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THE SUPERPOSITION PROPERTY FOR TAYLOR'S FUNCTIONAL CALCULUS

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This note is a continuation of [10] and it contains some functorial properties of the functional calculus with sections of an analytic space developed there. The spectral mapping theorem [12] and the superposition theorem in several variables spectral theory ([1], [4]) are particular cases of Theorem 1, respectively Theorem 2. The implicit function theorem of Arens and Calderon ([1], [6]) is extended to Taylor's joint spectrum.

The proofs are based on the topological-homological techniques initiated by J. L. Taylor in [14] and presented shortly, in the form used in this paper, in [10]. The reader is refered to [2], [8] and [13] for a more complete description of this theory. Some facts from several complex variables function theory, like Čech cohomology, a Künneth formula and the existence of the envelope of holomorphy for an open subset of $\mathbf{C}^m$, are also used in this paper. No integral representation formulas for analytic functions appear explicitly in the proofs.

It is interesting to remark that the superposition theorem for Taylor's functional calculus (Theorem 2 below) need for the proof a coordinateless argument, by passing from an open subset of $\mathbf{C}^n$ to its envelope of holomorphy, which is a Stein manifold. This may be the substitute for Arens and Calderon's Lemma from the classical spectral theory.

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First of all, let us recall some notation and results from [10]. Let X be an analytic finite dimensional Stein space and let M be a Fréchet $\mathcal{O}(X)$-module. Then there exists a closed subset $\sigma(X, M)$ of X with the following properties:

a) A Stein open subset V of X is disjoint of $\sigma(X, M)$ if and only if

$$\widehat{\mathrm{Tor}}_q^{\mathcal{O}(X)}(\mathcal{O}(V), M) = 0$$

for each $q \geqslant 0$, [10, Proposition 3.a.].

b) If $\mathscr{U}$ is a Stein covering of $\sigma(X, M)$, with finite dimensional nerve, then

$$\widehat{\mathrm{Tor}}_q^{\mathcal{O}(X)}(\mathscr{C}^{\cdot}(\mathscr{U}), M) \cong M \quad \text{for } q = 0,$$

and

$$= 0 \quad \text{for } q > 0,$$

where $\mathscr{C}^{\cdot}(\mathscr{U})$ is the alterning cochains Čech complex. Moreover, in this case there is a structure of Fréchet $\mathcal{O}(X')$-module on M which is compatible with the $\mathcal{O}(X)$--structure, given by the natural map

$$\mathcal{O}(X')\hat{\otimes}_{\mathcal{O}(X)} M \to \widehat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(\mathscr{C}^{\cdot}(\mathscr{U}), M) \cong M \tag{1}$$

where $X' = \bigcup_{U \in \mathscr{U}} U$, [10, Theorem 4]. Moreover, the map (1) does not depend on the covering $\mathscr{U}$ and it is functorial in X'.

If $X = \mathbf{C}^n$, if M is a Banach space, and if the structure of $\mathcal{O}(\mathbf{C}^n)$-module on M is given by a commuting n-tuple of linear bounded operators a, then $\sigma(X, M)$ coincides with Taylor's joint spectrum $\mathrm{Sp}(a, M)$, [14].

THEOREM 1. *Let X be a finite dimensional Stein space, let M be a Fréchet $\mathcal{O}(X)$-module and let $f : X' \to Y$ be a morphism of analytic spaces between an open subset X' of X which contains $\sigma(X, M)$ and a Stein space Y. Then*

$$\overline{f\sigma(X, M)} = \sigma(Y, M)$$

where M is endowed with the structure of Fréchet $\mathcal{O}(Y)$-module given by the map $f^* : \mathcal{O}(Y) \to \mathcal{O}(X')$ and by (1).

Proof. Let us factorize f by its graph:

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & Y \\ {\scriptstyle j}\big\downarrow\cap & & \big\uparrow{\scriptstyle p} \\ X' \times Y & \overset{i}{\hookrightarrow} & X \times Y \end{array} \tag{2}$$

and let us define $Y' = X \times Y$ and $f' = i \circ j$. The conclusion holds for the morphism p, [10, Theorem 5], so it is enough to prove the theorem for f' and Y' instead of f and Y (note that the set $f'\sigma(X, M)$ is closed). Let $\pi : X \times Y \to X$ be the projection map.

For each point $x_0 \in X'$ there exists an open Stein subset V, $x_0 \in V \subset Y'$, such that $(f')^{-1}V = \pi V$ is also Stein. Let $\mathscr{U}$ be a Stein open covering of X, with finite dimensional nerve. For each open subset $U = U_1 \cap \ldots \cap U_p$, $U_k \in \mathscr{U}$, there exists a morphism of complexes

$$B_{\bullet}^{\mathcal{O}(X)}(\mathcal{O}((f')^{-1}V), \mathcal{O}(U)) \to B_{\bullet}^{\mathcal{O}(Y')}(\mathcal{O}(V), \mathcal{O}(U))$$

induced by π, where the $\mathcal{O}(Y')$-structure on $\mathcal{O}(U)$ is given by $(f')^* : \mathcal{O}(Y') \to \mathcal{O}(X')$. By the Künneth formula [10, (1)], $\varphi^U_\bullet$ is a quasiisomorphism. Let denote by $K^U_\bullet$ its cone (which is exact).

Putting together all of $K^U_\bullet$'s one obtains an exact complex of complexes $K^\cdot_\bullet$ whose components are of the form $K_p^\cdot = E \hat{\otimes} \mathcal{C}^\cdot(\mathcal{U})$, where E is a Fréchet space. Hence $K_p^\cdot$ is $-\hat{\otimes}_{\mathcal{O}(X)} M$ acyclic by b), so that we have an exact sequence

$$\ldots \widehat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(K_1^\cdot, M) \to \widehat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(K_0^\cdot, M) \to 0$$

which, by the above construction and by b), is the cone of the morphism

$$B^{\mathcal{O}(X)}_\bullet(\mathcal{O}((f')^{-1}V), M) \to B^{\mathcal{O}(Y')}_\bullet(\mathcal{O}(V), M). \tag{3}$$

Therefore there are the isomorphisms

$$\widehat{\mathrm{Tor}}^{\mathcal{O}(X)}_\bullet(\mathcal{O}((f')^{-1}V), M) \to \widehat{\mathrm{Tor}}^{\mathcal{O}(Y')}_\bullet(\mathcal{O}(V), M). \tag{4}$$

Let us remark that the $\mathcal{O}(Y')$-module structure on M is given via $(f')^* : \mathcal{O}(Y') \to \mathcal{O}(X')$ and via the natural map (1).

The set $f'\sigma(X, M)$ being closed, for each point x of $X' \setminus \sigma(X, M)$ there exists an open Stein neighbourhood V of $f'(x)$ in Y', such that $(f')^{-1}V = \pi V$ is also Stein and disjoint from $\sigma(X, M)$. From (4) and a) we deduce that $V \cap \sigma(Y', M) = \emptyset$.

Conversely, if $x \in X' \setminus (f')^{-1}\sigma(Y', M)$, then there is a Stein open neighbourhood V of $f'(x)$ in Y', such that V is disjoint from $\sigma(Y', M)$. From (4) and a) we have $((f')^{-1}V) \cap \sigma(X, M) = \emptyset$. Therefore the equality

$$f'\sigma(X, M) = \sigma(Y', M) \cap f'(X') \tag{5}$$

holds.

By the functoriality of (1) with respect to X', (5) remains valid if X' is replaced by an open subset X'' of X, such that $X' \supset X'' \supset \sigma(X, M)$. Therefore if we verify that $\sigma(Y', M)$ is contained in $\overline{f'(X')}$, the proof will be complete.

Let V be a Stein open subset of Y' that is disjoint of $f'(X')$. Then, by the Künneth formula [10, (1)],

$$\widehat{\mathrm{Tor}}_q^{\mathcal{O}(Y')}(\mathcal{O}(V), \mathcal{O}(U)) = 0$$

for each $q \geqslant 0$ and $U = U_1 \cap \ldots \cap U_p$, $U_k \in \mathcal{U}$. Thus the complex of complexes

$$B^{\mathcal{O}(Y')}_\bullet(\mathcal{O}(V), \mathcal{C}^\cdot(\mathcal{U})) \tag{6}$$

is exact. The components of (6) are $-\hat{\otimes}_{\mathcal{O}(X)} M$ acyclic. Hence, with the same argument as above,

$$\widehat{\mathrm{Tor}}_q^{\mathcal{O}(Y')}(\mathcal{O}(V), M) = 0$$

for each $q \geqslant 0$, and, by a), $V \cap \sigma(Y', M) = \emptyset$. Q.E.D.

Let us denote M^f the $\mathcal{O}(Y)$-module M obtained from the $\mathcal{O}(X')$ structure via $f^*:\mathcal{O}(Y)\to\mathcal{O}(X')$, as before. While the previous theorem asserts the naturality of $\sigma(X, M)$ with respect to X, the next result proves the naturality of M^f in f, when the underling space is $\mathbf{C}^n$.

THEOREM 2. *Let M be a Fréchet $\mathcal{O}(\mathbf{C}^n)$-module and let $f: U\to\mathbf{C}^m$, $g: V\to\mathbf{C}$ be analytic maps, where $U\subset\mathbf{C}^n$, $V\subset\mathbf{C}^m$ are open subsets and $\sigma(\mathbf{C}^n, M)\subset U\subset f^{-1}V$, $\sigma(\mathbf{C}^m, M^f)\subset V$. Then*

$$(M^f)^g = M^{g\circ f}.$$

Proof. The set $\overline{f\sigma(\mathbf{C}^n, M)}$ is contained in V by Theorem 1, thus $(M^f)^g$ has a good meaning.

One can suppose U to be connected, otherwise M decomposes in a direct sum with respect to the components of U.

Consider the envelope of holomorphy $\tilde U$ of U, which exists by [7, Theorem 5.4.5.] and is a Stein manifold. There is a natural isomorphism $\mathcal{O}(\tilde U)\xrightarrow{\sim}\mathcal{O}(U)$, which induces a structure of Fréchet $\mathcal{O}(\tilde U)$-module on M. Then, using Theorem 1, $\sigma(\tilde U, M) = \sigma(U, M)$, by identifying U with an open subset of $\tilde U$.

Let us consider the following diagram with commutative squares:

$$\begin{array}{ccccccc}
\mathcal{O}(\mathbf{C})\hat\otimes M & \xrightarrow{g^*} & \mathcal{O}(V)\hat\otimes M & \longrightarrow & \widehat{\mathrm{Tor}}_0^{\mathcal{O}(\mathbf{C}^m)}(\mathcal{C}^\cdot(\mathcal{V}), M) & \xleftarrow{\sim} & M\\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \tilde f^*} & & \Big\downarrow{\scriptstyle \tilde f^*} & & \Big\downarrow{\scriptstyle \mathrm{id}}\\
\mathcal{O}(\mathbf{C})\hat\otimes M & \xrightarrow{(g\circ\tilde f)^*} & \mathcal{O}(f^{-1}V)\hat\otimes M & \longrightarrow & \widehat{\mathrm{Tor}}_0^{\mathcal{O}(\tilde U)}(\mathcal{C}^\cdot(f^{-1}\mathcal{V}), M) & \xleftarrow{\sim} & M
\end{array}$$

where $\mathcal{V}$ is an open Stein covering of V, with finite dimensional nerve, $\tilde f^*$ is the morphism induced by $\tilde f^*\otimes\mathrm{id}_M$, and $g\circ\tilde f$ means g composed with the restriction of $\tilde f$ to $\tilde f^{-1}V$. Note that $\tilde f^{-1}\mathcal{V}$ is still an open Stein covering of $\sigma(\tilde U, M)$, with finite dimensional nerve.

The multiplication map given by the first row in the diagram is that of $(M^f)^g$, while the second row gives the $M^{g\circ f}$ structure. Indeed, starting with $h\otimes m$ on the first row, the result is $(h\circ g)\cdot m$, where the product is made in the M^f-structure (that is m multiplied by h in $(M^f)^g$). On the other hand $h\otimes m$ goes on the second row to $(h\circ g\circ\tilde f)\cdot m$, where the product is that of $\mathcal{O}(\tilde U)$-module. Extending the function $h\circ g\circ\tilde f$ to $\tilde U$, we have

$$(h\circ g\circ\tilde f)\cdot m = (h\circ g\circ f)^\sim\cdot m,$$

by the functoriality of the functional calculus with respect to the restriction maps. Further

$$(h\circ g\circ f)^\sim\cdot m = (h\circ g\circ f)\cdot m$$

by definition, the last term being just the product between h and m in the $M^{g,f}$-structure. Q.E.D.

The main application of the superposition property in existence theorems is the implicit function theorem of Arens and Calderon ([1] and [6]). We shall present below a refinement of this theorem.

Let A be a commutative Banach algebra of linear operators on a Banach space M and suppose that the algebra A is closed with respect to the functional calculus considered before. Then there exists a closed subset $\Delta(A, M)$ in the maximal spectrum of A, such that

$$\mathrm{Sp}(a, M) = \hat{a}\, \Delta(A, M)$$

for each $a \in A^n$, $n \geq 1$, [11]. With this notation and assumptions, the following result holds:

THEOREM 3. *Let $a \in A^n$ and let h be a continuous function on $\Delta(A, M)$. Let S be the set $\{(h(x), \hat{a}_1(x), \ldots, \hat{a}_n(x)),\ x \in \Delta(A, M)\}$, considered in $\mathbf{C}^{n+1}$, with the coordinates $w, z_1, \ldots, z_n$.*

If F is an analytic function in a neighbourhood of S such that

$$F(h, \hat{a}_1, \ldots, \hat{a}_n) = 0 \quad \text{on } \Delta(A, M)$$

and

$$\partial F/\partial w \neq 0 \quad \text{on } S,$$

then there is an element b of A which satisfies

$$F(b, a_1, \ldots, a_n) = 0. \tag{8}$$

Moreover, there is at most one $b \in A$ such that $\hat{b} = h$ and such that (8) is fulfilled.

The proof of the existence is the same as the original one [1] and it uses Theorem 2. The uniqueness statement has a slight modified proof:

Let $b, b + c \in A$ which satisfy (8) and $\hat{b} = \hat{b} + \hat{c} = h$. Then $\hat{c} = 0$ on $\Delta(A, M)$. Let $R \in \mathcal{O}(\mathrm{Sp}((c, b, a), M))$ be the analytic function

$$R(\zeta, w, z) = \zeta^{-2}[F(w + \zeta, z) - F(w, z) - \zeta \partial F/\partial w] .$$

By applying the functional calculus to R one obtains

$$c[\partial F/\partial w(b, a) + cR(c, b, a)] = 0. \tag{9}$$

But

$$\mathrm{Sp}(cR(c, b, a) + \partial F/\partial w(b, a): M) =$$

$$= [cR(c, b, a) + \partial F/\partial w(b, a)]^{\wedge} \Delta(A, M) =$$

$$= [\partial F/\partial w(b, a)]^{\wedge} \Delta(A, M) =$$

$$= \mathrm{Sp}(\partial F/\partial w(b, a); M) =$$

$$= \partial F/\partial w\,(\mathrm{Sp}((b, a), M)) =$$

$$= \partial F/\partial w(S)$$

and the last set does not contain zero. Hence the factor which multiplies c in (9) is invertible, thus $c = 0$. Q.E.D.

By using this theorem, all classical corollaries (Silov's idempotent theorem the existence of radicals, logarithms,...) can be adapted for Taylor's joint spectrum

If the functional calculus commutes with the Gelfand transform, then there is exactly one element b whose existence was proved before. We don't know if the uniqueness always holds.

Let us finally remark that the superposition property can be proved also for other spaces endowed with a sheaf of Fréchet nuclear algebras. For example, the superposition holds for manifolds and $\mathscr{C}^{\infty}$-functions on them.

REFERENCES

1. Arens, R. F.; Calderon, A. P., Analytic functions of several Banach algebra elements, *Ann. of Math.*, (2) **62** (1955), 204–216.
2. *Astérisque*, Rev. de la Soc. Math. de France, Séminaire de Géométrie Analytique, 1974.
3. Bănică, C.; Stănăşilă, O., *Méthodes algebriques dans la théorie globale des espaces complexes*, Gauthier-Villars, 1977.
4. Bourbaki, N., *Théories spectrales*, Hermann, 1967.
5. Frisch, J., *Introduction à la géométrie analytique*, Univ. de Pisa, 1971.
6. Gamelin, T., *Uniform algebras*, Prentice Hall, N. J., 1969.
7. Hörmander, L., *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, 1966.
8. Helemskii, A. Ia., Homological methods in the analytic functional calculus of several operators in a Banach space, after Taylor (Russian), *Uspekhi Mat. Nauk.*, **36**: 1(1981), 127–172.
9. Mac Lane, S., *Homology*, Springer, 1963.
10. Putinar, M., Functional calculus with sections of an analytic space, *J. Operator Theory*, **4** (1980), 297–306.
11. Taylor, J. L., A joint spectrum for several commuting operators, *J. Functional Analysis*, **6** (1970), 172–191.

12. TAYLOR, J. L., Analytic functional calculus for several commuting operators, *Acta Math.*, **125**(1970), 1–38.
13. TAYLOR, J. L., Homology and cohomology for topological algebras, *Advances in Math.*, **9** (1972), 131–182.
14. TAYLOR, J. L., A general framework for a multi-operator functional calculus, *Advances in Math.*, **9**(1972), 183–252.
15. VASILESCU, F.-H., *Multi-dimensional analytic functional calculus* (Romanian), Ed. Academiei, București, 1979.

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