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INVARIANT OPERATOR RANGES FOR REFLEXIVE ALGEBRAS

KENNETH R. DAVIDSON

In this paper, we investigate the operator ranges invariant for a closed algebra $\mathscr{A}$ with a commutative subspace lattice. If $\mathscr{A}$ is large enough, in particular if it is reflexive, all the invariant operator ranges are described. Certain examples indicate that there are many "pathological" operator ranges in some other cases.

The study of invariant operator ranges was begun by Foiaş [3], who referred to them as paraclosed subspaces. The main result of this paper generalizes Foiaş's theorems on "full" algebras (algebras containing a masa). This paper also generalizes a paper of Ong [5] in which he describes the invariant operator ranges of certain nest algebras. Many properties of operator ranges are studied in [2].

Let $\mathscr{A}$ be a norm closed algebra on a Hilbert space $\mathscr{H}$. Let Lat$\mathscr{A}$ denote the lattice of invariant subspaces for $\mathscr{A}$. We will also identify this lattice with the set of orthogonal projections onto these subspaces. Let $\mathrm{Lat}_{1/2}\mathscr{A}$ denote the invariant operator ranges of $\mathscr{A}$. If $\mathscr{L}$ is a lattice, Alg$\mathscr{L}$ will denote the algebra of all operators leaving the elements of $\mathscr{L}$ invariant. An algebra is said to be reflexive if $\mathscr{A} = \mathrm{AlgLat}\mathscr{A}$. If P is a projection, $P^{\perp}$ will denote $I - P$.

If an algebra $\mathscr{A}$ contains a maximal abelian von Neumann algebra (masa) $\mathscr{M}$, then $\mathrm{Lat}\mathscr{A} \subseteq \mathrm{Lat}\mathscr{M}$; so the projections in Lat$\mathscr{A}$ commute. Conversely, if $\mathscr{L}$ is a commutative lattice, then Alg$\mathscr{L}$ contains a masa. With this in mind, we note that the following theorem applies to full algebras. In particular, it applies if $\mathscr{A}$ is reflexive and Lat$\mathscr{A}$ is commutative.

THEOREM. *Suppose $\mathscr{A}$ is a norm closed algebra such that $(\mathscr{A} \cap \mathscr{A}^*)''$ contains a masa $\mathscr{M}$ and $\mathscr{M}\mathscr{A}\mathscr{M} \subseteq \mathscr{A}$. Let $\mathscr{L} = \mathrm{Lat}\mathscr{A}$. Then the following are equivalent.*

(1) *η belongs to* $\mathrm{Lat}_{1/2}\mathscr{A}$.

(2) *η belongs to* $\mathrm{Lat}_{1/2}(\mathrm{Alg}\mathscr{L})$.

(3) *η is the range of* $\sum_{n=1}^{\infty} 2^{-n}(P_n - P_{n-1})$ *where* $\{P_n, n \geqslant 0\}$ *is an increasing sequence of projections in* $\mathscr{L}$.

(3′) *η is the range of an operator T in the norm-closed convex hull of* $\mathscr{L}$.

(4) *η is the range of a positive operator M in $\mathscr{M}$ such that the spectral projections* $E(\varepsilon, \|M\|]$ *belong to $\mathscr{L}$ for all* $\varepsilon \geqslant 0$.

(4′) *η is the range of an operator T in the weak operator closed convex hull of* $\mathscr{L}$.

Proof. It is routine to verify that (3) $\Rightarrow$ (3′) $\Rightarrow$ (4′) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We will prove (1) $\Rightarrow$ (3).

Let η be an invariant operator range for $\mathscr{A}$. A fortiori, η is invariant for $\mathscr{A} \cap \mathscr{A}^*$. By a theorem of Ong [4], η is invariant for $(\mathscr{A} \cap \mathscr{A}^*)''$ which contains $\mathscr{M}$. So by Foiaş [3], we may suppose that η is the range of a positive operator T in $\mathscr{M}$ with $0 \leqslant T \leqslant I$. Since $\mathscr{L}$ is also contained in $\mathscr{M}$, $\mathscr{L}$ commutes with T and all its spectral projections. Let $E_n = E(2^{-n}, 1]$ be the spectral projections of T. Let L_n be the least projection in $\mathscr{L}$ dominating E_n. (This is, in fact, the projection onto $\overline{\mathscr{A}E_n\mathscr{H}}$.)

We claim there is an integer k such that $L_n \leqslant E_{n+k}$ for all n. Once we have established this claim, we finish the proof as follows. Let

$$A = \sum_{n=1}^{\infty} 2^{-n}L_n = \sum_{n=1}^{\infty} 2^{1-n}(L_n - L_{n-1}).$$

It follows from the claim that $A \geqslant T \geqslant 2^{-k-1}A$. It is readily verified that the range of A equals the range of T (c.f. [2]). This establishes (1) $\Rightarrow$ (3).

If the claim is false, there are several possibilities:

Case 1. There exists an n_0 such that $E\{0\}L_{n_0} \neq 0$.

Case 2. There exists an n_0 such that L_{n_0} is not dominated by any E_k.

Case 3. There is a sequence $n_k \to \infty$ such that $E^{\perp}_{n_k+2k}L_{n_k} \neq 0$.

We exclude from Cases 2 and 3 any situation handled by a previous case.

Case 1. Since $\mathscr{A}E_{n_0}\mathscr{H}$ is dense in $L_{n_0}\mathscr{H}$, we conclude that $E\{0\}\mathscr{A}E_{n_0}\mathscr{H} \neq 0$. Now, $E_{n_0}\mathscr{H}$ is contained in the range η of T, so $\mathscr{A}E_{n_0}\mathscr{H}$ is contained in η. However $E\{0\}T = 0$, so $E\{0\}$ annihilates η. This contradiction rules out Case 1.

Case 2. As Case 1 is excluded, there must exist a sequence $n_k \to \infty$ so that $P_k = E_{n_{k+1}} - E_{n_k}$ satisfies $P_kL_{n_0} \neq 0$. As above, one gets $P_k\mathscr{A}E_{n_0} \neq 0$. We may suppose that $n_k \geqslant 4k$. Choose $A_k = P_kA_kE_{n_0}$ in $\mathscr{A}$ of norm one (recall $\mathscr{M}\mathscr{A}\mathscr{M} \subseteq \subseteq \mathscr{A}$). Set $A = \sum 2^{-k}A_k$ and choose unit vectors y_k in $E_{n_0}\mathscr{H}$ with $\|A_ky_k\| > 9/10$.

Inductively define $\varepsilon_k = 0$ or 1 and $w_k = \sum_{j=1}^{k} \varepsilon_j 4^{-j} y_j$ as follows: If $\| A_k w_{k-1} \| \geqslant \geqslant \frac{1}{2} \cdot 4^{-k}$, set $\varepsilon_k = 0$; otherwise, set $\varepsilon_k = 1$. Let $y = \sum_{j=1}^{\infty} \varepsilon_j 4^{-j} y_j$. Then

$$\| A_k y \| \geqslant | \| A_k w_{k-1} \| - \varepsilon_k 4^{-k} \| A_k y_k \| | - \| A_k \| \left\| \sum_{j=k+1}^{\infty} \varepsilon_j 4^{-j} y_j \right\| \geqslant$$

$$\geqslant 4^{-k} \left(\frac{9}{10} - \frac{1}{2} \right) - \sum_{j \geqslant k+1}^{\infty} 4^{-j} = 4^{-j}/15.$$

Now, $T \geqslant 2^{-n_0} E_{n_0}$ so $y = E_{n_0} y$ belongs to the range of T. However, $\| P_k A y \| = = \| 2^{-k} A_k y \| \geqslant 8^{-k}/15$. And since $P_k T \leqslant 2^{-n_k} P_k \leqslant 16^{-k} P_k$, one has $\| T^{-1} P_k A y \| \geqslant \geqslant 16^k \cdot 8^{-k}/15 = 2^k/15$. (Although T is not invertible, $T | P_k \mathscr{H}$ is invertible.) It follows that Ay is not in the range of T. This contradicts the invariance of η, and so rules out Case 2.

Case 3. As Case 2 is ruled out, for every n_k there is an n'_k such that $E_{n'_k}^{\perp} \mathscr{A} E_{n_k} = = 0$. By dropping to a subsequence, we may assume that $n_{k+1} > n'_k$. So,

$$(E_{n'_k} - E_{n_k+2k}) \mathscr{A} (E_{n_k} - E_{n_{k-1}}) = E_{n_k+2k}^{\perp} \mathscr{A} E_{n_k} \neq 0.$$

Let

$$P_k = E_{n_k} - E_{n_{k-1}}$$

and

$$Q_k = E_{n'_k} - E_{n_k+2k}.$$

Choose $A_k = Q_k A_k P_k$ in $\mathscr{A}$ of norm one, and choose unit vectors $y_k = P_k y_k$ with $\| A_k y_k \| > \frac{1}{2}$. Then $A = \sum_{k=1}^{\infty} 2^{-k} A_k$ belongs to $\mathscr{A}$. Since $TP_k \geqslant 2^{-n_k} P_k$, there is a (unique) vector $z_k = P_k z_k$ such that $T z_k = y_k$; and $\| z_k \| \leqslant 2^{n_k}$. Set

$$z = \sum_{k=1}^{\infty} 2^{-n_k - k} z_k.$$

So

$$y = Tz = \sum_{k=1}^{\infty} 2^{-n_k - k} y_k$$

belongs to η. Let

$$x = Ay = \sum_{k=1}^{\infty} 2^{-n_k - k} A_k y_k.$$

Then

$$Q_k x = 2^{-n_k - k} A_k y_k$$

and

$$TQ_k \leqslant 2^{-n_k - 2k} Q_k,$$

so the unique vector $w_k = Q_k w_k$ such that $Tw_k = Q_k x$ has norm

$$\|w_k\| \geqslant 2^{n_k + 2k} \|Q_k x\| \geqslant 2^{k-1}.$$

Thus $x = \sum_{k=1}^{\infty} \oplus Q_k x$ is not in the range of T. This again contradicts $A\eta \subseteq \eta$, so Case 3 is also eliminated. This establishes the claim. ▨

REMARKS. 1) One of the advantages of this theorem over the weaker version applying to algebras containing a masa is that it can be applied to smaller algebras such as algebras of compact operators. For example, the algebra of compact operators which are upper triangular with respect to a fixed basis satisfies the hypotheses. For many "discrete" commutative lattices, the compact operators in Alg$\mathscr{L}$ are weak* dense in Alg$\mathscr{L}$, and the compact operators in $\mathscr{L}'$ are weak* dense in $\mathscr{L}'$, so the theorem again applies.

2) The diagonal condition $(\mathscr{A} \cap \mathscr{A}^*)'' \supseteq \mathscr{M}$ is used only once to conclude that η is the range of some operator in $\mathscr{M}$. Thus, when $\mathscr{A}$ does not satisfy this condition, but does satisfy $\mathscr{M}\mathscr{A}\mathscr{M} \subseteq \mathscr{A}$ and Lat$\mathscr{A} \subseteq \mathscr{M}$, then every η in $\mathrm{Lat}_{1/2}\mathscr{A}$ of the form Ran(T) for some T in $\mathscr{M}$, in fact, is of the form given in (3). However, as the example below shows, there may be other invariant ranges even when $\mathscr{A}$ is large.

3) It is possible to give a "coordinatized" proof of this theorem in the separable case using Arveson's theory [1]. The algebra $\mathscr{A}$ can be represented on a partially ordered measure space $(X, \mu, \leqslant)$ so that $\mathscr{L}$ corresponds to essentially increasing sets and $\mathscr{M} = L^\infty(X, \mu)$. Every invariant operator range has the form Ran(M_h) where h is an essentially increasing function in $L^\infty(X, \mu, \leqslant)$. We omit the details, but η has the form Ran(M_f) for some $f \in L^\infty(X, \mu)$. The idea is to find a null set N so that $f^*(x) = \sup\{f(y) : y \leqslant x, y \notin N\}$ satisfies $f^* \leqslant kh$ for some $k > 0$. N is determined by showing that $G_k = \{(x, y) : y < x \text{ and } f(y) > kh(x)\}$ is marginally null for some k. When it is not, Arveson's theory provides a nonzero operator in $\mathscr{A}$ "supported" on G_k which plays the role of the A_k in the proof above.

EXAMPLE. The purpose of this example is to show that the diagonal condition $(\mathscr{A} \cap \mathscr{A}^*)'' \supseteq \mathscr{M}$ is essential. Let $\mathscr{T}^+$ and $\mathscr{T}_0^+$ be the upper triangular and strictly upper triangular operators with respect to a fixed orthonormal basis $\{e_n, n \geqslant 1\}$. We will show that $\mathrm{Lat}_{1/2}\mathscr{T}_0^+ \neq \mathrm{Lat}_{1/2}\mathscr{T}^+$, and as a result $\mathrm{Lat}_{1/2}\mathscr{T}_0^+$ is quite complicated. Note that by Remark 2, every $\mathscr{T}_0^+$ invariant range of a diagonal operator is the range of a diagonal with decreasing entries, and is invariant under $\mathscr{T}^+$ as well.

Suppose $\alpha_n \to 0$, $|\alpha_n| \leqslant 1/2$, and $\sum |\alpha_n|^2 = +\infty$; and $0 < \delta_n$ such that $\delta_n/\delta_{n-1} < < 2^{-n}$. Choose $n_k \to \infty$ such that

$$k^2 \leqslant \sum_{n > n_{k-1}}^{n_k} |\alpha_n|^2 < k^2 + 1.$$

Let $\{f_n, g_n, n \geqslant 1\}$ be an orthonormal basis for a Hilbert space, and set $x_n = f_n \oplus \oplus \alpha_n g_k$ for $n_{k-1} < n \leqslant n_k$. Let X be the closed span of $\{x_n : n \geqslant 1\}$.

Define $T : X \to H$ by $Tx_n = \delta_n e_n$ and extend by linearity.

$$\|T \sum a_n x_n\| = \|\sum \delta_n a_n e_n\| \leqslant \delta_1 (\sum |a_n|^2)^{1/2} \leqslant \delta_1 \|\sum a_n x_n\| .$$

So T is continuous. Since $\|x_n\| \leqslant 2$ and the distance to $\operatorname{span}\{x_j, j \neq n\}$ is at least 1, the projection P_n onto the span of $\{x_n\}$ with kernel $\operatorname{span}\{x_j, j \neq n\}$ has norm at most 2. Suppose A belongs to $\mathcal{T}_0^+$, with matrix (a_{ij}) where $a_{ij} = 0$ if $i \geqslant j$. Then

$$ATx_n = A\delta_n e_n = \sum_{i=1}^{n-1} \delta_n a_{in} e_i = T \sum_{i=1}^{n-1} \left(\frac{\delta_n}{\delta_i} a_{in} \right) x_i .$$

Let $\pi(A)x_n = \sum_{i=1}^{n-1} \frac{\delta_n}{\delta_i} a_{in} x_i$, and extend to a linear operator on X.

$$\|\pi(A)P_n\| \leqslant \left(\sum_{i=1}^{n-1} \frac{\delta_n}{\delta_i} \right) \max |a_{in}| \cdot \|P_n\| \leqslant 2^{2-n} \|A\| .$$

So $\|\pi(A)\| \leqslant \sum \|\pi(A)P_n\| \leqslant 4\|A\|$. Hence $\pi(A)$ is bounded and $AT = T\pi(A)$, showing that the range of T is invariant under A. Foiaş [3] has shown that such a bounded representation π exists for an algebra if and only if $\operatorname{Ran}(T)$ is invariant.

We will demonstrate that π cannot be extended to a representation of $\mathcal{T}^+$, which will show that $\operatorname{Ran}(T)$ is not invariant for $\mathcal{T}^+$. Since T is one to one on $\operatorname{span}\{x_n : n_{k-1} < n \leqslant n_k\}$, and these sets are taken to orthogonal ranges, T must be one to one. Choose (β_n) in ℓ^2 such that $\alpha_n \beta_n \geqslant 0$ and $\sum_{n > n_{k-1}}^{n_k} \alpha_n \beta_n \geqslant 1$ (e.g. $\beta_n = \bar{\alpha}_n / k^2$).

Choose λ_n such that $|\lambda_n| = 1$ and $\sum_{n > n_{k-1}}^{n_k} \bar{\lambda}_n \alpha_n \beta_n = 0$. Let U be the diagonal operator on $\{e_n\}$ with weights $\{\lambda_n\}$ and U_k diagonal with weights $\{\lambda_n, n \leqslant n_k ; 0, n > n_k\}$.

We have

$$U_k T x_n = \delta_n \lambda_n e_n = T \lambda_n x_n \quad \text{ for } n \leqslant n_k,$$

$$U_k T x_n = 0 \quad \text{ for } n > n_k.$$

So $U_k T = T\pi(U_k)$ where $\pi(U_k)x_n = \lambda_n x_n$ if $n \leqslant n_k$, $\pi(U_k)x_n = 0$ if $n > n_k$. Now let

$$y_k = \sum_{n=1}^{n_k} \bar{\lambda}_n \beta_n x_n = \sum_{n=1}^{n_k} \bar{\lambda}_n \beta_n f_n + \sum_{j=1}^{k} \left(\sum_{n > n_{j-1}}^{n_j} \bar{\lambda}_n \alpha_n \beta_n \right) g_j = \sum_{n=1}^{n_k} \bar{\lambda}_n \beta_n f_n.$$

So

$$\|y_k\|^2 = \sum_{n=1}^{n_k} |\beta_n|^2 \leqslant \sum_{n=1}^{\infty} |\beta_n|^2 = B.$$

But,

$$\pi(U_k) y_k = \sum_{n=1}^{n_k} \beta_n x_n = \sum_{n=1}^{n_k} \beta_n f_n + \sum_{j=1}^{k} \left(\sum_{n > n_{j-1}}^{n_j} \alpha_n \beta_n \right) g_k \,.$$

So

$$\|\pi(U_k) y_k\|^2 \geqslant \sum_{n=1}^{n_k} |\beta_n|^2 + k.$$

Hence

$$\|\pi(U_k)\| \geqslant (1 + k/B)^{1/2}$$

tends to infinity as $k \to \infty$.

Since U_k tend strongly to U, $\pi(U_k)$ would tend strongly to $\pi(U)$ if it exists. Thus $\pi(U)$ is unbounded. So the range of T is not invariant for U, and hence not for $\mathscr{T}^+$.

REMARK. The situation is quite different for lower triangular operators $\mathscr{T}^-$. The theorem shows that $\mathrm{Lat}_{1/2}\mathscr{T}^- = \mathrm{Lat}\mathscr{T}^-$, and it is not hard to show that $\mathrm{Lat}_{1/2}\mathscr{T}_0^- = \mathrm{Lat}\mathscr{T}^-$ also.

The analytic Toeplitz operators sit inside $\mathscr{T}^-$, and it would be interesting to characterize their invariant operator ranges. The situation is quite complicated however. If h belongs to H^∞, the range of T_h is invariant. Also, if w_n is a decreasing sequence of inner functions (w_{n+1} divides w_n), then by analogy to statement (3) of the theorem, $\left\{ \sum_{n=1} 2^{-n} h_n : \sum \|h_n\|^2 < \infty,\ h_n \in (w_n H^2 \ominus w_{n-1} H^2) \right\}$ is seen to be invariant. These two examples have little in common since if the outer factor of h is not invertible, $\mathrm{Ran}(T_h)$ does not contain wH^2 for any inner function w. More generally, if f_n belong to H^∞ and $\sum \|f_n\|_\infty < \infty$, then $\left\{ \sum_{n=1}^{\infty} f_n h_n : \sum \|h_n\|^2 < \infty \right\}$ is invariant for analytic Toeplitz operators. Are they all of this form? To attack this problem, it would seem to be helpful to know that invariant ranges come from operators in some tractable class.

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KENNETH R. DAVIDSON
Department of Matematics,
University of Waterloo,
Waterloo, N2L 3G1, Ontario,
Canada.

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