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COMPLETING MATRIX CONTRACTIONS

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The aim of this note is to describe all solutions of the following problem:

(*) Let $H = H_1 \oplus H_2$, $K = K_1 \oplus K_2$ be Hilbert spaces and $A \in \mathscr{L}(H_1, K_1)$, $B \in \mathscr{L}(H_2, K_1)$, $C \in \mathscr{L}(H_1, K_2)$ such that $A_r = (A, B)$ and $A_c = \begin{pmatrix} A \\ C \end{pmatrix}$ be contractions. Find all $X \in \mathscr{L}(H_2, K_2)$ such that $\tilde{A} = \begin{pmatrix} A & B \\ C & X \end{pmatrix}$ be a contraction.

Applications of this labelling to dual pairs of subspaces in a Kreĭn space, to dual pairs of accretive operators, as well as to extensions of positive unbounded operators are given. Thanks are due to Zoia Ceaușescu for helpful discussions on the subject of this paper.

1. MAIN THEOREM

We consider complex Hilbert spaces and we denote by $\mathscr{L}(H, K)$ the set of all (linear bounded) operators from the Hilbert space H into the Hilbert space K. For $T \in \mathscr{L}_1(H, K)$ (i.e. T is a contraction, that is $\|T\| \leqslant 1$) let $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = \overline{D_T(H)}$ be the defect operator, respectively the defect space of T. We shall use the following result, which is proved in this form in [5], Lemma 1.1.2.

LEMMA 1.1. *Let H and K be Hilbert spaces, and suppose that $H = H_1 \oplus H_2$ and $T_1 \in \mathscr{L}(H_1, K)$. Then the formula*

$$T = (T_1, D_{T_1^*}\Gamma) \tag{1.1}$$

establishes a one-to-one correspondence between all $T \in \mathcal{L}_1(H, K)$ *such that* $T|H_1 = T_1$ *and all* $\Gamma \in \mathcal{L}_1(H_2, \mathcal{D}_{T_1^*})$. *Moreover, the operators*

$$(1.2)\qquad \begin{cases} Z(T_1; T) = Z : \mathcal{D}_{T_1} \oplus \mathcal{D}_\Gamma \to \mathcal{D}_T \\ Z(D_{T_1} \oplus D_\Gamma) = (D_T|H_1) \oplus (P^{\mathcal{D}_T}_{\mathcal{D}_T \ominus \overline{D_T(H_1)}} D_T|H_2) \end{cases}$$

$$(1.3)\qquad \begin{cases} Z_*(T_1; T) = Z_* : \mathcal{D}_{\Gamma^*} \to \mathcal{D}_{T^*} \\ Z_*(D_{\Gamma^*} D_{T_1^*}) = D_{T^*} \end{cases}$$

are unitary operators.

(For $H_0 \subset H$, $P^H_{H_0}$ denotes the orthogonal projection of H onto H_0.) Let us note that the correspondence described in (1.1) first appeared in [7] and that formulas similar to (1.2) and (1.3) were used in [6] and [3].

In the proof of (1.1) a well known result on the factorization is used (namely $A^*A \leqslant B^*B$ if and only if $A = CB$, where C is a contraction; see for example [9], Theorem 1). For the reader's convenience we repeat here the proof from [5] for the formula (1.2). We will prove that

$$(1.2)' \qquad \|D_{T_1}h_1\| = \|D_T h_1\|, \qquad \text{for every } h_1 \in H_1,$$

and that

$$(1.2)'' \qquad \|D_\Gamma h_2\| = \|PD_T h_2\|, \qquad \text{for every } h_2 \in H_2$$

(where we put $P = P^{\mathcal{D}_T}_{\mathcal{D}_T \ominus \overline{D_T(H_1)}}$), which will imply (1.2). We have

$$\|D_T h_1\|^2 = \|h_1\|^2 - \|Th_1\|^2 = \|h_1\|^2 - \|T_1 h_1\|^2 = \|D_{T_1} h_1\|^2$$

for every $h_1 \in H_1$, which is (1.2)$'$. Now if $h \in H$ is written as $h = h_1 \oplus h_2$, with $h_1 \in H_1$ and $h_2 \in H_2$, then

$$\|D_T h\|^2 = \|h\|^2 - \|Th\|^2 = \|h_1\|^2 + \|h_2\|^2 - \|T_1 h_1 + D_{T_1^*}\Gamma h_2\|^2 =$$

$$= \|h_1\|^2 + \|h_2\|^2 - \|T_1 h_1\|^2 - \|\Gamma h_2\|^2 + \|T_1^*\Gamma h_2\|^2 - 2\mathrm{Re}\langle T_1 h_1, D_{T_1^*}\Gamma h_2\rangle =$$

$$= \|D_{T_1} h_1\|^2 + \|D_\Gamma h_2\|^2 + \|T_1^*\Gamma h_2\|^2 - 2\mathrm{Re}\langle D_{T_1} h_1, T_1^*\Gamma h_2\rangle =$$

$$= \|D_\Gamma h_2\|^2 + \|D_{T_1} h_1 - T_1^*\Gamma h_2\|^2,$$

where we used the fact that $D_{T_1^*} T_1 = T_1 D_{T_1}$ (cf. [17], Ch. I. (3.4)).

This formula implies that

$$\|PD_T h_2\|^2 = \inf_{h'_1 \in H_1} \|D_T h_2 + D_T h'_1\|^2 =$$

$$= \inf_{h'_1 \in H_1} \{\|D_\Gamma h_2\|^2 + \|D_{T_1} h'_1 - T_1^* \Gamma h_2\|^2\}.$$

But $T_1^*(\mathscr{D}_{T*}) \subset \mathscr{D}_{T_1}$, so $\inf_{h'_1 \in H_1} \|D_{T_1} h'_1 - T_1^* \Gamma h_2\| = 0$. It follows that

$$\|PD_T h_2\| = \|D_\Gamma h_2\|,$$

for every $h_2 \in H_2$, which is exactly (1.2)″.

Coming back to the problem (∗), we have from Lemma 1.1 that

(1.4) $$B = D_{A^*}\Gamma_1, \qquad \text{where } \Gamma_1 \in \mathscr{L}_1(H_2, \mathscr{D}_{A^*}),$$

and

(1.5) $$C = \Gamma_2 D_A, \qquad \text{where } \Gamma_2 \in \mathscr{L}_1(\mathscr{D}_A, K_2).$$

Moreover, the operators

$$Z_r = Z(A; A_r) : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \to \mathscr{D}_{A_r},$$

$$Z_{*r} = Z_*(A, A_r) : \mathscr{D}_{\Gamma_1^*} \to \mathscr{D}_{A_r^*},$$

$$Z_c = Z(A^*; A_c^*) : \mathscr{D}_{A^*} \oplus \mathscr{D}_{\Gamma_2^*} \to \mathscr{D}_{A_r^*}$$

and

$$Z_{*c} = Z_*(A^*; A_c^*) : \mathscr{D}_{\Gamma_2} \to \mathscr{D}_{A_c}$$

are unitary operators.

We need the following

LEMMA 1.2. *The operator* $D_{A_c^*} Z_c : \mathscr{D}_{A^*} \oplus \mathscr{D}_{\Gamma_2^*} \to K_1 \oplus K_2$ *has the matrix*

$$\begin{pmatrix} D_{A^*} & 0 \\ -\Gamma_2 A^* & D_{\Gamma_2^*} \end{pmatrix}.$$

Proof. First we have

$$D_{A_c^*} Z_c D_{A^*} = D_{A_c^*}^2 | K_1 = \begin{pmatrix} D_{A^*}^2 \\ -\Gamma_2 D_A A^* \end{pmatrix} = \begin{pmatrix} D_{A^*} \\ -\Gamma_2 A^* \end{pmatrix} D_{A^*},$$

which proves that the first column in the above matrix is correct. Put now $Q = P^{\mathscr{D}_{A_c^*}}_{\mathscr{D}_{A_c^*} \ominus \overline{D_{A_c^*}(K_1)}}$; we have

$$\langle D_{A_c^*} Z_c D_{\Gamma_2^*} k_2, \; k_1' \oplus k_2' \rangle = \langle Z_c D_{\Gamma_2^*} k_2, \; Q D_{A_c^*}(k_1' \oplus k_2') \rangle =$$

$$= \langle D_{\Gamma_2^*} k_2, \; Z_c^* Q D_{A_c^*} k_2' \rangle = \langle D_{\Gamma_2^*} k_2, \; D_{\Gamma_2^*} k_2' \rangle =$$

$$= \langle D_{\Gamma_2^*}^2 k_2, \; k_1' \oplus k_2' \rangle ,$$

for any $k_1' \in K_1$ and $k_2, k_2' \in K_2$ (where we wrote k_2' for $0 \oplus k_2' \in K_1 \oplus K_2$). This implies that $D_{A_c^*} Z_c D_{\Gamma_2^*} = D_{\Gamma_2^*}^2$, which completes the proof of the lemma.

Our main result is the following:

THEOREM 1.3. *The formula*

$$X = -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} . \tag{1.6}$$

establishes a one-to-one correspondence between all operators $X \in \mathscr{L}(H_2, K_2)$ *such that* $\tilde{A} = \begin{pmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & X \end{pmatrix}$ *is a contraction, and all* $\Gamma \in \mathscr{L}_1(\mathscr{D}_{\Gamma_1}, D_{\Gamma_2^*})$. *Moreover,* $\mathscr{D}_{\tilde{A}}$ *can be identified with* $\mathscr{D}_{\Gamma_2} \oplus \mathscr{D}_{\Gamma}$ *and* $\mathscr{D}_{\tilde{A}^*}$ *can be identified with* $\mathscr{D}_{\Gamma_1^*} \oplus \mathscr{D}_{\Gamma^*}$.

Proof. We apply Lemma 1.1 for $T_1 = A_c$. It follows that there exists a one-to-one correspondence between the contractions $\tilde{A} \in \mathscr{L}_1(H, K)$ with $\tilde{A} | H_1 = A_c$ and all the contractions $\Gamma_c \in \mathscr{L}_1(H_2, \mathscr{D}_{A_c^*})$. This correspondence is given by

$$\tilde{A} = (A_c, \; D_{A_c^*} \Gamma_c) . \tag{1.7}$$

Moreover, $Z^*(A_c; \tilde{A})$ and $Z_*^*(A_c; \tilde{A})$ are unitary operators between $\mathscr{D}_{\tilde{A}}$ and $\mathscr{D}_{A_c} \oplus \mathscr{D}_{\Gamma_c}$, respectively between $\mathscr{D}_{\tilde{A}^*}$ and $\mathscr{D}_{\Gamma_c^*}$. We have the following supplementary condition

$$D_{A^*}\Gamma_1 = P^K_{K_1} D_{A_c^*} \Gamma_c . \tag{1.8}$$

Using the definition of Z_c we have that

$$(Z_c | \mathscr{D}_{A^*}) D_{A^*} = D_{A_c^*} | K_1 .$$

If we denote by $\Gamma_c' = Z_c^* \Gamma_c$, the relation (1.8) becomes

$$\Gamma_1^* D_{A^*} = \Gamma_c'^* Z_c^* D_{A_c^*} | K_1 = (\Gamma_c'^* | \mathscr{D}_{A^*}) D_{A^*} ,$$

which means that there exists a one-to-one correspondence between the solutions $\tilde{A}$ of the problem (*) and the contractions $\Gamma_c' \in \mathscr{L}(H_2, \mathscr{D}_{A^*} \oplus \mathscr{D}_{\Gamma_2^*})$ which verify $\Gamma_c'^* | \mathscr{D}_{A^*} = \Gamma_1^*$. In this correspondence the spaces $\mathscr{D}_{\tilde{A}}$ and $\mathscr{D}_{\Gamma_2} \oplus \mathscr{D}_{\Gamma_c'}$ (resp. $\mathscr{D}_{\tilde{A}^*}$ and $\mathscr{D}_{\Gamma_c'^*}$) can be identified by the unitary operators $(Z_*^*(A^*; A_c^*) \oplus \oplus I)Z^*(A_c; \tilde{A})$ (resp. by $Z_c^* Z_*^*(A_c; \tilde{A})$). The above correspondence is given by

$$\tilde{A} = (A_c,\ D_{A_c^*} Z_c \Gamma_c'). \tag{1.7}'$$

It remains to describe all contractions $\Gamma_c' \in \mathscr{L}(H_2, \mathscr{D}_{A^*} \oplus \mathscr{D}_{\Gamma_2^*})$ which satisfy $\Gamma_c'^* | \mathscr{D}_{A^*} = \Gamma_1^*$; this can be done using again Lemma 1.1. We obtain then a one-to-one correspondence between all such contractions Γ_c' and all contractions $\Gamma^* \in \mathscr{L}_1(\mathscr{D}_{\Gamma_2^*}, \mathscr{D}_{\Gamma_1})$, which is given by

$$\Gamma_c'^* = (\Gamma_1^*, D_{\Gamma_1} \Gamma^*). \tag{1.9}$$

The operators $Z(\Gamma_1^*; \Gamma_c'^*)$ and $Z_*(\Gamma_1^*, \Gamma_c'^*)$ are unitary operators between $\mathscr{D}_{\Gamma_c'^*}$ and $\mathscr{D}_{\Gamma_1^*} \oplus \mathscr{D}_{\Gamma^*}$, respectively between $\mathscr{D}_{\Gamma_c'}$ and $\mathscr{D}_{\Gamma}$. Taking into consideration (1.7)′ and (1.9) a one-to-one correspondence results between the solutions $\tilde{A}$ of the problem (*) and the contractions $\Gamma \in \mathscr{L}_1(\mathscr{D}_{\Gamma_1}, \mathscr{D}_{\Gamma_2^*})$ given by

$$\tilde{A} = \left(A_c,\ D_{A_c^*} Z \begin{pmatrix} \Gamma_1 \\ \Gamma D_{\Gamma_1} \end{pmatrix}\right).$$

Using Lemma 1.2 we obtain that

$$\tilde{A} = \begin{pmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} \end{pmatrix}. \tag{1.10}$$

Finally, note that the operators

$$(Z_{*c}^* \oplus Z_*(\Gamma_1^*;\ \Gamma_c'^*))Z^*(A_c;\ \tilde{A}) : \mathscr{D}_{\tilde{A}} \to \mathscr{D}_{\Gamma_2} \oplus \mathscr{D}_{\Gamma} \tag{1.11}$$

and

$$Z(\Gamma_1^*;\ \Gamma_c'^*)Z_c^*(A_c;\ \tilde{A}) : \mathscr{D}_{\tilde{A}^*} \to \mathscr{D}_{\Gamma_1^*} \oplus \mathscr{D}_{\Gamma^*} \tag{1.12}$$

are unitary operators, which completes the proof of the theorem.

Let us make some historical remarks concerning the problem (*). The case $A = A^*$, $B = C^*$ was considered by M. G. Kreĭn in [11], in connection with self-adjoint extensions of positive unbounded operators (see Section 4 below). The existence part of (*) (which follows, of course, from our theorem) is implicitly contained in [13], [14], [17] or [2]. In [8] Ch. Davis quoted the names of W. M. Kahan, T. Kato and H. F. Weinberger in connection with the proof of the existence. A proof of the existence part of (*) can be found in [12]. In [16], B. Sz.-Nagy and C. Foiaş solved the problem (*) for $A = 0$ (their formula follows from Theorem 1.3). In [8] Ch. Davis discussed the problem (*) for A, B, and C being compact operators.

Here are some further remarks concerning Theorem 1.3.

REMARK 1. Taking into account the proof of (1.2), we have that $\mathscr{D}_{\Gamma_1}$ can be identified with $\mathscr{D}_{A_r} \ominus \overline{D_{A_r}(H_1)}$ and $\mathscr{D}_{\Gamma_2^*}$ can be identified with $\mathscr{D}_{A_c^*} \ominus \overline{D_{A_c^*}(K_1)}$. It foillows that the solutions of the problem (*) can be indexed by all contractions from $\mathscr{D}_{A_r} \ominus \overline{\mathscr{D}_{A_r}(H_1)}$ into $\mathscr{D}_{A_c^*} \ominus \overline{\mathscr{D}_{A_c^*}(K_1)}$.

REMARK 2. If one starts proving Theorem 1.3 by completing A_r (instead of A_c) to $\tilde{A}$, the correspondence which results will be the same as in the proof above. This implies some relations between the unitary operators (1.11), (1.12) and the analogous ones for A_r.

REMARK 3. For $A \in \mathscr{L}_1(H_1, K_1)$, take $H_2 = \mathscr{D}_{A^*}$, $K_2 = \mathscr{D}_A$, $\Gamma_1 = I_{\mathscr{D}_{A^*}}$, $\Gamma_2 = I_{\mathscr{D}_A}$. Theorem 1.3 implies then the known fact that the operator

$$J(A) = \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} : H_1 \oplus \mathscr{D}_{A^*} \to K_1 \oplus \mathscr{D}_A \tag{1.13}$$

is a unitary operator.

REMARK 4. With the notation of Theorem 1.3, consider the operators:

$$\begin{cases} \hat{\Gamma}_1 = \begin{pmatrix} I & 0 \\ 0 & \Gamma_1 \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus \mathscr{D}_{A^*} \\ \hat{\Gamma}_2 = \begin{pmatrix} I & 0 \\ 0 & \Gamma_2 \end{pmatrix} : K_1 \oplus \mathscr{D}_A \to K_1 \oplus K_2 \\ \hat{\Gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix} : H_1 \oplus \mathscr{D}_{\Gamma_1} \to K_1 \oplus \mathscr{D}_{\Gamma_2^*}. \end{cases} \tag{1.14}$$

Then (1.10) and (1.13) show that all the solutions of the problem (*) are of the form

$$\tilde{A} = \hat{\Gamma}_2 J(A) \hat{\Gamma}_1 + D_{\hat{\Gamma}_2^*} \hat{\Gamma} D_{\hat{\Gamma}_1}, \tag{1.10)'}$$

where $\hat{\Gamma}$ is as in (1.14) with $\Gamma \in \mathscr{L}_1(\mathscr{D}_{\Gamma_1}, \mathscr{D}_{\Gamma_2^*})$.

REMARK 5. Suppose in the problem (*) that A is selfadjoint and $B = C^*$. Theorem 1.3 shows in this case that the *selfadjoint* solutions $\tilde{A}$ to the problem (*) are in one-to-one correspondence with the selfadjoint part of $\mathscr{L}_1(\mathscr{D}_{\Gamma_1})$.

2. MAXIMAL DUAL PAIRS OF SUBSPACES

In this section we will apply Theorem 1.3 to the problem of extending dual pairs of subspaces in a Kreĭn space to maximal ones. Let us recall some terminology concerning Kreĭn spaces. A Kreĭn space $\{K, J\}$ is a Hilbert space K together with a hermitian unitary operator J on it. In a Kreĭn space, besides the usual scalar product $\langle \cdot, \cdot \rangle$, one considers also the (indefinite) inner product $[x, y] = \langle Jx, y \rangle$. If $J = J^+ - J^-$ is the Jordan decomposition of J (J^+ and J^- being orthogonal projections with $J^+ + J^- = I$), then $K^+ = J^+(K)$ and $K^- = J^-(K)$ are the positive and negative parts of K, respectively. A vector $x \in K$ is called positive (resp. negative) if $[x, x] \geqslant 0$ (resp. $[x, x] \leqslant 0$); two vectors $x, y \in K$ are called J-orthogonal if $[x, y] = 0$. For more notation and terminology on Kreĭn spaces see [4] or [2].

A *dual pair* of subspaces of K is a pair $\{\mathscr{M}, \mathscr{N}\}$ of closed subspaces of K such that $\mathscr{M}$ is positive, $\mathscr{N}$ is negative and $\mathscr{M}$ is J-orthogonal on $\mathscr{N}$ (that is $[x, y] = 0$ for every $x \in \mathscr{M}$ and $y \in \mathscr{N}$; we will write this as $\mathscr{M}[\perp]\mathscr{N}$). A dual pair $\{\mathscr{M}, \mathscr{N}\}$ is *maximal* if $\mathscr{M}$ is maximal positive and $\mathscr{N}$ is maximal negative. It is natural to ask the following question:

(**) For a given dual pair $\{\mathscr{M}, \mathscr{N}\}$ find all maximal dual pairs $\{\tilde{\mathscr{M}}, \tilde{\mathscr{N}}\}$ such that $\tilde{\mathscr{M}} \supset \mathscr{M}$ and $\tilde{\mathscr{N}} \supset \mathscr{N}$.

The existence part of the problem (**) was proved in [14] (see also [2]). That proof uses the notion of the *angular operator* associated with a subspace. For a positive subspace $\mathscr{M}$, the operator $T : \mathscr{M}^+ = J^+\mathscr{M} \to K^-$ defined by $T(J^+x) = J^-x$, $x \in \mathscr{M}$, is a contraction. The operator T is called the angular operator of $\mathscr{M}$ and we have that $\mathscr{M}$ is the graph of T. Conversely, if T is a contraction from $\mathscr{M}^+ \subset \mathscr{K}^+$ into K^-, then the graph of T is a positive subspace with the angular operator equal to T. The positive subspaces $\mathscr{M}' \supset \mathscr{M}$ correspond to contractive extensions T' of T. The angular operator for a negative subspace $\mathscr{N}$ (acting between $J^-\mathscr{N}$ and K^+) is defined analogously. In this correspondence the objects associated with T can be also described "geometrically". For example, if $\mathscr{M}$ is a positive subspace with the angular operator T, then the negative subspace associated with T^* is the only maximal negative subspace $\mathscr{N}$ which is J-orthogonal to $\mathscr{M}$ and satisfying $J^+\mathscr{N} \subset \mathscr{M}^+$. Also, $\ker D_T = J^+(\mathscr{M}^0)$, where $\mathscr{M}^0 = \mathscr{M} \cap \mathscr{M}^{[\perp]}$($\mathscr{M}^{[\perp]}$ is the J-orthogonal of $\mathscr{M}$), and $\mathscr{D}_T = J^+(\mathscr{M} \ominus \mathscr{M}^0)$.

Coming back to the problem (**), let $\{\mathscr{M}, \mathscr{N}\}$ be a dual pair of subspaces with angular operators $\{T, S\}$. The condition that $\mathscr{M}[\perp]\mathscr{N}$ means that $[x \oplus Tx, Sy \oplus y] = 0$ for every $x \in \mathscr{M}^+$ and $y \in \mathscr{N}^-$. This implies that

$$\langle x, Sy \rangle = \langle Tx, y \rangle, \quad \text{for every } x \in \mathscr{M}^+ \text{ and } y \in \mathscr{N}^-. \tag{2.1}$$

Write $T=\begin{pmatrix}T_1\\ T_2\end{pmatrix}$ and $S=\begin{pmatrix}S_1\\ S_2\end{pmatrix}$ with respect to the decompositions $K^- = \mathcal{N}^- \oplus \oplus (K^- \ominus \mathcal{N}^-)$ and $K^+ = \mathcal{M}^+ \oplus (K^+ \ominus \mathcal{M}^+)$ respectively. The relation (2.1) means that

$$T_1 = S_1^*. \tag{2.2}$$

From these considerations it follows that in order to describe all maximal positive subspaces $\tilde{\mathcal{M}}$ which contain $\mathcal{M}$ and are J-orthogonal on $\mathcal{N}$, one has to find all contraction completions of the matrix

$$\begin{pmatrix}T_1 & S_2^*\\ T_2 & ?\end{pmatrix}.$$

This is also the solution to the problem (**), because $\tilde{\mathcal{N}}$ is exactly $\tilde{\mathcal{M}}^{[\perp]}$ (from the maximality). Using Theorem 1.3 and Remark 1, we obtain

COROLLARY 2.1. *There exists a one-to-one correspondence between all solutions of the problem* (**) *and all contractions from* $\mathcal{D}_{S^*} \ominus \overline{D_{S^*}(\mathcal{M}^+)}$ *into* $\mathcal{D}_{T^*} \ominus \ominus \overline{D_{T^*}(\mathcal{N}^-)}$.

A "geometrical" variant of this corollary can be given. Similar results can be obtained for maximal uniform (resp. maximal strict uniform) dual pairs.

3. MAXIMAL DUAL PAIRS OF ACCRETIVE OPERATORS

A densely defined closed operator $T : \mathcal{D}(T) \to H$ is called *accretive* if

$$\mathrm{Re}\langle Th, h\rangle \geqslant 0, \qquad \text{for every } h \in \mathcal{D}(T).$$

A pair of accretive operators $\{T, S\}$ is called a *dual pair* if

$$\langle Th, k\rangle = \langle h, Sk\rangle, \qquad \text{for every } h \in \mathcal{D}(T) \text{ and } k \in \mathcal{D}(S). \tag{3.1}$$

A *maximal* dual pair of accretive operators is a self-explained expression. Consider the problem

(***) For a given dual pair $\{T, S\}$ of accretive operators, find all maximal dual pairs $\{\tilde{T}, \tilde{S}\}$ of accretive operators such that $\tilde{T}$ extends T and $\tilde{S}$ extends S.

The existence part of the problem (***) was solved in [14] and in [17], Ch. IV, Proposition 4.2. Extensions of a single accretive operator are considered in [13]; the study of all solutions in this case is made in [7]. In studying (***) one usually uses the Cayley transform; let us recall some facts about that (see for example [13] or [17], Ch. IV, Section 4).

For an accretive operator T denote by T^c its Cayley transform, i.e. the *contraction*

$$T^c = (T - I)(T + I)^{-1} : (T + I)\mathscr{D}(T) \to H. \tag{3.2}$$

We have that

$$T = (I + T^c)(I - T^c)^{-1}. \tag{3.3}$$

In this correspondence any accretive extension of T corresponds to a contractive extension of T^c and conversely. In particular T is maximal accretive if and only if $(T + I)\mathscr{D}(T) = H$. If T is maximal accretive, then T^* is also maximal accretive, so any maximal dual pair of accretive operators is of the form $\{T, T^*\}$.

Let now $\{T, S\}$ be a dual pair of accretive operators with Cayley transforms T^c and S^c. Consider $T^c = \begin{pmatrix} T_1^c \\ T_2^c \end{pmatrix}$ and $S_c = \begin{pmatrix} S_1^c \\ S_2^c \end{pmatrix}$ with respect with the decompositions $H = \mathscr{D}(S^c) \oplus (H \ominus \mathscr{D}(S^c)$ and $H = \mathscr{D}(T^c) \oplus (H \ominus \mathscr{D}(T^c))$ respectively; the condition (3.1) becomes

$$T_1^c = (S_1^c)^*.$$

This means that the problem (***) is equivalent to finding all contractive completions of the matrix

$$\begin{pmatrix} T_1^c & (S_2^c)^* \\ T_2^c & ? \end{pmatrix}.$$

Applying again Theorem 1.3 and Remark 1, we have

COROLLARY 3.1. *There exists a one-to-one correspondence between all solutions of the problem* (***) *and all contractions from* $\mathscr{D}_{T^{c*}} \ominus \overline{D_{T^{c*}}(\mathscr{D}(S^c))}$ *into* $\mathscr{D}_{S^{c*}} \ominus \overline{D_{S^{c*}}(\mathscr{D}(T^c))}$.

Similar results can be stated for *dissipative* operators. Note that the analogy between Section 2 and Section 3 is explained by Phillips' connections [14] between Kreĭn spaces and accretive operators.

4. EXTENSIONS OF POSITIVE OPERATORS

Let $T : \mathscr{D}(T) \to H$ a closed densely defined symmetric operator. Then T is called *positive* if

$$\langle Th, h\rangle \geq 0, \qquad \text{for every } h \in \mathscr{D}(T).$$

It follows that a closed densely defined operator is positive if and only if the pair $\{T, T\}$ is a dual pair of accretive operators, so Corollary 3.1 can be applied in this situation. Let us write $H_T = \mathscr{D}_{(T^c)_*} \ominus \overline{D_{(T^c)_*}(\mathscr{D}(T^c))}$. We have

COROLLARY 4.1. *There exists a one-to-one correspondence between all maximal accretive extensions of a positive operator T and the set $\mathscr{L}_1(H_T)$.*

Moreover, from Remark 5 following Theorem 1.3 one can infer that.

COROLLARY 4.2. *There exists a one-to-one correspondence between all positive selfadjoint extensions of a positive operator T and the selfadjoint part of $\mathscr{L}_1(H_T)$.*

REMARK 1. The order between Cayley transforms of positive selfadjoint extensions of T is exactly the order in the selfadjoint part of $\mathscr{L}_1(H_T)$. In this way Corollary 4.2 implies the celebrated Kreĭn theorem [11] or [15] (for a recent account see [1]):

"*Among positive selfadjoint extensions of a positive operator T there exist a "smallest" one T_0 and a "largest" one T_∞. The set of all positive extensions of T is exactly the set of positive operators V such that $T_0^c \leqslant V^c \leqslant T_\infty^c$*".

In the dictionary of Corollary 4.2, T_0 is obtained from $-I \in \mathscr{L}_1(H_T)$; this extension is called the von Neumann's (or Kreĭn's) extension. The extension T_∞ (called the Friedrichs' extension) is obtained from $I \in \mathscr{L}_1(H_T)$.

REMARK 2. From Corollary 4.1 it follows immediately Theorem 2 of [18]:

"*A positive operator T has a non-selfadjoint maximal accretive extension if and only if T has a non unique positive selfadjoint extension. This happens if and only if $H_T \neq \{0\}$*".

Other problems related to Kreĭn theory (for example the results in [10]) can be described in the present dictionary. Some of these matters will appear in a paper of T. Constantinescu and A. Gheondea (in preparation).

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Added in proofs. After this paper has been circulated as Preprint INCREST No.14/1981, we were informed about the following contributions to the problem (*):

1) In the paper "Norm-preserving dilations and their applications to optimal error bounds" (to appear in *Siam J. Numerical Analysis)*, Ch. Davis, W. M. Kahan and H. F. Weinberger obtained also the formula (1.6) and applied it to finite-dimensional approximations of a linear transformation, to numerical quadratures and to computation of eigenvalues of Hermitian matrices.

2) In the paper "On matrices whose entries are contractions" (Russian) which will appear in *Izv. Visch. Ucheb. Zaved. Matematica*, 7(230), 1981, Iu. L. Smulian and R. N. Ianovskii obtained a weaker form of Theorem 1.3. They also informed us that the problem (**) was considered by V. S. Rintzer in the paper "On the theory of extensions of dual pairs of subspaces" (Russian), which appeared in the proceedings of a conference held in Habarovsk in 1977.

3) In connection with [6] and [3], J. A. Ball obtained partial results (unpublished) on the problem (*).

4) H. Langer and B. Textorius used Theorem 1.3 for describing all generalized resolvent of a given "dual pair" of contractions.