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ALGEBRAIC MODELS FOR POSITIVE OPERATOR VALUED MEASURES

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1. INTRODUCTION

Dinculeanu and Foiaş [7] characterized the conjugate relation between probability measures in terms of the isomorphism of their algebraic models. Concerning other topics related to algebraic models, we refer to Dinculeanu and Foiaş [6] and Foiaş [8]. Schreiber, Sun and Bharucha-Reid [19] and Christensen and Bharucha-Reid [3, 4] investigated algebraic models for measures induced by stochastic processes and for measures on Banach spaces. Christensen [5] gave the definition of algebraic models for positive operator valued measures and used it to prove an extension theorem ([5, Theorem 3]) for a consistent family of positive operator valued measures indexed by a directed set.

In this paper we first introduce the concept of algebraic models for positive operator valued measures, which is slightly different from that defined by Christensen [5], on separable Hilbert spaces. Then, analogously to Dinculeanu and Foiaş [7], we give a characterization of the conjugate relation between positive operator valued measures by the isomorphism of their algebraic models. We also obtain a necessary and sufficient condition in order that a positive operator valued measure is conjugate to a spectral measure.

2. PRELIMINARIES

Let H be a Hilbert space and $(\Omega, \mathscr{A})$ be a measurable space. Let $B(H)$ be the set of bounded linear operators on H.

DEFINITION 2.1 (cf. Berberian [1, Definition 1 and Proposition 1]). A mapping $E : \mathscr{A} \to B(H)$ is called a *(normalized) positive operator valued measure* (*PO-measure*) if E satisfies the following conditions:

(i) For any $M \in \mathscr{A}$, $E(M) \geqslant 0$;

(ii) $E(\Omega) = I$ (the identity on H) and $E(\emptyset) = 0$;

(iii) For any pairwise disjoint sequence $\{M_n\}$ of sets in $\mathscr{A}$,

$$E\left(\bigcup_{n=1}^{\infty} M_n\right) = \text{w-}\lim_n \sum_{i=1}^{n} E(M_i),$$

where w-lim is the limit in the weak operator topology.

If $E(M)$ is an orthogonal projection for each $M \in \mathscr{A}$, then E is called a *spectral measure*.

Let $\mathscr{A}_0 = \{M \in \mathscr{A} : E(M)=0\}$. For any ($\mathscr{A}$-) measurable functions $f, g : \Omega \to \mathbf{C}$ (the complex numbers), we denote $f \sim g$ if $\{\omega \in \Omega : f(\omega) \neq g(\omega)\} \in \mathscr{A}_0$. Let

$$\Gamma_E^0 = \{f : \Omega \to \mathbf{C} \;:\; f \text{ is measurable and } |f| \sim 1\},$$

where 1 is the constant 1 function on Ω. Then $\sim$ is an equivalence relation in Γ_E^0 and Γ_E^0 is an abelian group whose multiplication and inverse are defined by pointwise product and complex conjugate respectively. Let $\Gamma_E = \Gamma_E^0/\sim$ (the set of equivalence classes of Γ_E^0 with respect to $\sim$). Γ_E is also an abelian group whose multiplication and inverse are naturally induced from those in Γ_E^0. For $f \in \Gamma_E^0$, we denote by $\bar{f}$ and $[f]$ the complex conjugate of f and the equivalence class in Γ_E containing f respectively. Let $S = \{x \in H : \|x\| = 1\}$. For any $x, y \in H$, define a complex valued measure $\mu_{x,y}$ on $(\Omega, \mathscr{A})$ by $\mu_{x,y}(M) = (E(M)x, y)$ $(M \in \mathscr{A})$ and a measure μ_x on $(\Omega, \mathscr{A})$ by $\mu_x = \mu_{x,x}$. If $x \in S$, then μ_x is a probability measure on $(\Omega, \mathscr{A})$.

In the sequel we always assume that H is *separable*. For any countable dense set $\{x_n\}$ in S, a probability measure μ is defined by

$$\mu(M) = \sum_{n=1}^{\infty} \mu_{x_n}(M)/2^n \quad (M \in \mathscr{A}).$$

Let P_E be the set of all such probability measures μ.

Take a measurable function $f : \Omega \to \mathbf{C}$ such that for some $c > 0$, $\{\omega \in \Omega : |f| > c\} \in \mathscr{A}_0$. Then we can define the "integral" $\int f \mathrm{d}E \in B(H)$ by

$$\left(\left(\int f \mathrm{d}E\right) x, y\right) = \int f \mathrm{d}\mu_{x,y} \quad (x, y \in H)$$

(cf. Berberian [1, Theorem 9]).

Now we give some lemmas. The proof of Lemma 2.2 is clear.

LEMMA 2.2. *Let* $f, g \in \Gamma_E^0$. *If* $f \sim g$, *then* $\int f \mathrm{d}E = \int g \mathrm{d}E$.

LEMMA 2.3. *For* $f \in \Gamma_E^0$, $\int f\mathrm{d}E = I$ *if and only if* $f \sim 1$.

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose that $\int f\mathrm{d}E = I$. If we write $f = g + \mathrm{i}h$, where g and h are real valued, then

$$\int g\mathrm{d}E + \mathrm{i}\int h\mathrm{d}E = \int f\mathrm{d}E = I,$$

and for every $x \in S$,

$$\left(\left(\int g\mathrm{d}E\right)x,\ x\right) + \mathrm{i}\left(\left(\int h\mathrm{d}E\right)x,\ x\right) = (x,\ x).$$

Since $\int g\mathrm{d}E$ and $\int h\mathrm{d}E$ are Hermitian (cf. Berberian [1, Theorem 10]),

$$\left(\left(\int h\mathrm{d}E\right)x,\ x\right) = 0.$$

We have

$$1 = (x,\ x) = \left(\left(\int f\mathrm{d}E\right)x,\ x\right) = \left(\left(\int g\mathrm{d}E\right)x,\ x\right) = \int g\mathrm{d}\mu_x =$$

$$= \int g^+\mathrm{d}\mu_x - \int g^-\mathrm{d}\mu_x \leqslant \int g^+\mathrm{d}\mu_x \leqslant \int |g|\mathrm{d}\mu_x \leqslant \int |f|\mathrm{d}\mu_x = (x,\ x) = 1,$$

hence $\int g^-\mathrm{d}\mu_x = 0$. This implies that $g = |g|$ (a.e. μ_x) and

$$\int g\mathrm{d}\mu_x = 1.$$

Since $|g| \leqslant 1$ (a.e. μ_x), it follows that $|g| = 1$ (a.e. μ_x), thus $h = 0$ (a.e. μ_x). Therefore $f = g = |g| = 1$ (a.e. μ_x) and $f \sim 1$.

By Lemma 2.2 we can define $\varphi_E : \Gamma_E \to B(H)$ by

$$\varphi_E([f]) = \int f\mathrm{d}E \quad ([f] \in \Gamma_E).$$

Lemma 2.3 means that $\varphi_E([f]) = I$ if and only if $[f] = [1]$.

3. ALGEBRAIC MODELS FOR PO-MEASURES

Let Γ be an abelian group and $\varphi : \Gamma \to B(H)$.

DEFINITION 3.1. (Γ, φ) is called an *algebraic system* if the following conditions hold:

(i) φ is *positive definite* (PD), that is, for any positive integer n, for any $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ and for any $\{x_1, \ldots, x_n\} \subset H$.

$$\sum_{i,j} (\varphi(\gamma_j^{-1}\gamma_i)\, x_i, x_j) \geqslant 0;$$

(ii) $\varphi(\gamma) = I$ if and only if $\gamma = e$ (the unit of Γ).

DEFINITION 3.2. Two algebraic systems (Γ, φ) and (Γ', φ') are said to be *isomorphic* if there exists an onto isomorphism $J : \Gamma \to \Gamma'$ such that $\varphi(\gamma) = \varphi'(J(\gamma))$ for all $\gamma \in \Gamma$.

Let E be a PO-measure on $(\Omega, \mathscr{A})$.

DEFINITION 3.3. An algebraic system (Γ, φ) is called an *algebraic model* for a PO-measure E if there exists an injective homomorphism $h : \Gamma \to \Gamma_E$ with the following properties:

(i) For each $\mu \in P_E$, $h(\Gamma)$ generates $L^2(\Omega, \mathscr{A}, \mu)$;

(ii) $\varphi(\gamma) = \varphi_E(h(\gamma))$ for every $\gamma \in \Gamma$.

PROPOSITION 3.4. *(Γ_E, φ_E) is an algebraic model for E.*

Proof. It suffices to show that $\varphi_E : \Gamma_E \to B(H)$ is PD (cf. Christensen [5, Proposition 3 and Theorem 2]). For each positive integer n, take $\{f_1, \ldots, f_n\} \subset \Gamma_E^0$ and $\{x_1, \ldots, x_n\} \subset H$ arbitrarily. Then

$$\sum_{i,j} (\varphi_E([f_j]^{-1}[f_i])\, x_i, x_j) = \sum_{i,j} \left(\left(\int f_i \bar{f}_j \mathrm{d}E\right) x_i, x_j\right) = \sum_{i,j} \int f_i \bar{f}_j \mathrm{d}\mu_{x_i, x_j}.$$

For any $\varepsilon > 0$, there exist simple functions $g_1, \ldots, g_n$ such that $|f_i(\omega) - g_i(\omega)| < \varepsilon$ $(i = 1, \ldots, n, \omega \in \Omega)$. We may write

$$g_i = \sum_{k=1}^{m} \alpha_{ik} \chi_{M_k} \quad (i = 1, \ldots, n),$$

where $\alpha_{ik} \in \mathbf{C}$ and χ_{M_k} is the characteristic function of the measurable set M_k

$(k = 1, \ldots, m)$. It follows that

$$\sum_{i,j} \int g_i \bar{g}_j \mathrm{d}\mu_{x_i, x_j} = \sum_k \sum_{i,j} \alpha_{ik} \bar{\alpha}_{jk} \mu_{x_i, x_j}(M_k) = \sum_k \sum_{i,j} (E(M_k)\, \alpha_{ik} x_i\, , \alpha_{jk} x_j) =$$

$$= \sum_k (E(M_k) (\sum_i \alpha_{ik} x_i), (\sum_j \alpha_{jk} x_j)) \geqslant 0.$$

Letting $\varepsilon \to 0$, then

$$\sum_{i,j} \int g_i \bar{g}_j \mathrm{d}\mu_{x_i, x_j}$$

converges to

$$\sum_{i,j} \int f_i \bar{f}_j \mathrm{d}\mu_{x_i, x_j} .$$

Therefore

$$\sum_{i,j} \int f_i \bar{f}_j \mathrm{d}\mu_{x_i, x_j} \geqslant 0.$$

Hence φ_E is PD.

For $M, N \in \mathscr{A}$, we denote $M \sim N$ if $\chi_M \sim \chi_N$. Then $\sim$ is an equivalence relation in $\mathscr{A}$. Let $\mathscr{A}_* = \mathscr{A}/\sim$ (the set of all equivalence classes of $\mathscr{A}$ with respect to $\sim$). $\mathscr{A}_*$ is a Boolean σ-algebra and E may be also considered as a mapping of $\mathscr{A}_*$ into $B(H)$. For $M \in \mathscr{A}$, let $[M]$ be the equivalence class in $\mathscr{A}_*$ containing M.

Let $(\Omega', \mathscr{A}')$ be a measurable space and $E' : \mathscr{A}' \to B(H)$ be a PO-measure.

DEFINITION 3.5. Two PO-measures E and E' are said to be *conjugate* if there exists an onto Boolean σ-isomorphism $\psi : \mathscr{A}_* \to \mathscr{A}'_*$ such that $E([M]) = E'(\psi[M])$ for every $M \in \mathscr{A}$.

The following theorem is an extension of a main result in Dinculeanu and Foiaş [7, Theorem 2] to the cases of positive operator valued measures.

THEOREM 3.6. *Two PO-measures are conjugate if and only if they possess isomorphic algebraic models.*

Proof. ($\Rightarrow$) Let $\psi : \mathscr{A}_* \to \mathscr{A}'_*$ be an onto Boolean σ-isomorphism realizing the conjugacy between E and E'. Take $\mu \in P_E$ and $\mu' \in P_{E'}$ such that

$$\mu = \sum_{n=1}^{\infty} \mu_{x_n}/2^n \quad \text{and} \quad \mu' = \sum_{n=1}^{\infty} \mu'_{x_n}/2^n ,$$

where $\{x_n\}$ is a countable dense sequence in the unit sphere S of H. Then $\mathscr{A}_*$ and $\mathscr{A}'_*$ are measure algebras of probability measures μ on $(\Omega, \mathscr{A})$ and μ' on $(\Omega', \mathscr{A}')$ respectively (cf. Halmos [9, p. 42)], and, since E and E' are conjugate, μ and μ'

are conjugate (Halmos [9, pp. 44—45], cf. Dinculeanu and Foiaş [7, Definition 1 and Remark 1]). By Dinculeanu and Foiaş [7, Proposition 2 and its proof]), there exists an onto isomorphism $h : \Gamma_E \to \Gamma_{E'}$ which is also a linear isometry from $L^2(\Omega, \mathscr{A}, \mu)$ onto $L^2(\Omega', \mathscr{A}', \mu')$ and satisfies the equality $\int f \mathrm{d}\mu = \int h(f) \mathrm{d}\mu'$ for every $[f] \in \Gamma_{E'}$ where we use same symbols f for square integrable functions f and its equivalence classes $[f]$ in $L^2(\Omega, \mathscr{A}, \mu)$. Fix $[f] \in \Gamma_E$ arbitrarily. For any $\varepsilon > 0$, choose a simple function $g = \sum \alpha_i \chi_{M_i}$ such that $|f(\omega) - g(\omega)| < \varepsilon$ for all $\omega \in \Omega$. For each i, take $M_i' \in \psi([M_i])$. Then $h(g) = \sum \alpha_i \chi_{M_i'}$ and $h(g)$ converges to $h(f)$ in $L^2(\Omega', \mathscr{A}', \mu')$ as $\varepsilon \to 0$. Hence, for any n, if $\varepsilon \to 0$, then

$$\int h(g) \mathrm{d}\mu'_{x_n} \to \int h(f) \mathrm{d}\mu'_{x_n}.$$

It is clear that if $\varepsilon \to 0$, then

$$\int g \mathrm{d}\mu_{x_n} \to \int f \mathrm{d}\mu_{x_n}.$$

Since E and E' are conjugate, it follows that

$$\int h(g)\, \mathrm{d}\mu'_{x_n} = \sum \alpha_i \mu'_{x_n}(M_i') = \sum \alpha_i (E'(M_i') x_n, x_n) = \sum \alpha_i (E(M_i) x_n, x_n) =$$

$$= \sum \alpha_i \mu_{x_n}(M_i) = \int g \mathrm{d}\mu_{x_n}.$$

Therefore we have

$$\int h(f) \mathrm{d}\mu'_{x_n} = \int f \mathrm{d}\mu_{x_n}$$

and

$$\left(\left(\int h(f) \mathrm{d}E'\right) x_n, x_n\right) = \left(\left(\int f \mathrm{d}E\right) x_n, x_n\right)$$

for all n. This implies that $\int h(f)\, \mathrm{d}E' = \int f \mathrm{d}E$, which in turn means that $\varphi_{E'}(h([f])) = \varphi_E([f])$. Thus, the algebraic models (Γ_E, φ_E) and $(\Gamma_{E'}, \varphi_{E'})$ are isomorphic.

($\Leftarrow$) Suppose that E and E' possess isomorphic algebraic models (Γ, φ) and (Γ', φ'). Let $\mu \in P_E$ and $\mu' \in P_{E'}$ be such that

$$\mu = \sum_{n=1}^{\infty} \mu_{x_n}/2^n \quad \text{and} \quad \mu' = \sum_{n=1}^{\infty} \mu'_{x_n}/2^n.$$

We may consider that $\Gamma \subset \Gamma_{E'}$, $\varphi = \varphi_E$ and $\Gamma' \subset \Gamma_{E'}$, $\varphi' = \varphi_{E'}$. Let $h : \Gamma \to \Gamma'$ be an onto isomorphism for which $\varphi_E(f) = \varphi_{E'}(h(f))$ for all $f \in \Gamma$. Consider the linear subspace $A \subset L^\infty(\Omega, \mathscr{A}, \mu)$ of the finite linear combinations $f = \sum \alpha_i f_i$ with $f_i \in \Gamma$ and $\alpha_i \in \mathbf{C}$. Then for $f = \sum \alpha_i f_i \in A$, we obtain

$$\int |f|^2 \mathrm{d}\mu = \int \sum_{i,j} \alpha_i \bar{\alpha}_j f_i \bar{f}_j \mathrm{d}\mu = \sum_{i,j} \alpha_i \bar{\alpha}_j \sum_{n=1}^{\infty} \frac{1}{2^n} \int f_i \bar{f}_j \mathrm{d}\mu_{x_n} =$$

$$= \sum_{i,j} \alpha_i \bar{\alpha}_j \sum_{n=1}^{\infty} \frac{1}{2^n} (\varphi_E(f_i f_j^{-1}) x_n, x_n) = \sum_{i,j} \alpha_i \bar{\alpha}_j \sum_{n=1}^{\infty} \frac{1}{2^n} (\varphi_{E'}(h(f_i)\, h(f_j)^{-1}) x_n, x_n) =$$

$$= \sum_{i,j} \alpha_i \bar{\alpha}_j \sum_{n=1}^{\infty} \frac{1}{2^n} \int h(f_i)\, \overline{h(f_j)}\, \mathrm{d}\mu'_{x_n} = \sum_{i,j} \alpha_i \bar{\alpha}_j \int h(f_i)\, \overline{h(f_j)}\, \mathrm{d}\mu' = \int |\sum \alpha_i h(f_i)|^2 \mathrm{d}\mu'.$$

Thus, if $f = \sum \alpha_i f_i = 0$ (a.e. μ), then $\sum \alpha_i h(f_i) = 0$ (a.e. μ'). Hence we can define a linear mapping $h : A \to L^\infty(\Omega', \mathscr{A}', \mu')$ by $h(f) = \sum \alpha_i h(f_i)$ ($f = \sum \alpha_i f_i \in A$). It is easy to observe that this h is multiplicative on A and satisfies the equality $\|h(f)\|_2 = \|f\|_2$ (the L^2-norm) ($f \in A$). By Dinculeanu and Foiaş [7, Theorem 1], h may be extended to a linear isometry of $L^2(\Omega, \mathscr{A}, \mu)$ onto $L^2(\Omega', \mathscr{A}', \mu')$, still denoted by h, for which $h(L^\infty(\Omega, \mathscr{A}, \mu)) = L^\infty(\Omega', \mathscr{A}', \mu')$ and $h(fg) = h(f)\, h(g)$ for each f, $g \in L^\infty(\Omega, \mathscr{A}, \mu)$, so that there exists a Boolean σ-isomorphism ψ of $\mathscr{A}_*$ onto $\mathscr{A}'_*$ such that $\mu(M) = \mu'(M')$ whenever $\psi([M]) = [M']$ ($M \in \mathscr{A}$) (Halmos [9, p. 45]). Now it is sufficient to show that this ψ realizes the conjugacy between E and E'. Let $M \in \mathscr{A}$ and $M' \in \mathscr{A}'$ be such that $\psi([M]) = [M']$. Then, by the construction of ψ, $h(\chi_M) = \chi_{M'}$. Since $\varphi_E(f) = \varphi_{E'}(h(f))$ for every $f \in \Gamma$, it follows that for any n and any $f \in A$ with $f = \sum \alpha_i f_i$,

$$\int f \mathrm{d}\mu_{x_n} = \sum \alpha_i \int f_i \mathrm{d}\mu_{x_n} = \sum \alpha_i ((\varphi_E(f_i)) x_n, x_n) =$$

$$= \sum \alpha_i ((\varphi_{E'}(h(f_i))) x_n, x_n) = \sum \alpha_i \int h(f_i)\, \mathrm{d}\mu'_{x_n} = \int h(f)\, \mathrm{d}\mu'_{x_n}.$$

Since Γ generates $L^2(\Omega, \mathscr{A}, \mu)$, there exists a sequence $\{f\} \subset A$ such that $f \to \chi_M$ in $L^2(\Omega, \mathscr{A}, \mu)$. Then $h(f) \to \chi_{M'}$ in $L^2(\Omega', \mathscr{A}', \mu')$ because h is isometric. Thus, for any n,

$$\int f \mathrm{d}\mu_{x_n} \to \int \chi_M \mathrm{d}\mu_{x_n} = (E(M) x_n, x_n)$$

and

$$\int h(f)\, \mathrm{d}\mu'_{x_n} \to \int \chi_{M'} \mathrm{d}\mu'_{x_n} = (E'(M') x_n, x_n).$$

Therefore $(E(M) x_n, x_n) = (E'(M') x_n, x_n)$. This implies that $E(M) = E'(M')$, and E and E' are conjugate. This completes the proof of Theorem 3.6.

COROLLARY 3.7. *Two PO-measures E and E' are conjugate if and only if (Γ_E, φ_E) and $(\Gamma_{E'}, \varphi_{E'})$ are isomorphic.*

If Ω is a compact Hausdorff space and $\mathscr{A}$ is the Borel field on Ω, then a PO-measure $E : \mathscr{A} \to B(H)$ is called *regular* if for each $x \in H$, the Borel measure μ_x is regular (cf. Berberian [1, Definition 14, Theorem 20 and Examples 3 (p. 88)]).

THEOREM 3.8. *Let (Γ, φ) be an algebraic system. Then there exist a compact abelian group G with $\mathscr{B}$ its Borel field and a regular PO-measure $E : \mathscr{B} \to B(H)$ such that (Γ, φ) is an algebraic model for E.*

Proof. Consider Γ as an discrete group. Then G, the dual group of Γ, is a compact abelian group. Since $\varphi : \Gamma \to B(H)$ is PD and $\varphi(e) = I$, by Nagy's dilation theorem [20] (cf. Itoh [11]), there exist a Hilbert space K containing H and a unitary representation $\pi : \Gamma \to B(K)$ such that $\varphi(\gamma)x = P\pi(\gamma)x$ $(x \in H,\ \gamma \in \Gamma)$, where P is the orthogonal projection of K onto H. For this π, by Stone's theorem (cf. Loomis [14, p. 147 and Remark 34E]), there exists a unique regular spectral measure $F : \mathscr{B} \to B(K)$ such that

$$\pi(\gamma) = \int \langle g, \gamma \rangle \mathrm{d}F(g)$$

$(\gamma \in \Gamma)$, where $\langle g, \gamma \rangle$ is the value of the character g at γ. Hence, for each $x \in H$,

$$(\varphi(\gamma)x, x) = (P\pi(\gamma)x, x) = \int \langle g, \gamma \rangle \mathrm{d}(PF(g)x, x).$$

Define $E : \mathscr{B} \to B(H)$ by $E(M) = PF(M)$ $(M \in \mathscr{B})$. Then E is a regular PO-measure. For any $\gamma \in \Gamma$, let $\langle \cdot, \gamma \rangle$ be the continuous function $g \to \langle g, \gamma \rangle$ or the equivalence class of this function in Γ_E. Define a mapping $h : \Gamma \to \Gamma_E$ by $h(\gamma) = \langle \cdot, \gamma \rangle$. Then h is a homomorphism. It is well-known that linear combinations of characters of G are dense in $C(G)$ (the Banach space of continuous functions on G with supremum norm). Since any $\mu \in P_E$ is regular, $C(G)$ is dense in $L^2(G, \mathscr{B}, \mu)$. Therefore, for every $\mu \in P_E$, $h(\Gamma)$ generates $L^2(G, \mathscr{B}, \mu)$. For any $x \in H$, we have

$$(\varphi(\gamma)x, x) = \int \langle g, \gamma \rangle \mathrm{d}\mu_x(g) = \left(\left(\int h(\gamma)\, \mathrm{d}E\right)x, x\right) = ((\varphi_E(h(\gamma)))x, x),$$

thus $\varphi(\gamma) = \varphi_E(h(\gamma))$. It remains to show that h is injective. If $h(\gamma) = [1]$, then for any $x \in S$, $h(\gamma) = 1$ (a.e. μ_x) and

$$(\varphi(\gamma)x, x) = \int h(\gamma)\, \mathrm{d}\mu_x = (x, x).$$

This implies that $\varphi(\gamma) = I$ and $\gamma = e$.

COROLLARY 3.9. *Every* PO-*measure is conjugate to a regular* PO-*measure on an compact abelian group.*

Proof. Let E be a PO-measure on $(\Omega, \mathscr{A})$. Then (Γ_E, φ_E) is an algebraic model for E (Proposition 3.4). By Theorem 3.8, (Γ_E, φ_E) is an algebraic model for a regular PO-measure E' on an compact abelian group. Hence, by Theorem 3.6, E and E' are conjugate.

4. ALGEBRAIC MODELS FOR SPECTRAL MEASURES

An algebraic system (Γ, φ) is called *canonical* if $\varphi : \Gamma \to B(H)$ is a unitary representation.

The following theorem is proved similarly to Theorem 3.8, hence we omit the proof.

THEOREM 4.1. *Let (Γ, φ) be a canonical algebraic system. Then there exist a compact abelian group G with $\mathscr{B}$ the Borel field of G and a regular spectral measure $E : \mathscr{B} \to B(H)$ such that (Γ, φ) is an algebraic model for E.*

THEOREM 4.2. *A* PO-*measure is conjugate to a spectral measure if and only if it possess es a canonical algebraic model.*

Proof. ($\Rightarrow$) Suppose that a PO-measure E on $(\Omega, \mathscr{A})$ is conjugate to a spectral measure E' on $(\Omega', \mathscr{A}')$. We show that the algebraic model (Γ_E, φ_E) of E is canonical. By Berberian [1, Theorem 15], for any $f, g \in \Gamma_{E'}$, we have

$$\varphi_{E'}(fg) = \int fg\,\mathrm{d}E' = \left(\int f\,\mathrm{d}E'\right)\left(\int g\,\mathrm{d}E'\right) = \varphi_{E'}(f)\,\varphi_{E'}(g).$$

Moreover, by Berberian [1, Theorem 10], we obtain

$$I = \varphi_{E'}([1]) = \left(\int f\,\mathrm{d}E'\right)\left(\int \bar{f}\,\mathrm{d}E'\right) = \varphi_{E'}(f)\,\varphi_{E'}(f)^{*}.$$

Thus, $(\Gamma_{E'}, \varphi_{E'})$ is canonical. Since (Γ_E, φ_E) and $(\Gamma_{E'}, \varphi_{E'})$ are isomorphic (Corollary 3.7), (Γ_E, φ_E) is canonical.

($\Leftarrow$) This is clear from Theorem 4.1 and Theorem 3.6.

It is not difficult to prove the following

COROLLARY 4.3. *Any spectral measure is conjugate to a regular spectral measure on an compact abelian group.*

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