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A NOTE ON COMMON INVARIANT SUBSPACES

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It follows from the celebrated paper of V. Lomonosov [5] that two commuting operators on a Banach space have a common invariant subspace if one of them is compact and nonzero. In the present note, we establish the following results which roughly say that two operators have a common invariant subspace if they are "almost commuting" and one of them is a nonzero compact operator.

THEOREM 1. *Let K be a nonzero compact operator on a Banach space $\mathscr{X}$ and T be a (bounded) operator on $\mathscr{X}$ for which there exist a bounded open set D containing $\sigma(T)$ (the spectrum of T), and an analytic function φ from D into D such that $TK = K\varphi(T)$. (In case that $\mathscr{X}$ is finite dimensional, we assume further that $0 \in \sigma(K)$.) Then there is a (nontrivial, closed) subspace of $\mathscr{X}$ which is invariant for both T and K.*

THEOREM 2. *If K and T are bounded operators on an infinite dimensional Banach space $\mathscr{X}$ such that K is compact and nonzero and if there exist a bounded open set D containing $\sigma(K)$ and an analytic function φ from D into D such that $KT = T\varphi(K)$, then there is a subspace of $\mathscr{X}$ which is invariant for both T and K.*

REMARK. The assumption that $0 \in \sigma(K)$ in case $\dim\mathscr{X} < \infty$ in Theorem 1 is essential in view of the following example. Let

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $KT = -TK$; but T and K do not have a common invariant subspace (except the trivial ones.)

It follows from [2; Corollary 1] that, under the assumption of Theorem 1, T has a hyperinvariant subspace (unless T is a scalar multiple of identity). However, this does not ensure the existence of a subspace invariant for both T and K. Also, from [3] it follows that, under an assumption slightly different from that of Theorem 2, T has a hyperinvariant subspace.

We remark that some related problems concerning existence of common invariant subspaces of two operators and their simultaneous triangulation have been studied by several authors, see [1], [4].

The proofs of Theorem 1 and Theorem 2 depend on the following key lemma.

LEMMA 3 [2, Theorem 3]. *If $\mathscr{A}$ is an algebra of operators in $\mathscr{L}(\mathscr{X})$ and $\mathscr{A}$ is an operator range (i.e., there exist a Banach space $\mathscr{Y}$ and a bounded linear map S from $\mathscr{Y}$ into $\mathscr{L}(\mathscr{X})$ such that $S\mathscr{Y} = \mathscr{A}$), and if there is a compact operator K such that $\mathscr{A}K \subseteq K\mathscr{A}$, then $\mathscr{A}$ has an invariant subspace. (Again, we assume $0 \in \sigma(K)$ if* $\dim\mathscr{X} < \infty$.)

Proof of Theorem 1. With no loss of generality, we may assume that $\|K\| < 1$. We write $H^\infty(D)$ for the Banach space of all holomorphic functions on D with the uniform norm $\|\cdot\|_\infty$ and $\ell^1(H^\infty(D))$ for the space of sequences $\{f_n\}$ of functions in $H^\infty(D)$ with

$$\|\{f_n\}\| = \sum_{n=0}^{\infty} \|f_n\|_\infty < \infty .$$

Let $F: \ell^1(H^\infty(D)) \to \mathscr{L}(\mathscr{X})$ be the map defined by

$$F(\{f_n\}) = \sum_{n=0}^{\infty} K^n f_n(T) .$$

(We agree that $K^0 = I$.) Let $\mathscr{A}$ be the range of F. Obviously, T and K are in $\mathscr{A}$.

For $f \in H^\infty(D)$, we define $f^{(n)}$ inductively as follows: $f^{(0)}(z) \equiv z$, $f^{(1)} = f$ and $f^{(n+1)}(z) = f^{(n)}(f(z))$. From $TK = K\varphi(T)$ we obtain $TK^n = K^n\varphi^{(n)}(T)$ for $n = 0,1,2, \ldots$ and hence $g(T)K^n = K^n g(\varphi^{(n)}(T))$ for all n and all g in $H^\infty(D)$. Now, for $\{f_n\}$, $\{g_n)$ in $\ell^1(H^\infty(D))$, we have

$$F(\{f_n\})F(\{g_n\}) = \sum_{n=0}^{\infty} \Big(\sum_{j+k=n} K^j f_j(T) K^k g_k(T)\Big) =$$

$$= \sum_{n=0}^{\infty} \sum_{j+k=n} K^n f_j(\varphi^{(k)}(T)) g_k(T) = F(\{h_n\})$$

where

$$h_n(z) = \sum_{j+k=n} f_j(\varphi^{(k)}(z)) g_k(z) .$$

Note that $\{h_n\} \in \ell^1(H^\infty(D))$; in fact, $\|\{h_n\}\| \leqslant \|\{f_n\}\| \, \|\{g_n\}\|$. Therefore the image of F is an algebra of operators in $\mathscr{L}(\mathscr{X})$.

It is easy to see that there is a positive number M such that $\|g(T)\| \leqslant M\|g\|_\infty$ for all g in $H^\infty(D)$. (See, e.g., the proof of Corollary 1 in [2].) Hence, for $\{f_n\} \in \ell^1(H(D))$, we have

$$\|F(\{f_n\})\| \leqslant \sum_{n=0}^{\infty} \|f_n(T)\| \leqslant M\|\{f_n\}\|.$$

Therefore $\mathscr{A}$ is the range of the bounded linear map F. In view of Lemma 3, it remains to show $\mathscr{A}K \subseteq K\mathscr{A}$. Now, for $\{f_n\} \in \ell^1(H^\infty(D))$,

$$F(\{f_n\})K = \sum_{n=0}^{\infty} K^n f_n(T)K =$$

$$= \sum_{n=0}^{\infty} K^{n+1} f_n(\varphi(T)) = KF(\{f_n \circ \varphi\}).$$

The proof is complete.

Proof of Theorem 2. In case $\varphi(0) \neq 0$, we write $\tilde{A}$ for the image of $A \in \mathscr{L}(\mathscr{X})$ in the quotient algebra $\mathscr{L}(\mathscr{X})/\mathscr{K}(\mathscr{X})$ (where $\mathscr{K}(\mathscr{X})$ stands for the set of all compact operators on $\mathscr{X}$) and hence obtain $0 = \tilde{K}\tilde{T} = \tilde{T}\varphi(\tilde{K}) = \tilde{T}\cdot\varphi(0)$. Thus, in this case, T is compact and we can make the conclusion from Theorem 1. Hence we may assume that $\varphi(0) = 0$. We may also assume that $\|T\| < 1$. Let $F: \ell^1(H^\infty(D)) \to \to \mathscr{L}(\mathscr{X})$ be the mapping defined by

$$F(\{f_n\}) = \sum_{n=0}^{\infty} T^n f_n(K)$$

and let $\mathscr{A}$ be the range of F. Obviously, $T, K \in \mathscr{A}$. By using the same argument as that in the proof of Theorem 1, we see that F is a bounded linear mapping from the Banach space $\ell^1(H^\infty(D))$ onto $\mathscr{A}$ which is a subalgebra of $\mathscr{L}(\mathscr{X})$.

Notice that there exists a positive number c (depending on D only) such that for every holomorphic function h with $h(0) = 0$ and $h(D) \subseteq D$, $|h(z)| \leqslant c|z|$. (In fact, let r, R be positive numbers such that $|z| \leqslant R$ for all $z \in D$ and $|z| \leqslant r$ implies $z \in D$. For $|z| \geqslant r$, obviously we have $|h(z)| \leqslant r^{-1}R\,|z|$. For $|z| \leqslant r$, by applying Schwarz's inequality, we obtain the same inequality. Take $c = r^{-1}R$.) In particular, we have $\varphi^{(n)}(z) = zg_n(z)$ where $\|g_n\|_\infty \leqslant c$ for $n = 0,1,2, \ldots$. (Here, $\varphi^{(n)}$ is defined by repeated composition of φ in the same way as $f^{(n)}$ defined in the proof of Theorem 1.) Hence

$$KF(\{f_n\}) = \sum_{n=0}^{\infty} KT^n f_n(K) =$$

$$= \sum_{n=0}^{\infty} T^n \varphi^{(n)}(K) f_n(K) = F(\{f_n g_n\})K$$

with $\|\{f_n g_n\}\| \leqslant c \|\{f_n\}\| < \infty$. Therefore we have $K\mathscr{A} \subseteq \mathscr{A}K$. Now the conclusion follows from Lemma 3.

Following [4], a collection of operators is said to be *simultaneously triangularizable* if there is a maximal chain of subspaces which are invariant for each operator in this collection. According to this definition, we have:

COROLLARY 4. *If K, T are compact operators on a Banach space $\mathscr{X}$ for which there is a bounded open set D containing $\sigma(T)$ and an analytic function φ from D into D such that $TK = K\varphi(T)$ and if K is quasinilpotent, then $\{T, K\}$ is simultaneously triangularizable and for every noncommutative polynomial p in two variables, $\sigma(p(K, T)) \subseteq p(\sigma(K), \sigma(T))$.*

Proof. That the second conclusion follows from the first one was shown in [4; Theorem 3.2] by means of Ringrose's Theorem. It remains to show that $\{T, K\}$ is simultaneously triangularizable.

By Zorn's lemma, we see that there exists a maximal chain $\mathscr{C}$ of common invariant subspaces of $\{T, K\}$. We proceed to show that $\mathscr{C}$ is a maximal chain of subspaces, i.e., for each $\mathscr{M}$ in $\mathscr{C}$, if $\mathscr{M}_-$ denotes the closed linear span of those subspaces which are properly contained in $\mathscr{M}$, then the dimension of $\mathscr{M}/\mathscr{M}_-$ is at most one.

Suppose the contrary that $\dim \mathscr{M}/\mathscr{M}_- \geqslant 2$. Let $\hat{T}$, $\hat{K}$ denote the operators on $\mathscr{M}/\mathscr{M}_-$ induced by T, K respectively. Then it is not difficult to show that $\hat{T}\hat{K} = \hat{K}\varphi(\hat{T})$. The existence of a subspace invariant for both $\hat{T}$ and $\hat{K}$ is guaranteed by Theorem 1 if $\hat{K} \neq 0$. In case $\hat{K} = 0$, any invariant subspace of $\hat{T}$ would be invariant for both $\hat{K}$ and $\hat{T}$. Let $\hat{\mathscr{N}}$ be a proper common invariant subspace of $\hat{K}$ and $\hat{T}$ and let $\mathscr{N}$ be its preimage in $\mathscr{X}$. Then $\mathscr{N} \notin \mathscr{C}$ and $\mathscr{C} \cup \{\mathscr{N}\}$ is a chain of subspaces invariant for both T and K. This contradicts the maximality of $\mathscr{C}$. The proof is complete.

The following example shows that two "almost commuting" operators may not have a common invariant subspace if the compactness assumption is dropped.

EXAMPLE. Let $\{e_n: -\infty < n < \infty\}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$ and U be the bilateral shift defined by $Ue_n = e_{n+1}$ and S be the backward weighted shift defined by $Se_{n+1} = \lambda^n e_n$ where $\lambda = \exp(2\pi i\theta)$ with θ irrational. Then $\lambda US = SU$ but S and U do not have a common invariant subspace.

To prove this, let $T = SU$. Then $Te_n = \lambda^n e_n$ for all n. Suppose that $\mathscr{M}$ is a subspace invariant for both U and S. Then $\mathscr{M}$ is also invariant for T. We claim that $\mathscr{M}$ is actually reducing for T. In fact, for a given unit vector x in $\mathscr{M}$ and $\varepsilon > 0$, we can choose some positive integers N and k such that $\left\| x - \sum_{n=-N}^{N} (x, e_n)e_n \right\| < \varepsilon$ and $|\lambda^k - \bar{\lambda}| < N^{-1}\varepsilon$. Now for $-N \leqslant n \leqslant N$,

$$T^k e_n - T^* e_n = (\lambda^{kn} - \bar{\lambda}^n)e_n$$

with $|\lambda^{kn} - \bar{\lambda}^n| \leqslant N | \lambda^k - \bar{\lambda}| < \varepsilon$. By a 3ε-argument, we obtain $\|T^k x - T^* x\| \leqslant 3\varepsilon$. Therefore $T^* \mathcal{M} \subseteq \mathcal{M}$.

Thus every invariant subspace of T is reducing and therefore must be the closed linear span of a subset of $\{e_n : -\infty < n < \infty\}$. Of course no such subspace can be invariant for both U and S except the trivial ones.

Finally, we remark that Theorem 1 still holds if the identity $TK = K\varphi(T)$ is replaced by $KT = \varphi(T)K$ and, similarly, Theorem 2 holds if $KT = T\varphi(K)$ is replaced by $TK = \varphi(K)T$.

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