, if A satisfies a stronger condition called (II), then the ideal structure of $\mathcal{O}_A$ was determined [9, Theorem 2.5]. In particular if A is irreducible, and not a permutation matrix, then $\mathcal{O}_A$ is simple. Thus if the $n \times n$ matrix C has entries $C(i,j) \equiv 1$, where $n \geqslant 2$, then the C^*-algebra $\mathcal{O}_C$, which is also called $\mathcal{O}_n$, is simple (see also [6]). Here we give an elementary proof of the simplicity of $\mathcal{O}_n$, inspired by [13] :

THEOREM 2.1. *$\mathcal{O}_n$ is simple for $n \geqslant 2$.*

Proof. Let $P = \int \alpha(t)\mathrm{d}t$ denote the faithful conditional expectation of $\mathcal{O}_n$ onto $\mathcal{F}_n = \mathcal{F}_C$, which is a UHF algebra with faithful trace τ. Let φ denote the averaging map or averaging process given by $\varphi(x) = \frac{1}{n}\sum_{i=1}^{n} S_i^* x S_i$, $x \in \mathcal{O}_n$. Then as observed in [1] it is easy to show that

$$\tau \circ P(x) = \lim_{m\to\infty} \varphi^m(x), \quad x \in \mathcal{O}_n$$

if $n \geqslant 2$. Thus if J is a non-zero ideal in $\mathcal{O}_n$, take $0 \neq x \in J_+$. Then $\tau \circ P(x) \in J$ as φ leaves ideals invariant. Hence J contains a non-zero scalar as $\tau \circ P$ is a faithful state on $\mathcal{O}_n$. Hence $J = \mathcal{O}_n$.

REMARK. The above argument is similar to Powers' argument [23] on simplicity of the reduced C^*-algebra of the free group on n generators. In fact $\mathcal{O}_n^F$ can be regarded as the reduced C^*-algebra of the free semigroup on n-generators [16]. The same argument can be adapted to show that if $\mathscr{A}$ is the C^*-algebra generated by any n-non-zero isometries S_i satisfying $\sum_{i=1}^{n} S_i S_i^* = 1$, then $\mathscr{A}$ is simple. Following [13], take a unitary S with spectrum equal to the whole circle, and let γ be the strongly continuous automorphism group of $C^*(S)$ such that $\alpha(t)$ denote the restriction of $1 \otimes \gamma(t)$ on $\mathscr{A} \otimes C^*(S)$ to $C^*(S_i \otimes S : i \in \Sigma)$ so that $\alpha(t)(S_i \otimes S) = t(S_i \otimes S)$. Then the argument in the above theorem shows that $C^*(S_i \otimes S)$ is simple. Hence by taking a character χ of $C^*(S)$ and mapping $C^*(S_i \otimes S)$ onto $C^*(S_i)$ by $1 \otimes \chi$ it follows that $C^*(S_i)$ is simple. Uniqueness follows as in [13].

THEOREM 2.2. *If A is an irreducible $n \times n$ permutation matrix, then $\mathcal{O}_A$ is canonically isomorphic to $M_n \otimes C(\mathbf{T})$.*

Proof. We take $A(i, i-1) = 1 = A(1, n)$, $i = 2, \ldots, n$ and $A(i,j) = 0$ otherwise. Then $\{\bar{S}_i : i \in \Sigma\}$ are operators on F_A such that

$$\bar{S}_i^* \bar{S}_i = \bar{S}_{i-1} \bar{S}_{i-1}^* + \bar{\Omega} \otimes \bar{\Omega}, \quad i \in \Sigma,$$

and where the indices are taken mod n. Let $H_n = \bar{S}_n \bar{S}_n^* F_A$, $i \in \Sigma$, and consider $W : F_A \to H_n \oplus \ldots \oplus H_n$, ($n$-copies) given by

$$W\xi = \bigoplus_{i=1}^{n} S_n S_{n-1} \ldots S_i S_i^* \xi, \quad \xi \in F_A,$$

so that

$$W^* \left(\bigoplus_{i=1}^{n} \xi_i \right) = \sum_{i=1}^{n} S_i S_i^* \ldots S_n^* \xi_i, \quad \xi_i \in H_n.$$

Then $W^* W = 1$, WW^* is a finite rank projection and so W is a Fredholm partial isometry which induces an isomorphism α of $Q(F_A)$ onto $M_n \otimes Q(H_n)$. If e_{ij} denote the canonical matrix units in M_n, it is easily checked that

$$\alpha(S_i) = e_{i-1,i} \otimes 1 \quad i \neq 1$$

$$\alpha(S_1) = e_{n,1} \otimes S_n \ldots S_1.$$

Thus α takes $\mathcal{O}_A$ onto $M_n \otimes C^*(S_n \ldots S_1) \cong M_n \otimes C(\mathbf{T})$, as $\sigma(S_n \ldots S_1) = \mathbf{T}$.

It follows that if A is a $n \times n$ permutation matrix, then $\mathcal{O}_A$ is isomorphic to $B \otimes C(\mathbf{T})$, where B is a finite dimensional C^*-algebra.

If $A \in M_n(\mathbf{N})$, an $n \times n$ matrix with entries in $\mathbf{N}$, let A' [12] be the associated, strong shift equivalent, zero-one matrix defined as follows. The state space Σ' of A' is

$$\{(i,j,k) : i, k \in \Sigma,\ 1 \leqslant j \leqslant A(i,k)\},$$

and $A'((i,j,k), (i',j',k')) = 1$ if $k = i'$ and 0 otherwise.

If A is a zero-one matrix, and $p \in \mathbf{N}$, we let

$$F_i = F_A^i + F_A^{i+p} + F_A^{i+2p} + \ldots, \quad \text{for } 0 \leqslant i < p.$$

Then if $\mu \in \mathcal{M}_A^p$, S_μ leaves F_i invariant, and let T_μ denote the restriction of S_μ to F_1. Let $\pi : \mathcal{B}(F_1) \to Q(F_1)$ denote the quotient map, and set $T_\mu = \pi(T_\mu)$, $\mu \in \mathcal{M}_A^p$.

THEOREM 2.3. *If $A \in M_n(\mathbf{N})$ and $p \geqslant 1$, there is a natural isomorphism between* $\mathcal{O}_{(A^p)'}$ and $C^*(T_\mu : \mu \in \mathcal{M}_A^p)$.

Proof. The state space Σ' of $(A^p)' = B$ can be identified with $\mathcal{M}_A^{p+1}$, and for $\mu = (\mu_1, \ldots, \mu_{p+1})$, $\nu = (\nu_1, \ldots, \nu_{p+1}) \in \mathcal{M}_A^{p+1}$, $B(\mu, \nu) = 1$ if $\mu_{p+1} = \nu_1$, and 0 otherwise. There is a unitary map U of $\bigoplus_{m=1}^{\infty} F_B^m$ onto $\bigoplus_{m=1}^{\infty} F_A^{mp+1}$ given by

$$e_{\mu^1} \otimes \ldots \otimes e_{\mu^m} \to e_{\mu_1^1} \otimes \ldots \otimes e_{\mu_p^1} \otimes e_{\mu_1^2} \otimes \ldots \otimes e_{\mu_p^2} \otimes \ldots \otimes e_{\mu_p^{m-1}} \otimes$$

$$\otimes\, e_{\mu_1^m} \otimes \ldots \otimes e_{\mu_{p+1}^m}$$

for $\mu^i = (\mu_1^i, \ldots, \mu_{p+1}^i) \in \mathcal{M}_A^{p+1}$, $i = 1, \ldots, m$, and $(\mu^1, \ldots, \mu^m) \in \mathcal{M}_B^m$. To avoid confusion, we write $\bar{R}_\mu$ etc. ($\mu \in \mathcal{M}_A^{p+1} = \Sigma'$) for the shift operators on F_B. Then

$$U\bar{R}_\mu f = \bar{S}_{\mu_1} \ldots \bar{S}_{\mu_{p+1}} \bar{S}_{\mu_{p+1}}^* Uf,$$

for $f \in \bigoplus_{m=1}^{\infty} F_B^m$, $\mu = (\mu_1, \ldots, \mu_{p+1}) \in \mathcal{M}_A^{p+1}$. Thus going to the Calkin algebras, U induces an isomorphism of $\mathcal{O}_B = C^*(R_\mu : \mu \in \Sigma')$ onto $C^*(\pi(\bar{T}_\mu \bar{T}_i \bar{T}_i^*) : \mu \in \mathcal{M}_A^p, i \in \Sigma) = C^*(T_\mu : \mu \in \mathcal{M}_A^p)$, by elementary computations, and where we have denoted the restriction of $\bar{S}_i \bar{S}_i^*$ to F_1 by $\bar{T}_i \bar{T}_i^*$.

In particular taking $p = 1$, it is seen that if A is a zero-one matrix then $\mathcal{O}_A \simeq \mathcal{O}_{A'}$. Thus if $A \in \mathcal{M}_n(\mathbf{N})$, we can define unambiguously $\mathcal{O}_A$ to be $\mathcal{O}_{A'}$.

3. WEIGHTED SHIFT ALGEBRAS

We will now modify the construction of $\mathcal{O}_A$ from Fock space in order to build up inductive limits of such algebras.

Let A be a zero-one $n \times n$ matrix, and $p \geqslant 1$. Define subspaces $F_0, \ldots, F_{p-1}$ of F_A as in §2 by $F_j = F_A^j \oplus F_A^{j+p} \oplus F_A^{j+2p} \oplus \ldots$ so that $F_A = F_0 \oplus \ldots \oplus F_{p-1}$, and let p_j denote the projection on F_j. Let $\Sigma(p)=\Sigma\times\{1, 2, \ldots, p\}$ and we will identify $\{1, 2, \ldots, p\}$ with the group $\mathbf{Z}_p=\mathbf{Z}/p\mathbf{Z}$, in the obvious way. For $\alpha=(i,j)\in\Sigma(p)$, we let $\bar{R}_\alpha$ denote the partial isometry on F_A with initial projection in F_{j-1} and range projection in F_j given by $\bar{R}_\alpha=\bar{S}_i p_{j-1}$. Let $\mathcal{O}^F(A,p)$ denote the C^*-algebra generated by $\{\bar{R}_\alpha : \alpha\in\Sigma(p)\}$, so that $\mathcal{O}^F(A,p)\supseteq\mathcal{O}^F_A\supseteq\mathcal{K}(F_A)$, since $\bar{S}_i=\sum_{j=1}^{p}\bar{R}_{(i,j)}$. We let R_α denote the image of $\bar{R}_\alpha$ in $\mathcal{O}^F(A,P)/\mathcal{K}(F_A)$. If $\Sigma(p)$ is ordered by $(i,j)\leqslant(i',j')$ if $j<j'$ or if $j=j'$ and $i\leqslant i'$, let $B=A(p)$ denote the $np\times np$ matrix on the state space $\Sigma(p)$ given by

$$B=A(p)=\begin{pmatrix} 0 & & & A \\ A & 0 & & \\ \cdot & \cdot & & \cdot \\ & \cdot & \cdot & \\ 0 & & A & 0 \end{pmatrix} \tag{3.1}$$

Then $\bar{R}^*_\alpha\bar{R}_\alpha=\sum_{\beta\in\Sigma(p)}B(\alpha,\beta)\,\bar{R}_\beta\bar{R}^*_\beta$ if $\alpha\notin\Sigma\times\{p\}$ and $\bar{R}^*_\alpha\bar{R}_\alpha=\sum_{\beta\in\Sigma(p)}B(\alpha,\beta)\bar{R}_\beta\bar{R}^*_\beta+ +\Omega\otimes\bar{\Omega}$, if $\alpha\in\Sigma\times\{p\}$. Thus $R^*_\alpha R_\alpha=\sum B(\alpha,\beta)R_\beta R^*_\beta$, $\alpha\in\Sigma(p)$.

$\mathcal{O}^F(A,p)$ can be regarded as a weighted shift C^*-algebra. If $\gamma\in\ell_\infty$, $i\in\Sigma$, let $\bar{R}_i(\gamma)$ denote the bounded operator of F_A given by

$$\bar{R}_i(\gamma)e_{i_1}\otimes\ldots\otimes e_{i_m}=\begin{cases}\gamma_m e_i\otimes e_{i_m}\otimes\ldots\otimes e_{i_m} & \text{if } A(i,i_1)=1\\ 0 & \text{otherwise}\end{cases} \tag{3.2}$$

for $(i_1,\ldots,i_m)\in\mathcal{M}^m_A$. If γ is of period p, i.e. $\gamma_m=\gamma_{m+p}$ for all m, then $R_i(\gamma)$ can be represented as

$$\begin{pmatrix} 0 \;\ldots\ldots & 0 & \gamma_p S_{(i,p)} \\ \gamma_1 S_{(i,1)} & & 0 \\ & & \vdots \\ 0 \;\ldots\ldots & \gamma_{p-1}S_{(i,p-1)} & 0 \end{pmatrix}$$

on $F_0\oplus\ldots\oplus F_{p-1}$. Then $\mathcal{O}^F(A,p)$ is the C^*-algebra generated by $\{R_i(\gamma) : i\in\Sigma\}$ as γ ranges over all sequences of period p, (cf. [3, 4] for the case $n=1$). Notice also that if one takes the matrix of the full shift $A(i,j)\equiv 1$, and unbounded weights in (3.2) and project onto Boson and Fermion Fock space, one gets the Bose and Fermi creation operators, [18].

Let $\bar{T}_{(i,j)}=\bigoplus_{r=0}^{p-1}\bar{R}_{(i,j+r)}\in\mathcal{B}(F_A\oplus\ldots\oplus F_A)$, for $(i,j)\in\Sigma(p)$, and let $\pi:\mathcal{B}(\oplus F_A)\to Q(\oplus F_A)$ denote the quotient map, and $T_{(i,j)}=\pi(\bar{T}_{(i,j)})$. Then there is a natural isomorphism between $\mathcal{O}_{A(p)}$ and $C^*(T_\alpha : \alpha \in \Sigma(p))$. Let $\Omega' = \Omega \oplus \ldots \oplus \Omega \in F_A \oplus \ldots \oplus F_A$, ($p$-copies), and let Ω'' denote the vacuum of $F_{A(p)}$, with shift operators $\{\bar{N}_\alpha : \alpha \in \Sigma(p)\}$ on $F_{A(p)}$. Then $V : \bigoplus_{k=1}^{\infty} F^k_{A(p)} \to$ $\to \oplus \left(\bigoplus_{k=1}^{\infty} F^k_A \right)$ (p-copies) given by $\bar{N}_\mu \Omega'' \to \bar{T}_\mu \Omega'$, $\mu \in \mathcal{M}^k_{A(p)}$, $k \geqslant 1$ gives a unitary which induces an isomorphism of $Q(F_{A(p)})$ onto $Q(\oplus F_A)$ taking N_α onto $\bar{T}_\alpha$ $\alpha \in \Sigma(p)$. The coordinate map $\bar{T}_{(i,j)} \to \bar{R}_{(i,j)}$ then induces a homomorphism of $\mathcal{C}_{A(p)}$ onto $C^*(R_\alpha : \alpha \in \Sigma(p))$, (which is an isomorphism if A satisfies (I), since then $A(p)$ also satisfies (I)).

By §2, there is an action $\bar{\alpha}$ (respectively α) of $\mathbf{Z}_p = \{1, 2, \ldots, p\}$ on $\mathcal{O}^F_A$, (respectively $\mathcal{O}_A$) given by $\bar{\alpha}(q) : \bar{S}_i \to \gamma^q \bar{S}_i$ (respectively $\alpha(q) : S_i \to \gamma^q S_i$) where $\gamma = \exp(2\pi i/p)$, $q \in \{1, 2, \ldots, p\}$. Let $\bar{\theta} : \mathcal{O}^F_A \to C^*(\mathcal{O}^F_A, \mathbf{Z}_p) \subseteq M_p(\mathcal{B}(F_A))$ denote the canonical embedding in the crossed product, so that $\bar{\theta}(\bar{S}_i) = \bar{S}_i \oplus \gamma \bar{S}_i \oplus \ldots \oplus \gamma^{p-1} \bar{S}_i$, and let U be the unitary in $C^*(\mathcal{O}^F_A, \mathbf{Z}_p)$ implementing $\bar{\alpha}$, with spectral projections E_j, and spectral decomposition $U = \sum_{j=0}^{p-1} \gamma^j E_j$ so that $\theta(\bar{S}_i) E_j = E_{j+1} \theta(\bar{S}_i)$. Then let $\bar{L}_{(i,j)} = \bar{\theta}(S_i) E_j \in \mathcal{B}(H)$, if $H = F_A \oplus \ldots \oplus F_A$ (p-copies), and $C^*(\mathcal{O}^F_A, \mathbf{Z}_p) \simeq$ $\simeq C^*(L_\alpha : \alpha \in \Sigma(p))$. Similarly, one can define $L_{(i,j)} = \theta(S_i) E_j \in Q(H)$ and $C^*(\mathcal{O}_A, \mathbf{Z}_p) = C^*(L_\alpha : \alpha \in \Sigma(p))$. Then put $\bar{M}_{(i,j)} = \bigoplus_{r=0}^{p-1} \bar{L}_{(i,j+r)} \in \mathcal{B}(H \oplus \ldots \oplus H)$ (p-copies), and if $\pi : \mathcal{B}(\oplus H) \to Q(\oplus H)$ is the quotient map, put $M_{(i,j)} = \pi(\bar{M}_{(i,j)})$. Then there is a canonical isomorphism between $\mathcal{C}_{A(p)}$ and $C^*(M_\alpha : \alpha \in \Sigma(p))$. In fact let $\Omega' = (\Omega \oplus \ldots \oplus \Omega)/\sqrt{p} \in H$, and $\Omega'' = \Omega' \oplus \ldots \oplus \Omega' \in H \oplus \ldots \oplus H$ (p-copies). Define a Fredholm partial isometry V from $H \oplus \ldots \oplus H$ into $F_{A(p)}$ by $V : \bar{M}_{\alpha_1} \ldots \bar{M}_{\alpha_m} \Omega'' \to \bar{S}_{\alpha_1} \ldots \bar{S}_{\alpha_m} \Omega$, where Ω here is the vacuum in $F_{A(p)}$, $\bar{S}_\alpha$ are the shifts on $F_{A(p)}$ and $\alpha_i \in \Sigma(p)$, $m \geqslant 1$. (Details are left to the reader). Thus V induces an isomorphism of $Q(H \oplus \ldots \oplus H)$ onto $Q(F_{A(p)})$ taking M_α onto S_α, $\alpha \in \Sigma(p)$. Taking the coordinate map $M_\alpha \to L_\alpha$ induces a homomorphism of $\mathcal{O}_{A(p)}$ onto $C^*(\mathcal{O}_A, \mathbf{Z}_p)$, (which is an isomorphism if A satisfies (I), cf. [11]).

The following will shed some light on the relationship between the crossed product $C^*(\mathcal{O}_A, \mathbf{Z}_p)$ and the fixed point algebra $\mathcal{O}_A^{\mathbf{Z}_p}$.

LEMMA 3.1. *If A is a zero-one matrix, A^p is flow equivalent to $A(p)$.*

Proof. Let $C_i = [C^i_{r,s}]^p_{r,s=1}$, $D_i = [D^i_{r,s}]^p_{r,s=1}$ denote the $p \times p$ matrices $i = 1, \ldots, p-1$ where

$$C^i_{r,s} = 0 \quad \text{if } r \neq s, \qquad C^i_{r,r} = 1 \quad \text{if } r \neq i+1, \qquad C^i_{i+1,i+1} = A^i$$

and

$$D^i_{r,r-1} = 1 \text{ if } 1 < r \leqslant i+1, \quad D^i_{r,r-1} = A \text{ if } r > i+1,$$

$$D^i_{1,p} = A, \quad \text{and } D_{r,s} = 0 \text{ otherwise.}$$

Then $A(p) = C_1 D_1$, $D_1 C_1 = C_2 D_2$, ... and $D_{p-1} C_{p-1} = E = [E_{r,s}]^p_{r,s=1}$, where $E_{r,r-1} = 1$ if $r = 2, \ldots, p$, $E_{1,p} = A^p$, and $E_{r,s} = 0$ otherwise. Thus E is strong shift equivalent to $A(p)$. It is a routine matter using operation (ii) of [21, Theorem] to show that E is flow equivalent to A^p.

Now suppose A and its transpose satisfy (I), then so do $A(p)$, $(A^p)'$ and their transposes. In which case by [9, Theorem 2.4] and the above Lemma, $\mathcal{O}_{A(p)}$ and $\mathcal{O}_{A^p}$ are stably isomorphic. This will also be shown directly in §5. Note that by [11] the fixed point algebra $\mathcal{O}_A^{\mathbf{Z}_p}$ is isomorphic to $C^*(S_\mu : \mu \in \mathscr{M}_A^p)$ which by Theorem 2.3 is isomorphic to $\mathcal{O}_{A^p}$. Thus in this case, $\mathcal{O}_{A(p)} \simeq C^*(\mathcal{O}_A, \mathbf{Z}_p)$ and $\mathcal{O}_A^{\mathbf{Z}_p} \simeq \mathcal{O}_{A^p}$ are stably isomorphic.

From now on we assume that A satisfies (I). Let p_1, p_2 be positive integers with $p_1 | p_2$, and let $q = p_2/p_1$. Then we have a natural embedding $\bar{e}$ of $\mathcal{O}^F_{A(p_1)}$ in $\mathcal{O}^F_{A(p_2)}$, since every sequence of period p_1 is also of period p_2. If $\alpha = (i,j) \in \Sigma(p_1)$, we let $A(\alpha) = \{(i, rp_1 + j),\ r = 0, 1, \ldots, q-1\} \subseteq \Sigma(p_2)$. Then under this embedding $\bar{R}_\alpha$ in $\mathcal{O}^F_{A(p_1)}$ is mapped onto $\sum_{\beta \in A(\alpha)} \bar{R}_\beta$ in $\mathcal{O}^F_{A(p_2)}$. Then in the quotient algebras, $\mathcal{O}^F_{A(p_1)}/\mathscr{K}(F_A)$ is mapped into $\mathcal{O}^F_{A(p_2)}/\mathscr{K}(F_A)$ with R_α mapped onto $\sum_{\beta \in A(\alpha)} R_\beta$.

If we embed $\mathbf{Z}_{p_1}$ in $\mathbf{Z}_{p_2}$ by identifying $x \bmod p_1$ with $xq \bmod p_2$, $x \in \mathbf{Z}$, then identifying $C^*(\mathcal{O}_A, \mathbf{Z}_{p_i})$ with $\mathscr{F}(\mathbf{Z}_{p_i}, \mathcal{O}_A)$, the C^*-algebra of $\mathcal{O}_A$ valued functions on $\mathbf{Z}_{p_i}$, with twisted convolution, the function $f \in \mathscr{F}(\mathbf{Z}_{p_1}, \mathcal{O}_A)$ is mapped onto $\bar{f}$ in $\mathscr{F}(\mathbf{Z}_{p_2}, \mathcal{O}_A)$, given by

$$\bar{f}(g) = \begin{cases} f(g) & \text{if } g \in \mathbf{Z}_{p_1} \subseteq \mathbf{Z}_{p_2} \\ 0 & \text{if } g \in \mathbf{Z}_{p_2} \setminus \mathbf{Z}_{p_1}. \end{cases}$$

Then the embeddings $\mathcal{O}_{A(p_1)} \subseteq \mathcal{O}_{A(p_2)}$ and $C^*(\mathcal{O}_A, \mathbf{Z}_{p_1}) \subseteq C^*(\mathcal{O}_A, \mathbf{Z}_{p_2})$ are compatible with the previous identifications.

Let $p = \{p_i : i = 1, 2, \ldots\}$ be a sequence of integers such that $p_i | p_{i+1}$. Let $\mathcal{O}_{A(p)}$ denote the inductive limit of $\{\mathcal{O}_{A(p_i)} : i = 1, 2, \ldots\}$ under the natural embeddings of $\mathcal{O}_{A(p_i)}$ in $\mathcal{O}_{A(p_{i+1})}$, which can be regarded as a subalgebra of Calkin algebra $\mathscr{B}(F_A)/\mathscr{K}(F_A)$. Note that if G denotes the discrete subgroup of the circle generated by $\{e^{2\pi i/p_j} : j = 1, 2, \ldots\}$ then $\mathcal{O}_{A(p)}$ can be identified with the crossed product $C^*(\mathcal{O}_A, G)$ under the natural gauge action of §2. If A is an aperiodic matrix, then $A(p)$ is irreducible for finite p, in which case $\mathcal{O}_{A(p)}$ will be simple for any generalized integer p.

If $A \in M_n(\mathbf{N})$ we let $\mathcal{O}_{A(p)} = \mathcal{O}_{A'(p)}$. This is unambiguous because if A is such a matrix and p is finite then $A'(p) = A(p)'$ and so by Theorem 2.3, $\mathcal{O}_{A(p)} \simeq \mathcal{O}_{A'(p)}$, for finite p, and hence for all sequences considered above. In particular, if $A(i, j) \equiv 1$, we can consistently write $\mathcal{O}_{n(p)}$ for the corresponding $\mathcal{O}_{A(p)}$.

These algebras $\mathcal{O}_{n(p)}$ are related to some others appearing in the literature. If p is finite, $\mathcal{O}_{n(p)}$ can be identified as the twisted tensor product $\mathbf{C}^p \times \mathcal{O}_n$ of $\mathbf{C}^p$ by $\mathcal{O}_n$ acting by cyclic permutation of the factors. In fact if $\{e_{ij}\}$ are canonical matrix units in M_p, let $u = \sum_{i=1}^{p} e_{i,i-1}$. Then the twisted tensor product $\mathbf{C}^p \times \mathcal{O}_n$ is defined in [8] to be the C^*-subalgebra of $M_p \otimes \mathcal{O}_n$ generated by $\mathbf{C}^p \otimes 1$ and $\{u \otimes S_i : i \in \Sigma\}$. Then $\mathbf{C}^p \times \mathcal{O}_n$ is also generated by the np partial isometries $\{e_{jj}u \otimes S_i : (i, j) \in \Sigma(p)\}$ which generate $\mathcal{O}_{n(p)}$. Renault [Chapter III.2] has expressed $\mathcal{O}_n$ as a groupoid C^*-algebra. First let G_n be the groupoid corresponding to the transformation group $\bigoplus_{-\infty}^{\infty} \mathbf{Z}_n$ acting pointwise on $\prod_{-\infty}^{\infty} \mathbf{Z}_n$ with the product topology. Let $\mathbf{Z}$ act on G_n by a shift, σ, in the natural way, and form $G_n \times_\sigma \mathbf{Z}$ the semi-direct product of groupoids [24, 1.1.7]. Let $O_n^0 = \{u \in \prod \mathbf{Z}_n : u_i = 0 \text{ if } i < 0\}$ which is a subset of the unit space of G_n. Then the Cuntz groupoid O_n is the reduction of $G_n \times_\sigma \mathbf{Z}$ to O_n^0 [24]. Let $c_0 : G_n \times_\sigma \mathbf{Z} \to \mathbf{Z}_p$ be the cocycle given by $c_0(g, m) = m \bmod p$ for $g \in G_n$, $m \in \mathbf{Z}$, and c be the restriction of c_0 to O_n. Then the C^*-algebra $C^*(O_n \times_c \mathbf{Z}_p)$ of the skew products of groupoids (as defined in [24, 1.1.6]) is isomorphic by [24, 2.5.7] to the C^*-crossed product $C^*(C^*(O_n), \mathbf{Z}_p)$, where $\mathcal{O}_n = C^*(O_n)$ the C^*-algebra of the groupoid O_n, and the action of $\mathbf{Z}_p$ on $\mathcal{O}_n$ is seen to the canonical defined in §2.

The analogous algebras in the case $n = 1$ are those of Bunce and Deddens [4]. If $n = 1$, $A = 1$, then $\mathcal{O}^F(1, p)/\mathcal{K}(\ell_2) \simeq M_p \otimes \mathcal{C}(\mathbf{T})$ by [4]. Thus by Theorem 2.2, $\mathcal{O}^F(1, p)/\mathcal{K}(\ell_2) \simeq \mathcal{O}_A$ where A is an irreducible $p \times p$ permutation matrix. We thus write $\mathcal{O}_{1(p)}$ for $\mathcal{O}^F(1, p)/\mathcal{K}(\ell_2) \simeq \mathcal{O}_A \simeq M_p \otimes \mathcal{C}(\mathbf{T})$, if p is finite, and if $p = \{p_i\}_{i=1}^{\infty}$ with $p_i | p_{i+1}$, we let $\mathcal{O}_{1(p)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{O}_{1(p_i)}}$. Note that $\mathcal{O}_{1(p)}$ can also be described as $C^*(\mathbf{T}, G)$ (or $C^*(\mathcal{O}_1, G)$) where G is the torsion subgroup of $\mathbf{T}$ associated with p as defined previously, and where G acts by rotations [19, page 248].

4. COMPUTING THE K-GROUPS OF WEIGHTED SHIFT ALGEBRAS

From now on we assume that A is a zero-one matrix, satisfying (I), with $n \geq 2$. The quotient map from $\mathbf{Z}^n$ onto $\mathbf{Z}^n/(1 - A^t)\mathbf{Z}^n$ will be denoted by $[\cdot]$, and the canonical basis for $\mathbf{Z}^n$ by $\{f_i : i \in \Sigma\}$. In [9], $\mathrm{K}_0(\mathcal{O}_A)$ was computed to be isomorphic to $\mathbf{Z}^n/(1 - A^t)\mathbf{Z}^n$, and generated by the equivalence classes of $P_i = S_i S_i^*$, $i \in \Sigma$, which are identified with $[f_i]$, $i \in \Sigma$. Moreover in [9, 10], $K_1(\mathcal{O}_A)$ was shown to be isomorphic to $\operatorname{Ker}(1 - A^t)$. If $\sum_{i=1}^{n} x_i f_i \in \operatorname{Ker}(1 - A^t)$, with $x_i \in \mathbf{Z}$, then taking the equivalence class of $\sum x_i S_i$ (suitably interpreted and modified via unitaries in $\mathcal{F}_A$) essentially gives a bijection between $\operatorname{Ker}(1 - A^t)$ and $K_1(\mathcal{O}_A)$.

If p_1 and p_2 are positive integers with $p_1 | p_2$, then $e : \mathcal{O}_{A(p_1)} \to \mathcal{O}_{A(p_2)}$ induces maps $e_* : K_*(\mathcal{O}_{A(p_1)}) \to K_*(\mathcal{O}_{A(p_2)})$. We already know from §3 that $\mathcal{O}_{A(p_i)}$ is stably isomorphic to $\mathcal{O}_{A^{p_i}}$, $i = 1, 2$ (at least if A^t also satisfies (I)). In which case, $K_0(\mathcal{O}_{A(p_i)}) \simeq \mathbf{Z}^n/(1 - A^{tp_i})\mathbf{Z}^n$, and $K_1(\mathcal{O}_{A(p_i)}) \simeq \operatorname{Ker}(1 - A^{tp_i})$. (Moreover if $A(i, j) \equiv 1$, then $\mathcal{O}_{A^{p_i}}$ is isomorphic to $\mathcal{O}_{n^{p_i}}$ and so $K_0(\mathcal{O}_{n(p_i)}) = \mathbf{Z}/(n^{p_i} - 1)\mathbf{Z}$, and $K_1(\mathcal{O}_{n(p_i)}) = 0$.) In order to compute e_* after these identifications, we first make these identifications a bit more precise.

LEMMA 4.1. *With the identifications* $K_0(\mathcal{O}_{A(p_i)}) \simeq \mathbf{Z}^n/(1 - A^{tp_i})\mathbf{Z}^n$, $K_1(\mathcal{O}_{A(p_i)}) \simeq \operatorname{Ker}(1 - A^{tp_i})$, $e_* : \mathbf{Z}^n/(1 - A^{tp_1})\mathbf{Z}^n \to \mathbf{Z}^n/(1 - A^{tp_2})\mathbf{Z}^n$ *takes* $[x]$ *onto* $[(1 + A^{tp_1} + A^{t2p_1} + \ldots + A^{t(p_2 - p_1)})x]$, $x \in \mathbf{Z}^n$, *and* $e_* : \operatorname{Ker}(1 - A^{tp_1}) \to \operatorname{Ker}(1 - A^{tp_2})$ *is the inclusion map.*

Proof. Let $p \geqslant 1$, and let X, Y denote the invertible $np \times np$ matrices given by

$$X = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ A & 1 & & & \vdots \\ A^2 & A & 1 & & \vdots \\ \vdots & \vdots & & \ddots\ 1 & 0 \\ A^{p-1} & A^{p-2} & \cdots & A & 1 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 1 & 0 & \cdots & 0 & A \\ 0 & 1 & & & A^2 \\ & & \ddots & & \vdots \\ \vdots & & & 1 & A^{p-1} \\ 0 & & \cdots & 0 & 1 \end{pmatrix}.$$

Then

$$X[1 - A(p)]Y = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & 1 - A^p \end{pmatrix}.$$

Thus Y^t induces an isomorphism f of $\mathbf{Z}^{np}/(1 - A(p))^t\mathbf{Z}^{np}$ onto $\mathbf{Z}^n/(1 - A^{tp})\mathbf{Z}^n$, given by $f : [x] \to [(Yx)_0]$, $x \in \mathbf{Z}^{np}$, where y_0 denotes the last n coordinates of y in $\mathbf{Z}^{np}$. Thus $f : [(x_1, \ldots, x_p)] \to [A^t x_1 + A^{t2} x_2 + \ldots A^{t(p-1)} x_{p-1} + x_p]$, $x_i \in \mathbf{Z}^n$, and $f^{-1} : [z] \to [(0, \ldots, 0, z)]$, $z \in \mathbf{Z}^n$. Similarly X^t induces an isomorphism g of

$\operatorname{Ker}(1 - A^{tp_1})$ onto $\operatorname{Ker}(1 - A(p)^t)$. Specifically if $x \in \operatorname{Ker}(1 - A^{tp})$ then $g(x) = (A^{t(p-1)}x, A^{t(p-2)}x, \ldots, A^t x, x)$ is the corresponding element of $\operatorname{Ker}(1 - A(p)^t)$.

For $p = p_1, p_2$, let f_1, f_2, g_1, g_2 etc. denote the corresponding objects. Define $\bar{\varphi} : \mathbf{Z}^{np_1} \to \mathbf{Z}^{np_2}$ by $\bar{\varphi}(x) = (x, \ldots, x)$ (p_2/p_1 copies), $x \in \mathbf{Z}^{np_1}$. Then $\bar{\varphi}(1 - A(p_1)^t) = (1 - A(p_2)^t)\bar{\varphi}$, so that we have an induced map φ from $\mathbf{Z}^{np_1}/(1 - A(p_1)^t)\mathbf{Z}^{np_1}$ into $\mathbf{Z}^{np_2}/(1 - A(p_2)^t)\mathbf{Z}^{np_2}$. Moreover since e maps $S_\alpha S_\alpha^*$ onto $\sum_{\beta \in A(\alpha)} S_\beta S_\beta^*$, $\alpha \in \Sigma(p)$ it is seen that $e_* = \varphi$. We now have the diagram

$$\begin{array}{ccc} \mathbf{Z}^n/(1 - A^{tp_1})\mathbf{Z}^n & & \mathbf{Z}^n/(1 - A^{tp_2})\mathbf{Z}^n \\ \downarrow f_1^{-1} & & \uparrow f_2 \\ \mathbf{Z}^{np_1}/(1 - A(p_1)^t)\mathbf{Z}^{np_1} & \xrightarrow[\varphi]{} & \mathbf{Z}^{np_2}/(1 - A(p_2)^t)\mathbf{Z}^{np_2}. \end{array}$$

Then for $x \in \mathbf{Z}^n$,

$$f_2 \varphi f_1^{-1}[x] = f_2 \varphi[(0, \ldots, 0, x)] = f_2[z],$$

where $z = (z_1, \ldots, z_{p_2})$, $z_i \in \mathbf{Z}^n$, $z_i = x$ if $i = 0 \bmod p_1$ and 0 otherwise. But

$$f_2[z] = [(A^{tp_1} + A^{t2p_1} + \ldots + A^{t(p_2 - p_1)} + 1)x]$$

so that

$$f_2 \varphi f_1^{-1}[x] = [(1 + A^{tp_1} + \ldots + A^{t(p_2 - p_1)})x],$$

which completes the identification for $\mathbf{K}_0$.

If $(x_\alpha) \in \mathbf{Z}^{np_1}$, then

$$e\Big(\sum_{\alpha \in \Sigma(p_1)} x_\alpha S_\alpha\Big) = \sum_{\beta \in \Sigma(p_2)} \bar{\varphi}(x)_\beta S_\beta.$$

It then follows that $e_* : \operatorname{Ker}(1 - A(p_1)^t) \to \operatorname{Ker}(1 - A(p_2)^t)$ is the restriction of $\bar{\varphi}$. We now have the diagram

$$\begin{array}{ccc} \operatorname{Ker}(1 - A^{tp_1}) & & \operatorname{Ker}(1 - A^{tp_2}) \\ \downarrow g_1 & & \uparrow g_2^{-1} \\ \operatorname{Ker}(1 - A(p_1)^t) & \xrightarrow[\bar{\varphi}]{} & \operatorname{Ker}(1 - A(p_2)^t). \end{array}$$

Then for $x \in \operatorname{Ker}(1 - A^{tp_1}) \subseteq Z^n$,

$$g_2^{-1}\bar{\varphi}g_1(x) = g_2^{-1}\bar{\varphi}(A^{t(p_1-1)}x, A^{t(p_1-2)}x, \ldots, A^t x, x) = g_2^{-1}(z),$$

where $z = (z_1, \ldots, z_{p_2})$, and $z_i = A^{t(p_1-j)}x$ if $i = j \bmod p_1$, $1 \leqslant j \leqslant p_1$. But $g_2^{-1}(z) = x$, so that $g_2^{-1}\varphi g_1(x) = x$, which completes the identification for K_1.

LEMMA 4.2. *With the identifications* $K_0(\mathcal{O}_{n(p_i)}) = \mathbf{Z}/(n^{p_i} - 1)\,\mathbf{Z}$, $e_* : \mathbf{Z}/(n^{p_1} - 1)\mathbf{Z} \to \mathbf{Z}/(n^{p_2} - 1)\mathbf{Z}$ *takes* x *mod*$(n^{p_1} - 1)$ *to* $x(n^{p_2} - 1)/(n^{p_1} - 1)$ *mod*$(n^{p_2} - 1)$, $x \in \mathbf{Z}$.

Proof. If $A(i,j) \equiv 1$, then $A^p(i,j) \equiv n^{p-1}$. Then by Lemma 4.1, $K_0(\mathcal{O}_{n(p)}) = \mathbf{Z}^n/(1 - A^p)\mathbf{Z}^n$. We have by [20] the diagonalization

$$V(1 - A^p)W = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & & & & 1-n^p \end{pmatrix}$$

where V, W are nonsingular $n \times n$ matrices over $\mathbf{Z}$ with

$$V = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \qquad V^{-1} = \begin{pmatrix} 1 & & & 0 \\ -1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Thus $\mathbf{Z}^n/(1 - A^p)\mathbf{Z}^n$ is identified with $\mathbf{Z}/(n^p - 1)\mathbf{Z}$ and the remainder of the identification is as in Lemma 4.1.

THEOREM 4.3. a). *If* $p = (p_i)_{i=1}^\infty$ *is a sequence of integers with* $p_i | p_{i+1}$, *then* $K_0(\mathcal{O}_{A(p)}) \cong \varinjlim(\mathbf{Z}^n/(1 - A^{tp_i})\mathbf{Z}^n, \varphi_i)$, *where* $\varphi_i : \mathbf{Z}^n/(1 - A^{tp_i})\mathbf{Z}^n \to \mathbf{Z}^n/(1 - A^{tp_{i+1}})\mathbf{Z}^n$ *is given by*

$$[x] \to [(1 + A^{tp_i} + A^{t2p_i} + \ldots + A^{t(p_{i+1}-p_i)})x], \quad x \in \mathbf{Z}^n$$

and $K_1(\mathcal{O}_{A(p)}) \simeq \operatorname{Ker}(1 - A^{tp_j})$, *for some* j.

b). $$K_0(\mathcal{O}_{n(p)}) \simeq \varinjlim(\mathbf{Z}/(n^{p_i} - 1)\mathbf{Z}, \psi_i)$$

where $\psi_i : \mathbf{Z}/(n^{p_i} - 1)\mathbf{Z} \to \mathbf{Z}/(n^{p_{i+1}} - 1)\mathbf{Z}$ *is given by* x *mod*$(n^{p_i} - 1) \to x(n^{p_{i+1}} - 1)/(n^{p_i} - 1)$ *mod*$(n^{p_{i+1}} - 1)$, $x \in \mathbf{Z}$. *Moreover* $K_1(\mathcal{O}_{n(p)}) = 0$.

Proof. This is an immediate consequence of continuity of K_*, Lemmas 4.1 and 4.2, and that $\varinjlim(\operatorname{Ker}(1 - A^{tp_i}), \theta_i) = \operatorname{Ker}(1 - A^{tp_j})$ for some j, if θ_i are the inclusion maps.

REMARK. If $n = 1$, and $p = (p_i)$ is a sequence of positive integers as usual, then as noted previously, the relevant algebras are the weighted shift algebras of Bunce and Deddens [4]. Then $\mathcal{O}_{1(p_i)} \simeq M_{p_i}(\mathscr{C}(\mathbf{T}))$. Now $K_0(\mathscr{C}(\mathbf{T})) \simeq \mathbf{Z}$, and $K_1(\mathscr{C}(\mathbf{T})) \simeq \mathbf{Z}$, with the constant function in $\mathscr{C}(\mathbf{T})$, and the map $f: z \to z$ in $\mathscr{C}(\mathbf{T})$ generating K_0 and K_1 respectively. Then $K_j(\mathcal{O}_{1(p_i)}) \simeq \mathbf{Z}$. Since the embedding of $\mathcal{O}_{1(p_i)}$ in $\mathcal{O}_{1(p_{i+1})}$ is unital, $K_0(\mathcal{O}_{1(p_i)}) \simeq \mathbf{Z}$ maps into $K_0(\mathcal{O}_{1(p_{i+1})}) \simeq \mathbf{Z}$ by $m \to (p_{i+1}/p_i)m$, $m \in \mathbf{Z}$. Moreover, by [4, Proof of Theorem 2], the unitary $u = \begin{pmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{pmatrix}$ of $M_{p_i}(\mathscr{C}(\mathbf{T}))$ maps to

$$\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ u & 0 & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & u \end{pmatrix} \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & & 0 \end{pmatrix}.$$

Then since the unitary group of $M_{p_{i+1}}$ is connected, the embedding of $K_1(\mathcal{O}_{1(p_i)}) \simeq \mathbf{Z}$ into $K_1(\mathcal{O}_{1(p_{i+1})}) \simeq \mathbf{Z}$ is the identity map. Thus $K_0(\mathcal{O}_{1(p)}) \simeq \varinjlim(\mathbf{Z}, p_{i+1}/p_i) \equiv \{a/p_i : a \in \mathbf{Z},\ i = 1, 2, \dots\}$, a subgroup of the rationals, and $K_1(\mathcal{O}_{1(p)}) \simeq \mathbf{Z}$.

COROLLARY 4.4. *Let $p = \{p_i\}$, $q = \{q_i\}$ be sequences of integers such that $p_i | p_{i+1}$, $q_i | q_{i+1}$, and $n \geqslant 1$. Then $\mathcal{O}_{n(p)}$ is isomorphic to $\mathcal{O}_{n(q)}$ if and only if for each i there exists j such that $p_i | q_j$ and $q_i | p_j$.*

Proof. Suppose that for each i there exists a j such that $p_i | q_j$ and $q_i | p_j$. Then $\mathcal{O}_{n(p_i)} \subseteq \mathcal{O}_{n(q_j)} \subseteq \mathcal{O}_{n(q)}$, $\mathcal{O}_{n(q_i)} \subseteq \mathcal{O}_{n(p_j)} \subseteq \mathcal{O}_{n(p)}$ and so $\mathcal{O}_{n(p)} = \mathcal{O}_{n(q)}$.

Conversely suppose $\mathcal{O}_{n(p)} \simeq \mathcal{O}_{n(q)}$. Then if $n \geqslant 2$, by Theorem 4.3 $\lim \mathbf{Z}/(n^{p_i} - 1)\mathbf{Z} \simeq \lim \mathbf{Z}/(n^{q_i} - 1)\mathbf{Z}$. Hence for each i there exists j such that $(n^{p_i} - 1) | (n^{q_j} - 1)$. Suppose $p_i \nmid q_j$. Then $q_j = rp_i + s$, where $r \geqslant 1$, $0 < s < p_i$, with $r, s \in \mathbf{Z}$. Then, for some integer x

$$(n^{p_i} - 1)x = n^{rp_i + s} - 1 = n^{rp_i} - 1 + n^{rp_i}(n^s - 1).$$

Hence $(n^{p_i} - 1)y = n^{rp_i}(n^s - 1)$, for some integer y. Let $z = n^{p_i} - 1$; $zy = (1 - z)^r(n^s - 1)$, and so $zw = n^s - 1$ for some integer w. Thus $w = (n^s - 1)/(n^{p_i} - 1)$, which means $0 < w < 1$, a contradiction. Thus $p_i | q_j$. The case $n = 1$ is an immediate corollary of the computation of $K_0(\mathcal{O}_{1(p)})$ in the preceeding remark.

We have thus recovered the result of Bunce and Deddens [4] on the classification of the weighted shift algebras $\mathcal{O}_{1(p)}$, through an examination of K_0.

For completeness we note that information about the Ext groups of the class of algebras considered in this paper can be obtained from the Universal Coefficient Theorem in [26]. Also if p is finite, $\mathrm{Ext}^1\, \mathcal{O}_{A(p)} \simeq \mathbf{Z}^n/(1 - A^p)\mathbf{Z}^n$, and if $p_1 | p_2$, $e^* : \mathbf{Z}^n/(1 - A^{p_2})\mathbf{Z}^n \to \mathbf{Z}^n/(1 - A^{p_1})\mathbf{Z}^n$ is simply $[x] \to [x]$, $x \in \mathbf{Z}^n$. This can be shown directly from the identification of [12] or from Lemma 4.1 and [10, 3.1, 4.1]. Similarly for Ext^0.

5. THE STABLE ALGEBRAS

Let A be an aperiodic $n \times n$ matrix, and $p = (p_i)$ a sequence of positive integers with $p_i | p_{i+1}$ as usual. Here we show that the stable algebra $\mathcal{O}_{A(p)}$ can be expressed as a crossed product of an AF algebra and an automorphism. In §4 we have shown that $K_0(\mathcal{O}_{A(p)}) \simeq \varinjlim (\mathbf{Z}/(1 - A^{{}^t p_i})\mathbf{Z}^n)$. In a similar fashion (by generalizing to not necessarily zero-one matrices) one can show that if p/p_1 denotes $(p_i/p_1)_{i=1}^{\infty}$, then

$$K_0(\mathcal{O}_{A^{p_1}(p/p_1)}) \simeq \lim \mathbf{Z}^n/(1 - A^{p_1})^{{}^t(p/p_1)}\mathbf{Z}^n.$$

Thus

$$K_0(\mathcal{O}_{A(p)}) \simeq K_0(\mathcal{O}_{A^{p_1}(p/p_1)}).$$

We also show in this section that $\mathcal{O}_{A(p)}$ and $\mathcal{O}_{A^{p_1}(p/p_1)}$ are in fact stably isomorphic but not in general isomorphic.

Let p be a positive integer. We first recall from [12, 9] how to express the stable algebra of $\mathcal{O}_A$ (and $\mathcal{O}_{A(p)}$) as a crossed product of an AF algebra by the integers. Let X_A be the compact space $\{(x_i) \in \Sigma^{\mathbf{Z}} : A(x_i, x_{i+1}) = 1\}$, with the shift $\sigma = \sigma_A$ defined on X_A by $\sigma(x)_i = x_{i+1}$, and with $\mathcal{G}(X_A)$ the group of uniformly finite homeomorphisms of X_A[12, 9]. Let $\theta : \Sigma(p) \to \Sigma$ be given by $\theta(i, j) = i$, and let Θ from $X_{A(p)}$ onto X_A be the continuous map given by $\Theta(x)_i = \theta(x_i)$. Define continuous injections $h_1, \ldots, h_p$ from X_A into $X_{A(p)}$ such that $\Theta h_j = \mathrm{id}_{X_A}$, and $[h_j(x)]_0 = (x_0, j)$ $x \in X_A$, $i = 1, \ldots, p$. Then $X_{A(p)}$ is the disjoint union $\bigcup_{j=1}^{p} h_j(X_A)$, and in this way we can identify $\mathcal{G}(X_{A(p)})$ with the direct sum $\mathcal{G}(X_A) \oplus \ldots \oplus \mathcal{G}(X_A)$, ($p$-copies). If F is a subset of X_A, we let $W^u(F)$ denote its unstable manifold [9], with the inductive limit topology. If F is a shift invariant countable subset of X_A, let $\mathcal{G}(W^u(F))$ be the group of homeomorphisms of $W^u(F)$ such that the coordinates affected by any g in $\mathcal{G}(W^u(F))$ are bounded on the right. Now let F be a countable shift invariant subset of X_A such that $W^u(F)$ is dense in X_A, and such that the orbit under $\mathscr{G}(X_A)$ of each $x \in W^u(F)$ is infinite. Let $F^1 = \Theta^{-1}F$, which is a countable shift invariant subset of $X_{A(p)}$. Then $W^u(F^1)$ is the disjoint union $\bigcup_{j=1}^{p} h_j W^u(F)$, and we can identify $\mathscr{G}(W^u(F^1))$ with $\mathscr{G}(W^u(F)) \oplus \ldots \oplus \mathscr{G}(W^u(F))$ (p-copies). Thus $W^u(F^1)$ is dense in $X_{A(p)}$, and orbits of points of $W^u(F^1)$ under $\mathscr{G}(W^u(F^1))$ are infinite. Moreover Θ maps $W^u(F^1)$ onto $W^u(F)$, and induces Θ^*, say, from $\mathscr{C}_0(W^u(F))$ into $\mathscr{C}_0(W^u(F^1)) \simeq \simeq \mathscr{C}_0(W^u(F)) \oplus \ldots \oplus \mathscr{C}_0(W^u(F))$, given by $\Theta^*(f) = f \oplus \ldots \oplus f$, which intertwines the shifts. Define a group homomorphism π from $\mathscr{G}(W^u(F))$ into $\mathscr{G}(W^u(F^1)) = \mathscr{G}(W^u(F)) \oplus \ldots \oplus \mathscr{G}(W^u(F))$ given by $\pi(g) = (g, \ldots, g)$. Let U denote the canonical representation of $\mathscr{G}(W^u(F))$ in the crossed product $\mathscr{A}(A) = C^*(W^u(F), \mathscr{G}(W^u(F)))$, so that $\mathscr{A}(A) = \text{lin}\{fU(g) : f \in \mathscr{C}_0(W^u(F)), g \in \mathscr{G}(W^u(F))\}$. Then $\mathscr{A}(A(p)) = \mathscr{A}(A)) \oplus \ldots \oplus \mathscr{A}(A)$ (p-copies). Let $\mathscr{I}(A)$ denote that ideal generated by $U(g)P(W) - U(h)P(W)$, where $P(W)$ is the characteristic function of a compact open subset W of $W^u(F)$ and $g, h \in \mathscr{G}(W^u(F))$ with $g|_W = h|_W$. Then $\mathscr{I}(A(p)) \simeq \simeq \mathscr{I}(A) \oplus \ldots \oplus \mathscr{I}(A)$. Let $\bar{\mathscr{F}}_A = \mathscr{A}(A)/\mathscr{I}(A)$ and define a $*$-homomorphism φ of $\bar{\mathscr{F}}_A$ into $\bar{\mathscr{F}}_{A(p)} \simeq \bar{\mathscr{F}}_A \oplus \ldots \oplus \bar{\mathscr{F}}_A$ by $\varphi(x) = (x, \ldots, x)$. Since the shift normalises $\mathscr{G}(W^u(F))$ and leaves $W^u(F)$ invariant, there is an induced shift denoted by σ_A or σ on $\bar{\mathscr{F}}_A$, and $C^*(\bar{\mathscr{F}}_A, \sigma)$ is written as $\bar{\mathscr{O}}_A$. With the above identification of $\bar{\mathscr{F}}_{A(p)}$ as $\bar{\mathscr{F}}_A \oplus \ldots \oplus \bar{\mathscr{F}}_A$ one has

$$(5.1) \qquad \sigma_{A(p)}(x_1, \ldots, x_p) = (\sigma_A(x_p), \sigma_A(x_1), \ldots, \sigma_A(x_{p-1})), \quad x_i \in \bar{\mathscr{F}}_A.$$

Thus φ intertwines with the shifts on $\bar{\mathscr{F}}_A$ and $\bar{\mathscr{F}}_{A(p)}$, and so induces a $*$-homomorphism $\tilde{\varphi}$ of $\bar{\mathscr{O}}_A$ into $\bar{\mathscr{O}}_{A(p)}$. Similarly, if $p_1 | p_2$, there is a natural homomorphism, also denoted by $\tilde{\varphi}$ of $\bar{\mathscr{O}}_{A(p_1)}$ into $\bar{\mathscr{O}}_{A(p_2)}$. Let

$$\mathscr{L}(F) = \bigcup_{x \in F} \bigcup_{m \in N} \{(y_k) \in \Sigma^{Z_-} : y_k = x_k,\ k \leqslant -m,\ A(y_k, y_{k+1}) = 1,\ k < 0\},$$

and $Z(a) = \{(y_k) \in W^u(F) : (y_k)_{k \leqslant 0} = a\}$, $a \in \mathscr{L}(F)$. Similarly one defines $\mathscr{L}(F^1)$, and $Z(a)$ for $a \in \mathscr{L}(F^1)$. Let $\psi_1, \ldots, \psi_p : \mathscr{L}(F) \to \mathscr{L}(F^1)$ be the maps such that $[\psi_i(y)]_0 = (y_0, i)$, and $\theta[\psi_i(y)]_j = y_j$, for all $j \leqslant 0$. Suppose $a \in \mathscr{L}(F)$ and $\mu = (\mu_1, \ldots, \mu_k) \in \in \mathscr{M}_A^k$ with $A(a_0, \mu_1) = 1$. Set $Z(a, \mu) = \{y \in Z(a) : (y_1, \ldots, y_k) = \mu\}$. If also $a' \in \mathscr{L}(F)$ and $v = (v_1, \ldots, v_k) \in \mathscr{M}_A^k$ with $A(a'_0, v_1) = 1$, let $u(a', v, a, \mu) : Z(a, \mu) \to Z(a', v)$ be the uniformly finite dimensional homeomorphism defined by

$$(u(a', v, a, \mu)y)_l = y_l, \quad l \geqslant k, \quad y \in Z(a, \mu).$$

If u is a uniformly finite dimensional homeomorphism of a compact open subset B of $W^u(F)$ onto a compact open subset C of $W^u(F)$, let $\tilde{u}$ denote its image in $\mathcal{F}_A$ which is a partial isometry with range projection $P(B)$ and support projection $P(C)$.

In [12, 9] it was shown that $\bar{\mathcal{O}}_A$ is isomorphic to $\mathcal{O}_A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the compact operators on a separable infinite dimensional Hilbert space.

LEMMA 5.1. *If p_1, p_2 are positive integers with $p_1 | p_2$, then there is a commutative diagram:*

$$\begin{array}{ccc} \bar{\mathcal{O}}_{A(p_1)} & \xrightarrow{\alpha_1} & \mathcal{O}_{A(p_1)} \otimes \mathcal{K} \\ \downarrow \tilde{\varphi} & & \downarrow e \otimes 1 \\ \bar{\mathcal{O}}_{A(p_2)} & \xrightarrow{\alpha_2} & \mathcal{O}_{A(p_2)} \otimes \mathcal{K} \end{array}$$

where $e : \mathcal{O}_{A(p_1)} \to \mathcal{O}_{A(p_2)}$ is the embedding of §3, *and α_1 and α_2 are isomorphisms.*

Proof. We follow the construction of [12, 9] for the isomorphism α_1, α_2, and choose them carefully so that the diagram commutes. For simplicity, we only give details in the case $p_1 = 1$, $p_2 = p$.

For all $i \in \Sigma$, choose $a(i) \in \mathcal{L}(F)$ such that $A(a(i)_0, i) = 1$. If $\alpha = (i, k) \in \Sigma \times \{1, \ldots, p\}$, let

$$a(\alpha) = \begin{cases} \psi_{k-1}(a(i)) & \text{if } k \neq 1 \\ \psi_p(a(i)) & \text{if } k = 1. \end{cases}$$

Then $a(\alpha) \in \mathcal{L}(F^1)$ and $B(a(\alpha_0), \alpha) = 1$, if $B = A(p)$. For all $i, j \in \Sigma$ such that $A(i, j) = 1$, define $a'_{ij} \in \mathcal{L}(F)$ by

$$(a'_{ij})_r = \begin{cases} a(i)_{r+1} & r \leqslant 1 \\ i & r = 0\,. \end{cases}$$

Then if $k \in \{1, \ldots, p\}$, and $\alpha = (i, k)$, $\beta = (j, k-1)$, let $a'_{\alpha\beta} = \psi_k(a'_{ij}) \in \mathcal{L}(F^1)$, so that $(a'_{\alpha\beta})_0 = \alpha$. Then let $a''_{ij} = a(i)$, $a''_{\alpha\beta} = a(\alpha)$. Then $\sigma^{-1}Z(a'_{ij}, j) = Z(a''_{ij}, (i, j))$ and $\sigma^{-1}Z(a'_{\alpha\beta}, \beta) = Z(a''_{\alpha\beta}, (\alpha, \beta))$. Then

$$S_i = \sum_j A(i, j)\, P(Z(a(i), (i, j)))\sigma^{-1}\tilde{u}(a'_{ij}, j; a(j), j), \quad i \in \Sigma$$

generate $\mathcal{O}_A$, and

$$S_\alpha = \sum_\beta B(\alpha, \beta) P(Z(a(\alpha), (\alpha, \beta)))\sigma^{-1}\tilde{u}(a'_{\alpha\beta}, \beta;\, a(\beta), \beta), \quad \alpha \in \Sigma(p)$$

generate $\mathcal{O}_{A(p)}$. Then under the homomorphism $\tilde{\varphi}$, S_i gets mapped to

$$\sum_j A(i,j)\{\sum_k P(Z(a,(i,k),(i,k),(j,k-1)))\sigma^{-1}(\sum_r \tilde{u}(a'_{(i,r)(j,r-1)},(j,r-1);$$

$$a(j,r-1),(j,r-1)))\} =$$

$$= \sum_{j,k} A(i,j)P[Z(a(i,k),(i,k),(j,k-1))]\sigma^{-1}\tilde{u}(a'_{(i,k)(j,k-1)},(j,k-1);$$

$$a(j,k-1),(j,k-1)) =$$

$$= \sum_{k=1}^{p} S_{(i,k)} = e(S_i)$$

(with $k-1$, $r-1$ taken mod p), where e is as in §3. Letting

$$B = \bigcup_{i\in\Sigma} Z(a(i),i),$$

and

$$B^1 = \Theta^{-1}B = \bigcup_{\alpha\in\Sigma(p)} S(a(\alpha),\alpha),$$

it follows that α_1, α_2 constructed as in [12] yield the required commutative diagram.

REMARK 5.2. In [9] Cuntz used the isomorphism $\mathcal{K}\otimes\mathcal{O}_A \simeq C^*(\bar{\mathcal{F}}_A,\sigma)$ to determine $K_*(\mathcal{O}_A)$ from the exact sequence of Pimsner and Voiculescu [22]. The commutative diagram of the above Lemma can be used together with the exact sequence of [22] to give an alternative proof of the identification of the embedding of $K_*(\mathcal{O}_{A(p_1)})$ in $K_*(\mathcal{O}_{A(p_2)})$, which we have already done in §4. One could also use the following immediate consequence of Lemma 5.1:

THEOREM 5.3.

$$\mathcal{K}\otimes\mathcal{O}_{A(p)} \simeq C^*(\bar{\mathcal{F}}_{A(p)},\mathbf{Z}), \quad \textit{where } \bar{\mathcal{F}}_{A(p)} = \lim_{\to} \bar{\mathcal{F}}_{A(p_i)}.$$

If $\bar{\mathcal{F}} = \bar{\mathcal{F}}_A$, $\sigma = \sigma_A$, we let $\bar{\mathcal{F}}^p = \bar{\mathcal{F}}\oplus\ldots\oplus\bar{\mathcal{F}}$, ($p$-copies) which we identify with $\bar{\mathcal{F}}_{A(p)}$ as before. Under this identification, $\sigma_p = \sigma_{A(p)}$ on $\bar{\mathcal{F}}^p$ is given by (5.1). If $\alpha\in\operatorname{Aut}(\bar{\mathcal{F}})$, the $*$-automorphism group of $\bar{\mathcal{F}}$, define $\theta = \theta(p,\alpha)\in\operatorname{Aut}(\bar{\mathcal{F}}^p)$ by

$$\theta(x_1,\ldots,x_p) = (\theta(x_p),x_1,\ldots,x_{p-1}), \quad x_i\in\bar{\mathcal{F}}.$$

Define also $\delta = \delta_p\in\operatorname{Aut}(\bar{\mathcal{F}}^p)$ by

$$\delta(x_1,\ldots,x_p) = (x_1,\sigma(x_2),\ldots,\sigma^{p-1}(x_p)), \quad x_i\in\bar{\mathcal{F}}.$$

Then if $\theta_p = \theta(p, \sigma^p)$, $\theta_p\delta = \delta\sigma_p$ and so

$$\mathcal{K} \otimes \mathcal{O}_{A(p)} \simeq C^*(\bar{\mathcal{F}}^p, \sigma_p) \simeq C^*(\bar{\mathcal{F}}^p, \theta_p)$$

using δ. Let u be the unitary in the multiplier algebra of $C^*(\bar{\mathcal{F}}, \sigma)$ satisfying $\sigma(x) = uxu^*$, $x \in \bar{\mathcal{F}}$. Then $C^*(\bar{\mathcal{F}}, \sigma) = \overline{\text{lin}}\{au^r : a \in \bar{\mathcal{F}}, \quad r \in \mathbf{Z}\}$. Define $\pi : \bar{\mathcal{F}}^p \to \to M_p(\bar{\mathcal{F}})$ by

$$\pi(x) = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_p \end{pmatrix},$$

and $v = v(p) = [v_{ij}] \in M_p[\mathcal{M}(C^*(\bar{\mathcal{F}}, \sigma))]$ by $v_{r, r-1} = 1$ if $r \neq 1$, $v_{1,p} = u^p$, $v_{rs} = 0$ otherwise. Then $\pi\theta_p(x) = v\pi(x)v^*$, $x \in \bar{\mathcal{F}}^p$, and so

$$\begin{aligned} C^*(\bar{\mathcal{F}}^p, \theta_p) &= \overline{\text{lin}}\{\pi(a)v^r \;:\; a \in \bar{\mathcal{F}}^p, \; r \in \mathbf{Z}\} = \\ &= M_p[\overline{\text{lin}}\{au^{pr} \;:\; a \in \bar{\mathcal{F}}, \; r \in \mathbf{Z}\}] = M_p[C^*(\bar{\mathcal{F}}, \sigma^p)]. \end{aligned} \tag{5.2}$$

In particular, this says $\mathcal{K} \otimes \mathcal{O}_{A(p)} \simeq M_p[\mathcal{K} \otimes \mathcal{O}_{A^p}]$, since we can identify $(\bar{\mathcal{F}}, \sigma^p)$ with $(\bar{\mathcal{F}}_{(A^p)'}, \sigma)$, and so $\mathcal{O}_{A(p)}$ is stably isomorphic with $\mathcal{O}_{A^p}$ as in §3. We would also like to draw the reader's attention to [12, Remark 3.9] on which this identification is based.

If $p_1 | p_2$, we have $\varphi : \bar{\mathcal{F}}^{p_1} \to \bar{\mathcal{F}}^{p_2}$ given by $\varphi(x) = (x, \ldots, x)$, ($p_2/p_1$-copies), $x \in \bar{\mathcal{F}}^{p_1}$. This was defined previously when $p_1 = 1$. Define $\varphi' : \bar{\mathcal{F}}^{p_1} \to \bar{\mathcal{F}}^{p_2}$ by

$$\varphi'(x_1, \ldots, x_{p_1}) =$$

$$= (x_1, \ldots, x_{p_1}, \sigma^{-p_1}(x_1), \ldots, \sigma^{-p_1}(x_{p_1}), \ldots, \sigma^{-(p_2-p_1)}(x_1), \ldots, \sigma^{-(p_2-p_1)}(x_{p_1})),$$

if $x_i \in \bar{\mathcal{F}}$. Then $\varphi\delta = \delta\varphi'$, $\varphi'\theta_{p_1} = \theta_{p_2}\varphi'$, and so φ extends to a map also denoted by φ of $C^*(\bar{\mathcal{F}}^{p_1}, \sigma_{p_1})$ into $C^*(\bar{\mathcal{F}}^{p_2}, \sigma_{p_2})$, which is as in Lemma 5.1, and φ' extends to a map of $C^*(\bar{\mathcal{F}}^{p_1}, \theta_{p_1})$ into $C^*(\bar{\mathcal{F}}^{p_2}, \theta_{p_2})$.

If $p = (p_i)$ is a sequence of positive integers with $p_i | p_{i+1}$, we let $\varphi_i, \varphi_i' : \bar{\mathcal{F}}^{p_i} \to \bar{\mathcal{F}}^{p_{i+1}}$, $\sigma_i = \sigma_{p_i}$, $\theta_i = \theta_{p_i}$, $\delta_i = \delta_{p_i} \in \text{Aut}(\bar{\mathcal{F}}^{p_i})$, $v_i = v(p_i)$ be the corresponding objects. Then

$$\begin{aligned} \mathcal{K} \otimes \mathcal{O}_{A(p)} &\simeq \varinjlim(C^*(\bar{\mathcal{F}}^{p_i}, \sigma_{p_i}), \varphi_i) \simeq && \text{by Lemma 5.1} \\ &\simeq \varinjlim(C^*(\bar{\mathcal{F}}^{p_i}, \theta_{p_i}), \varphi_i') \simeq && \text{using } \{\delta_i, \; i = 1, 2 \ldots\} \\ &\simeq \varinjlim M_{p_i}[C^*(\bar{\mathcal{F}}^{p_i}, \sigma^{p_i})] && \text{using (5.2).} \end{aligned}$$

Let 1 denote the identity in the multiplier algebra $\mathcal{M}(\mathcal{F})$, and $e = (1, 0, \ldots, 0) \in \mathcal{M}(\bar{\mathcal{F}}^{p_1})$, which we can regard also as a multiplier on $\varinjlim(C^*(\bar{\mathcal{F}}^{p_i}, \theta_i), \varphi_i') = \mathcal{A}$, say. We show that $e\mathcal{A}e \simeq \mathcal{O}_{A^{p_1}(p_i/p_1)} \otimes \mathcal{K}$. Now

$$e[M_{p_i}(C^*(\mathcal{F}, \sigma^{p_i}))]e \simeq M_{p_i/p_1}[C^*(\mathcal{F}, \sigma^{p_i})] \simeq C^*(\mathcal{F}^{p_i/p_1}, \beta_i),$$

where $\beta_i = \theta(p_i/p_1, \sigma^{p_i})$. We determine the embedding $C^*(\mathcal{F}^{p_i/p_1}, \beta_i) \to C^*(\mathcal{F}^{p_{i+1}/p_1}, \beta_{i+1})$ under φ_i' after the identifications have been made. If $\pi_1 : \mathcal{F}^{p_i/p_1} \to M_{p_i/p_1}(\mathcal{F})$ is defined by

$$\pi_1(x_1, \ldots, x_{p_i/p_1}) = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_{p_i/p_1} \end{pmatrix}, \quad x_j \in \bar{\mathcal{F}}$$

and $w_i = [w_{rs}] \in M_{p_i/p_1}(\mathcal{M}(C^*(\mathcal{F}, \sigma^{p_i})))$ is given by $w_{r,r-1} = 1$ if $r \neq 1$, $w_{1,q} = u^{p_i}$ if $q = p_i/p_1$, and $w_{rs} = 0$ otherwise, then

$$C^*(\mathcal{F}^{p_i/p_1}, \beta_i) = \text{lin}\{\pi_1(a)w^r : a \in \bar{\mathcal{F}}^{p_i/p_1}, r \in \mathbf{Z}\} =$$

$$= M_{p_i/p_1}(C^*(\mathcal{F}, \sigma^{p_i})).$$

Define $\varphi_i'' : \mathcal{F}^{p_i/p_1} \to \bar{\mathcal{F}}^{p_{i+1}/p_1}$ by

$$\varphi_i''(x_1, \ldots, x_q) =$$

$$= (x_1, \ldots, x_q, \sigma^{-p_i}(x_1), \ldots, \sigma^{-p_i}(x_q), \ldots, \sigma^{-(p_{i+1}-p_i)}(x_1), \ldots, \sigma^{-(p_{i+1}-p_i)}(x_q))$$

where $q = p_i/p_1$, $x_j \in \bar{\mathcal{F}}$. Then $\varphi_i''\beta_i = \beta_{i+1}\varphi_i''$, and so φ_i'' extends to a $*$-homomorphism from $C^*(\bar{\mathcal{F}}^{p_i/p_1}, \beta_i)$ into $C^*(\mathcal{F}^{p_{i+1}/p_1}, \beta_{i+1})$ also denoted by φ_i''. Now $\varphi_i'(v_i^{p_1}) = v_{i+1}^{p_1}$, and $\varphi_i''(w_i) = w_{i+1}$. From this, commutativity of the following diagram becomes clear:

$$\begin{array}{ccc} eC^*(\bar{\mathcal{F}}^{p_i}, \theta_i)e & \longrightarrow & C^*(\bar{\mathcal{F}}^{p_i/p_1}, \beta_i) \\ \downarrow \varphi_i' & & \downarrow \varphi_i'' \\ eC^*(\bar{\mathcal{F}}^{p_{i+1}}, \theta_{i+1})e & \longrightarrow & C^*(\mathcal{F}^{p_{i+1}/p_1}, \beta_{i+1}) \end{array}$$

where the horizontal arrows are given by the identifications noted above, and which take $v_i^{p_1}$ onto w_i. Details are left to the reader. Thus $e\mathcal{A}e \simeq \varinjlim(C^*(\mathcal{F}^{p_i/p_1}, \beta_i), \varphi_i'') \simeq$

$\simeq \mathcal{O}_{A^{p_1}(p/p_1)} \otimes \mathcal{K}$, using the identification of $(\overline{\mathcal{F}}, \sigma^{p_1})$ with $(\overline{\mathcal{F}}_{A^{p_1}}, \sigma_{(A^{p_1})})$. But Brown's theorem [2] shows that $e\mathcal{A}e$ is stably isomorphic to $\mathcal{A}$. We summarize all this by:

THEOREM 5.4. *If A is an aperiodic $n \times n$ matrix, and $p = (p_i)$ is a sequence of integers with $p_i | p_{i+1}$ then $\mathcal{O}_{A(p)}$ is stably isomorphic to $\mathcal{O}_{A^{p_1}(p/p_1)}$.*

Note that in general $\mathcal{O}_{A(p)}$ is not isomorphic to $\mathcal{O}_{A^{p_1}(p/p_1)}$, e.g. the image of the identity of $\mathcal{O}_{n(p)}$ in $K_0(\mathcal{O}_{n(p)})$ has order $n-1$, whilst the image of the identity of $\mathcal{O}_{n^{p_1}(p/p_1)}$ in $K_0(\mathcal{O}_{n^{p_1}(p/p_1)})$ has order $n^{p_1} - 1$.

An alternative proof of Theorem 5.4 can be obtained by embedding $C^*(\mathcal{O}_A^{\mathbf{Z}_{p_1}}, G(p)/\mathbf{Z}_{p_1})$ as a corner of $C^*(\mathcal{O}_A, G(p))$ as in [25].

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