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GEOMETRICAL MEANS OF EIGENVALUES

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1. INTRODUCTION

Let T be a bounded linear operator on a complex Banach space X. The spectrum of T will be denoted by $\sigma(T)$, and the spectral radius by $|T|_\sigma$. By the essential spectrum we mean the spectrum of T modulo the compact operators. The essential spectral radius $|T|_\omega$ cuts the spectrum $\sigma(T)$ into two parts. We shall be interested in the outer part

$$\Lambda(T) = \{\lambda \in \sigma(T) : |\lambda| > |T|_\omega\}.$$

It is well-known that this set is at most countable and consists of isolated eigenvalues of finite multiplicity. We refer to the introduction of the paper [6] for a more thorough explanation of these facts. So we can denote the eigenvalues in $\Lambda(T)$ in such a way that

$$|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \ldots ,$$

where each eigenvalue is counted according to its multiplicity (which is the dimension of the range of the corresponding spectral projection). If there are only n $(= 0, 1, 2, \ldots)$ such eigenvalues, including multiplicities, we put formally

$$|\lambda_{n+1}(T)| = |\lambda_{n+2}(T)| = \ldots = |T|_\omega.$$

Let U be the unit ball of X. For each $n = 1, 2, \ldots$ let $e_n(T)$ be the infimum of all $\varepsilon > 0$ such that the set $T(U)$ can be covered by n balls of radius ε, with centres arbitrary in X. These are the *entropy numbers* of A. Pietsch [3], p. 168. It was noted in [6], Corollary 3.6 that for each fixed $n = 1, 2, \ldots$ it holds

$$|T|_\sigma = \lim_{N\to\infty} e_n(T^N)^{1/N}. \tag{1}$$

Recently B. Carl [1] has defined $g_n(T) = \inf_k k^{1/2n}\, e_k(T)$, and called it the *n-th entropy modulus* of T. A natural geometric interpretation of these quantities comes from considering the n-dimensional (complex) volume of the coverings of the set

$T(U)$ by finite number of balls of equal radii. He also established in [1] some algebraic properties of these quantities, among them the submultiplicativity property $g_n(ST) \leqslant g_n(S)g_n(T)$ valid for each $n = 1, 2, \ldots$ and every S, T (this is easily verified directly from the definition of g_n). It is then a matter of algebraic calculation to see that the limits

$$G_n(T) = \lim_{N\to\infty} g_n(T^N)^{1/N}$$

exist for each fixed $n = 1, 2, \ldots$. The purpose of this paper is to prove that

$$|\lambda_1(T) \ldots \lambda_n(T)|^{1/n} = G_n(T). \tag{2}$$

Before embarking on our proof some partial results concerning (2) should be recorded. First of all, the inequality

$$|\lambda_1(T) \ldots \lambda_n(T)|^{1/n} \leqslant g_n(T) \tag{3}$$

was discovered for compact operators by B. Carl [1] using a similar argument as in Proposition 1 of B. Carl and H. Triebel [2]. Next, (3) was extended to an arbitrary operator T in [6], Proposition 3.5. Thus we already know that the inequality "$\leqslant$" in (2) holds, and the proof of it will not be repeated here. (In fact, it is just this result that yields (1) immediately.) Knowing these facts formula (2) was raised up as a conjecture in a conversation with B. Carl (May 1980) and was quoted also in [6]. Later, the problem was discussed in [7] and the converse inequality "$\geqslant$" in (2) was shown to hold in some special cases. The present improvement consists in refining the ball coverings by suitable parallelotope lattices. This geometric idea makes it possible to settle the problem in general. The details are done in the next section.

Note that both (1) and (2) can be regarded as generalizations of the classical Beurling-Gelfand formula for the spectral radius. Other related results can be found in [6] or [7].

2. THE MAIN RESULT

THEOREM. *Let T be a bounded linear operator on a complex Banach space X. Then*

$$|\lambda_1(T) \ldots \lambda_n(T)|^{1/n} = \lim_{N\to\infty} g_n(T^N)^{1/N} \tag{4}$$

holds for each fixed $n = 1, 2, \ldots$.

Proof. As explained in the preceding section it is enough to prove the inequality "$\geqslant$" in (4). For simplicity we shall write $\lambda_j = \lambda_j(T)$. If $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n|$ then the desired inequality is obvious by $g_n(T) \leqslant e_1(T)$ and (1).

So we may suppose there is an index $m < n$ such that

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_m| > |\lambda_{m+1}| = |\lambda_{m+2}| = \ldots = |\lambda_n| .$$

Let P be the spectral projection of T corresponding to the set $\{\lambda_1, \ldots, \lambda_m\} = \{\lambda \in \Lambda(T) : |\lambda| > |\lambda_n|\}$. Then $PT = TP$. Let $Y = P(X)$. So $\dim Y = m$ and Y is invariant under T. Let $T_Y = T|Y$. Since $T^N = PT^N + (I - P)T^N$ we have by the lipschitzian continuity of the entropy numbers ([3], p. 168)

$$e_k(T^N) \leqslant e_k(PT^N) + \|(I - P)T^N\|,$$

and then as in the proof of Theorem 3.3 in [7] we conclude that

$$\begin{aligned} g_n(T^N) &\leqslant k^{1/2n} e_k(PT^N) + k^{1/2n} \|(I - P)T^N\| \leqslant \\ &\leqslant k^{1/2n} \|P\| e_k(T_Y^N) + k^{1/2n} \|(I - P)T^N\| \end{aligned} \tag{5}$$

for each k, $N = 1, 2, \ldots$. This estimate reduces the problem to a finite dimensional situation.

Let $\varepsilon > 0$ be given. Since $|(I - P)T|_\sigma = |\lambda_n|$ we have for $N \geqslant N_\varepsilon$

$$\|(I - P)T^N\| \leqslant (|\lambda_n| + \varepsilon)^N. \tag{6}$$

Now we estimate $k^{1/2n} e_k(T_Y^N)$ from above, for some appropriately chosen k. Let $v_1, \ldots, v_m$ be a basis of Y relative to which T_Y has a Jordan form with the diagonal $\{\lambda_1, \ldots, \lambda_m\}$. Consider Y as a $2m$-dimensional real linear space with the basis $v_1, iv_1, \ldots, v_m, iv_m$. Let C be a real parallelotope with edges parallel to these axes. Consider a lattice of translates of the parallelotope $(|\lambda_n| + \varepsilon)^N C$, with translation vectors integer linear combinations of the edge vectors.

Let $Y = \oplus Y_j$ be the decomposition of Y according to the Jordan blocks, the (complex) dimension of Y_j being d(j). Analogously write $C = \oplus C_j$, where C_j is a parallelotope in Y_j. Let $T_j = T_Y|Y_j$, and $\lambda_{(j)}$ be the eigenvalue in the j-th block. The N-th power of T_Y can be calculated blockwise, and

$$\|T_j^N\| \leqslant (|\lambda_{(j)}| + \varepsilon)^N \quad \text{for } N \geqslant N_\varepsilon .$$

Let U_Y and U_j be the unit balls in Y and Y_j, respectively. Then $U_Y \subset \text{const} \cdot \oplus U_j$. (We denote by const a quantity independent of N. Here const $= \max_j \|P_j\|$, where P_j is the projection in Y sending the parallelotope C onto C_j.) Hence

$$T_Y^N(U_Y) \subset \text{const} \cdot \oplus T_j^N(U_j).$$

Moreover, as noted above $T_j^N(U_j) \subset (|\lambda_{(j)}| + \varepsilon)^N U_j$. So the translates of $(|\lambda_n| + \varepsilon)^N C_j$ in the lattice, which intersect const $\cdot T_j^N(U_j)$, surely lie in a ball of radius

$$\text{const} \cdot (|\lambda_{(j)}| + \varepsilon)^N + \text{const} \cdot (|\lambda_n| + \varepsilon)^N \leqslant \text{const} \cdot (|\lambda_{(j)}| + \varepsilon)^N .$$

(by $|\lambda_{(j)}| \geqslant |\lambda_n|$). Since the interiors of these translates are not overlapping the number of them does not exceed

$$\frac{\text{const}\cdot(|\lambda_{(j)}|+\varepsilon)^{2d(j)N}V(U_j)}{(|\lambda_n|+\varepsilon)^{2d(j)N}V(C_j)}$$

(V denoting the real volume). Thus $T_Y^N(U_Y)$ can be covered by

$$k \leqslant \text{const}\cdot\frac{[(|\lambda_1|+\varepsilon)\dots(|\lambda_m|+\varepsilon)]^{2N}}{(|\lambda_n|+\varepsilon)^{2mN}}$$

translates of $(|\lambda_n|+\varepsilon)^N C$, hence (since $C \subset \text{const}\cdot U_Y$) also by

$$(7)\quad k \leqslant \text{const}\cdot\frac{[(|\lambda_1|+\varepsilon)\dots(|\lambda_m|+\varepsilon)]^{2N}}{(|\lambda_n|+\varepsilon)^{2mN}} = \text{const}\cdot\frac{[(|\lambda_1|+\varepsilon)\dots(|\lambda_n|+\varepsilon)]^{2N}}{(|\lambda_n|+\varepsilon)^{2nN}}$$

translates of $\text{const}\cdot(|\lambda_n|+\varepsilon)^N U_Y$ (by $|\lambda_{m+1}| = \dots = |\lambda_n|$). It follows that for this k we have

$$(8)\qquad e_k(T_Y^N) \leqslant \text{const}\cdot(|\lambda_n|+\varepsilon)^N.$$

Using (5), (6) and the k from (7) and (8) we arrive at

$$g_n(T^N) \leqslant \text{const}\cdot[(|\lambda_1|+\varepsilon)\dots(|\lambda_n|+\varepsilon)]^{N/n},$$

hence

$$G_n(T) \leqslant [(|\lambda_1|+\varepsilon)\dots(|\lambda_n|+\varepsilon)]^{1/n}.$$

We finish the proof letting $\varepsilon \to 0$.

REMARK. For X a real Banach space and $T \in B(X)$ let $X^c = X + iX$ be the complexification with $\|x+iy\| = \inf\{\sum |c_\alpha|\cdot\|x_\alpha\| : c_\alpha \text{ complex}, x_\alpha \in X, x + iy = \sum c_\alpha x_\alpha \text{ (finite sum)}\}$, $x, y \in X$. Note that this norm lies between $\max(\|x\|, \|y\|)$ and $\|x\|+\|y\|$, and obviously $\|x+iy\| = \|x-iy\|$. Let $T^c \in B(X^c)$ be the extension of T. Then $\|T\| \leqslant \|T^c\| \leqslant 2\|T\|$. As usual we define $\sigma(T) = \sigma(T^c)$ and then $\Lambda(T) = \Lambda(T^c)$. Also the quantities $e_k(T)$, $g_n(T)$, and $G_n(T)$ can be defined for the real operator T analogously as before, only replacing the exponent $1/2n$ in the definition of g_n by $1/n$ (for the real volume). Then

$$g_n(T^c) = \inf_k k^{1/2n} e_k(T^c) \leqslant \inf_k k^{1/n} e_{k^2}(T^c) \leqslant 2g_n(T).$$

So by Theorem we get

$$|\lambda_1(T)\dots\lambda_n(T)|^{1/n} = G_n(T^c) \leqslant G_n(T).$$

Also the converse inequality $G_n(T) \leqslant |\lambda_1(T)\dots\lambda_n(T)|^{1/n}$ holds. To see this let us note first that $\sigma(T^c)$ is symmetric with respect to the real axis (this is easily verified by a straightforward calculation). Moreover, for the eigenvalues in $\Lambda(T^c)$ we even have a symmetry between the corresponding spectral subspaces. More precisely, a vector $x + iy$ $(x, y \in X)$ belongs to the spectral subspace corresponding to an eigenvalue λ if and only if $x - iy$ belongs to the spectral subspace of $\bar{\lambda}$ (by a criterion given in [4], p. 418). It follows that the spectral projection $P \in B(X^c)$ corresponding to the eigenvalues of T^c greater in absolute value than $|\lambda_n|$ is of the form $P = Q^c$ where $Q \in B(X)$, $Q^2 = Q$. Now consider the part of T^c restricted to $P(X^c)$. This operator will have a Jordan form relative to same base $\{v_\alpha\}$ which (in view of the above mentioned symmetry) can be supposed to contain with $v' + iv''$ also $v' - iv''$ ($v', v'' \in X$ and v'' being zero if the eigenvalue in the corresponding block is real). Then

$$\|(T^c)^N(v'_\alpha \pm iv''_\alpha)\| \leqslant (|\lambda_\alpha| + \varepsilon)^N \quad \text{for } N \geqslant N_\varepsilon.$$

Hence $\|T^N v'_\alpha\|$ and $\|T^N v''_\alpha\|$ are $\leqslant (|\lambda_\alpha| + \varepsilon)^N$. Since the vectors $\{v'_\alpha, v''_\alpha\}$ form a real base in $Y = Q(X)$, the above proof applies.

3. SOME COMMENTS

1. Formula (4) shows that the estimate (3) is asymptotically a good one (like the classical estimate $|T|_\sigma \leqslant \|T\|$).

2. If $ST = TS$ then the inequality $|\lambda_j(ST)| \leqslant |\lambda_j(S)| \cdot |\lambda_j(T)|$ is true with $j = 1$ but matrices like

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

show that it does not longer hold for $j \geqslant 2$. The true inequalities were found in [6], Theorem 5.1. The above Theorem yields immediately another result of this kind.

COROLLARY. *If* $ST = TS$ *then*

$$|\lambda_1(ST) \ldots \lambda_n(ST)| \leqslant |\lambda_1(S) \ldots \lambda_n(S)| \cdot |\lambda_1(T) \ldots \lambda_n(T)|$$

holds for each $n = 1, 2, \ldots$.

Note that this as well as analogous additive inequalities can also be derived from Lemma 5.3 of [6].

3. In view of the importance of the entropy moduli it would be interesting to know if g_n are continuous on $B(X)$. It is only known that $g_1(T) = \|T\|$ but for $n > 1$ these functions are not lipschitzian even if X is a Hilbert space [1].

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