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ON EQUIVALENCE OF TOEPLITZ OPERATORS

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Dedicated to Professor Paul R. Halmos on his 65 th birthday.

1. INTRODUCTION

For f in L^∞ of the unit circle ∂D, the Toeplitz operator T_f is the operator on the Hilbert space H^2 of the unit disk D, given by $T_f h = Pfh$, where P is the orthogonal projection of $L^2(\partial D)$ onto H^2. (As usual, we identify H^2 as a subspace of $L^2(\partial D)$ by $\sum_{n=0}^{\infty} a_n z^n \Leftrightarrow \sum_{n=0}^{\infty} a_n e^{in\theta}$ where $\sum |a_n|^2 < \infty$.) Toeplitz operators have been studied widely for the past thirty years, but, with two major exceptions, little seems to be known about the similarity or unitary equivalence of Toeplitz operators. The first of these exceptions is the computation of spectral measure for self-adjoint Toeplitz operators ([12], [13], [9], [10], [11]). The other exception is the striking work of Clark and Morrel on the similarity of Toeplitz operators with rational symbol ([3], [4], [5], [6]). In this paper we give an easy sufficient condition for unitary equivalence, appeal to the work of Rosenblum to show that it is not necessary, and prove a strong converse for some cases in which the symbol f is in H^∞. We recall that u is an inner function if u is in H^2 and $|u(e^{i\theta})| = |\lim_{r\to 1^-} u(re^{i\theta})| = 1$ for almost all θ. (Hoffman's book [8] is a good reference for Hardy spaces and factorization theory.) If u is an inner function, we say the *order of* u is n if u is a finite Blaschke product of order n, otherwise we say the order of u is infinity. We will show (Theorem 1) that if h is in $L^\infty(\partial D)$ and u is inner, then $T_{h\circ u} \cong \bigoplus_n T_h$ where $n =$ order u (possibly infinity). (We will write $\cong$ for unitary equivalence and $\sim$ for similarity.) It follows that if u and v are inner functions of the same order then $T_{h\circ u} \cong T_{h\circ v}$.

If $\mathscr{S}$ is a set of operators, let $\mathscr{S}'$ denote the set of operators that commute with everything in $\mathscr{S}$, and $\mathscr{S}'' = (\mathscr{S}')'$. We will also show (Theorem 2) that if f and g are in H^∞ such that for some u and v inner $\{T_f\}' = \{T_u\}'$ and $\{T_g\}' = \{T_v\}'$ then $T_f \cong T_g$ if and only if $T_f \sim T_g$ if and only if there is h in H^∞ and inner functions,

w_1, w_2, of the same order, such that $f = h \circ w_1$ and $g = h \circ w_2$. As a consequence of this, we derive the same conclusion if g is any function in H^∞ and f is a function in H^∞ with the property that for some α in $\mathbf{C}$, the inner factor of $f - \alpha$ is a non-constant finite Blaschke product.

Theorem 2 is a converse of Theorem 1 in a special case, but the converse of Theorem 1 is not true in general. Rosenblum, [12], shows that if f is real valued on ∂D, $f \in L^\infty$ and for each λ in $\mathbf{R}$, $\Gamma_\lambda = \{e^{i\theta} : f(e^{i\theta}) \leqslant \lambda\}$ is a closed subarc of ∂D (up to a set of measure 0) then T_f is unitarily equivalent to the operator of multiplication by x on $L^2(\mathbf{R}, \mu)$ where $d\mu = \pi^{-1} \sin\left(\frac{1}{2} |\Gamma_\lambda|\right) d\lambda$. Now multiplication by x on $L^2(\mathbf{R}, p_1 d\lambda)$ is unitarily equivalent to multiplication by x on $L^2(\mathbf{R}, p_2 d\lambda)$ if and only if $\{x : 0 < p_1(x) < \infty\} = \{x : 0 < p_2(x) < \infty\}$ (except a set of measure zero). It follows that if f_1 and f_2 are any two functions in L^∞ that satisfy the Rosenblum hypothesis and such that $\sigma(T_{f_1}) = \sigma(T_{f_2})$ then $T_{f_1} \cong T_{f_2}$. For example, let $f_1(e^{i\theta}) = \cos\theta$ and $f_2(e^{i\theta}) = 1$ for $0 < \theta < \pi$ and -1 for $-\pi < \theta < 0$, then $T_{f_1} \cong T_{f_2}$. Certainly there are no functions h in L^∞ and u_1 and u_2 inner such that $f_j = h \circ u_j$.

Moreover, Clark and Morrel ([6]) show that if $f_1(z) = 2z + z^{-1}$ and f_2 is the conformal map of D onto the interior of the ellipse $f_1(\partial D)$, then $T_{f_1} \sim T_{f_2}$, but since T_{f_2} is subnormal and T_{f_1} is not (see [1]), $T_{f_1} \not\cong T_{f_2}$. Thus the equivalence of similarity and unitary equivalence is a special feature of the self-adjoint case and some analytic cases.

Douglas Clark has pointed out that Theorem 1 and a result from [5] provide an example of a non-normal Toeplitz operator with a discontinuous symbol that is similar to a normal operator. Let α be a non-zero point in the open disk, and let $f(z) = z^{-1}(z - \alpha)(1 - \bar{\alpha}z)^{-1}$. By [5], T_f is similar to a normal operator, N. If u is an inner function of infinite order, then $T_{f \circ u} \cong \bigoplus_1^\infty T_f \sim \bigoplus_1^\infty N$. I would like to thank both Professor Clark and the referee for making helpful suggestions.

2. THE RESULTS

We first make a few observations about $h \circ u$ when h is in $L^\infty(\partial D)$ and u is inner. If h has Fourier series $\sum_{-\infty}^{\infty} a_n e^{in\theta}$ and P_N is the N^{th} Cesaro mean of h (i.e. $P_N(e^{i\theta}) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} a_k e^{ik\theta}$) then $P_N \to h$ and $P_N \circ u \to h \circ u$ in the weak-star topology on $L^\infty(\partial D)$. This means that $T_{P_N} \to T_h$ and $T_{P_N \circ u} \to T_{h \circ u}$ in the weak operator topology [14, p. 294].

THEOREM 1. *Suppose h is in $L^\infty(\partial D)$ and u is an inner function of order n (where n is a positive integer or ∞). Then*

$$T_{h\circ u} \cong \bigoplus_n T_h .$$

COROLLARY. *If h is in $L^\infty(\partial D)$ and u and v are inner functions of the same order, then $T_{h\circ u} \cong T_{h\circ v}$.*

Proof. The order of u is the dimension of the orthogonal complement of uH^2 in H^2 : let $\{w_k\}_{k=1}^n$ be an orthonormal basis for $(uH^2)^\perp$. Then $\{w_k u^m\}_{m=0}^{\infty}{}_{k=1}^{n}$ is an orthonormal basis for H^2. Let $U : \bigoplus_n H^2 \to H^2$ be the unitary operator given by

$$U \sum_{k=1}^{n} \oplus \left(\sum_{m=0}^{\infty} b_{km} e^{im\theta} \right) = \sum_{k=1}^{n} \sum_{m=0}^{\infty} b_{km} w_k u^m$$

where $\sum \sum |b_{km}|^2 < \infty$.

Now since $U^{-1}A_jU \to U^{-1}BU$ in the weak operator topology if and only if $A_j \to B$ in the weak operator topology, by the remarks preceeding the theorem, we may assume h is a trigonometric polynomial, say $h(e^{i\theta}) = \sum_{-l}^{l} a_j e^{ij\theta}$, so $h\circ u = \sum_{-l}^{l} a_j u^j$.

Now

$$T_{h\circ u}\left(\sum_{k=1}^{n} \sum_{m=0}^{\infty} b_{km} w_k u^m \right) = P\left(\sum_{j=-l}^{l} a_j u^j \right)\left(\sum_{m=0}^{\infty} \sum_{k=1}^{n} b_{km} w_k u^m \right) =$$

$$= P\left(\sum_{m=0}^{\infty} \sum_{k=1}^{n} \sum_{j=-l}^{l} a_j b_{km} w_k u^{m+j} \right) = \sum_{m=0}^{\infty} \sum_{k=1}^{n} \sum_{\substack{j=-l \\ j\geqslant -m}}^{l} b_{km} a_j w_k u^{m+j}$$

and for each k,

$$T_h\left(\sum_{m=0}^{\infty} b_{km} e^{im\theta} \right) = P\left(\sum_{m=0}^{\infty} \sum_{j=-l}^{l} b_{km} a_j e^{i(m+j)\theta} \right) = \sum_{m=0}^{\infty} \sum_{\substack{j=-l \\ j\geqslant -m}}^{l} b_{km} a_j e^{i(m+j)\theta}.$$

That is, $T_{h\circ u}Ux = U\left(\sum_{k=1}^{n} \oplus T_h x_k \right)$ for all $x = (x_1, x_2, \ldots)$ in $\bigoplus_n H^2$, and $T_{h\circ u} \cong \bigoplus_n T_h$.

The corollary follows immediately since, when u and v each have order n, $T_{h\circ u} \cong \bigoplus_n T_h \cong T_{h\circ v}$. ▨

We now state and prove a strong converse for certain analytic Toeplitz operators. The reader may consult [7] or [2] for information on commutants of analytic Toeplitz operators.

THEOREM 2. *Suppose f and g are in H^∞ and there are inner functions u and v such that $\{T_f\}' = \{T_u\}'$ and $\{T_g\}' = \{T_v\}'$. Then the following are equivalent:*

(1) $T_f \cong T_g$,

(2) $T_f \sim T_g$,

(3) *There are functions h in H^∞ and w_1, w_2 inner such that $f = h \circ w_1$, $g = h \circ w_2$ and* order $w_1 =$ order w_2.

Proof. (3) implies (1) is a special case of Theorem 1, and (1) implies (2) holds a priori, so we only need to prove (2) implies (3).

By hypothesis, there are inner functions u and v so that $\{T_f\}' = \{T_u\}'$ and $\{T_g\}' = \{T_v\}'$. We claim that u and v have the same order: let $n =$ order u and $m =$ order v. From Theorems 1 and 2 of [2] (pages 4 and 5), we have $\{T_f\}'' = \{T_u\}'' = \{T_{h\circ u} : h \in H^\infty\}$ and $\{T_g\}'' = \{T_{h\circ v} : h \in H^\infty\}$. Now $\ker(T^*_{h\circ u})$ is $(\varphi H^2)^\perp$ where φ is the inner factor of $h \circ u$. Since $\varphi = \tilde{\varphi} \circ u$ where $\tilde{\varphi}$ is the inner factor of h ([16], Theorem 1.2, page 260), either $\dim(\ker T^*_{h\circ u}) = 0$ or $\dim(\ker T^*_{h\circ u}) =$ order $\tilde{\varphi} \geqslant$ order $u = n$. Since $T_u \in \{T_f\}''$ and $\dim(\ker T^*_u) = n$, we have

$$n = \min\{\dim(\ker A^*) : \ker A^* \neq (0) \text{ and } A \in \{T_f\}''\}.$$

Similarly

$$m = \min\{\dim(\ker B^*) : \ker B^* \neq (0) \text{ and } B \in \{T_g\}''\}.$$

Let S be an operator that implements the similarity of T_f and T_g, say $S^{-1}T_fS = T_g$. It follows that A is in $\{T_f\}''$ if and only if $S^{-1}AS$ is in $\{T_g\}''$ and, since $\dim(\ker S^*A^*S^{*-1}) = \dim(\ker A^*)$, we have $n = m$.

Now S induces an algebra isomorphism $J : H^\infty \circ u \to H^\infty \circ v$ given by $J(h \circ u) = \tilde{h} \circ v$ where $T_{\tilde{h}\circ v} = S^{-1}T_{h\circ u}S$. Let $w = J(u)$. Since u is a sequential weak-star generator of $H^\infty \circ u$, w is a sequential weak-star generator of $H^\infty \circ v$. Moreover, since $T_w = S^{-1}T_uS$, we have $\sigma(w) = \sigma(T_w) = \sigma(T_u) = \sigma(u) = \overline{u(D)} = \bar{D}$ (where $\sigma(\varphi)$ is the spectrum of φ as an element of the normed algebra H^∞). Thus $\overline{w(D)} = \bar{D}$.

Thus there are polynomials P_k so that $P_k(w) \to v$ and since $w = q \circ v$, we have $P_k \circ q(v) \to v$ (weak-star). Since v is an inner function this means $P_k \circ q \to z$ so that q is a sequential weak-star generator of H^∞, and $\overline{q(D)} = \overline{q(v(D))} = \overline{w(D)}$. It follows from [15] (pages 520 and 521) that this means q is a Möbius transformation of D onto D, and that w is an inner function of order n.

Since T_f is in $\{T_f\}''$, there is h in H^∞ such that $f = h \circ u$. Since J is weak-star continuous we have $Jf = Jh \circ u = h \circ w$, but we have $S^{-1}T_fS = T_g$ so $Jf = g$. Therefore, we have $f = h \circ u$ and $g = h \circ w$ where h is in H^∞ and order $u =$ order w so the conclusion follows with $w_1 = u$ and $w_2 = w$. ▨

We now obtain two corollaries, the first of which replaces the commutant hypotheses on T_f and T_g by a factorization hypothesis on f.

COROLLARY 1. *Suppose f and g are in H^∞ and for some α in* **C** *the inner factor of $f - \alpha$ is a non-constant finite Blaschke product. Then the conclusion of Theorem 2 holds.*

Proof. As before we only need prove (2) implies (3). Let S be an invertible operator with $S^{-1}T_f S = T_g$. Since the kernel of $T^*_{f-\alpha}$ is $[(f-\alpha)H^2]^\perp$ which is finite dimensional by hypothesis, we have $[(g-\alpha)H^2]^\perp = \ker(T^*_{g-\alpha}) = S^*(\ker T^*_{f-\alpha})$ is also finite dimensional and the inner factor of $g - \alpha$ is also a finite Blaschke product. The corollary of Theorem 5 of [2] (page 19) says that there are finite Blaschke products u and v such that $\{T_f\}' = \{T_u\}'$ and $\{T_g\}' = \{T_v\}'$. We now apply Theorem 2.

COROLLARY 2. *If m_1 and m_2 are positive integers and f and g satisfy the hypothesis of Theorem 2 or Corollary 1, then the following are equivalent:*

(1) $\bigoplus_{m_1} T_f \cong \bigoplus_{m_2} T_g$;

(2) $\bigoplus_{m_1} T_f \sim \bigoplus_{m_2} T_g$;

(3) *There are functions h in H^∞ and w_1, w_2 inner such that $f = h \circ w_1$, $g = h \circ w_2$, and $m_1 \cdot \mathrm{order}\, w_1 = m_2 \cdot \mathrm{order}\, w_2$.*

Proof. Let u be a finite Blaschke product of order m_1 and v be a finite Blaschke product of order m_2. By Theorem 1, $\bigoplus_{m_1} T_f \cong T_{f\circ u}$ and $\bigoplus_{m_2} T_g \cong T_{g\circ v}$ and $f\circ u$ and $g \circ v$ satisfy the same hypotheses as do f and g. ▨

3. CONCLUSION

Clearly this paper is only a start on the equivalence problem for Toeplitz operators. It does, however, suggest pertinent questions to be considered. First, for which operators does the converse to Theorem 1 hold? Does it hold for all non-normal Toeplitz operators? Second, for which analytic Toeplitz operators is similarity the same as unitary equivalence? Does the conclusion of Theorem 2 follow from just f, g in H^∞? Resolution of these questions will probably further illuminate the structure of Toeplitz operators.

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