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LIE GROUPS OVER THE FIELD OF RATIONAL FUNCTIONS, SIGNED SPECTRAL FACTORIZATION, SIGNED INTERPOLATION, AND AMPLIFIER DESIGN

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INTRODUCTION

This paper concerns a Lie group of $2n \times 2n$ matrices over the field $\mathscr{R}$ of functions on the unit circle with rational continuations to the complex plane. The group we will study consists of all such matrices which satisfy

$$g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and is denoted by $\mathscr{R}U(n, n)$. We also investigate a particular semigroup $\mathscr{R}U^+(n, n)$ consisting of members of $\mathscr{R}U(n, n)$ which satisfy certain analyticity properties. The study of $\mathscr{R}U^+(n, n)$ is closely bound up with classical Nevanlinna-Pick interpolation theory extended to Grassmannian valued functions.

Although the paper is entirely mathematical the motivation for it is physical. The desire was to build the mathematical machinery appropriate for a systematic theory of amplifier design. So the last two sections of the paper concern an optimization problem which physically amounts to the design of a (linearized) transistor amplifier with maximum gain over all frequencies. While we were not able to fully solve the problem we do make a significant reduction, and it is reasonable to believe that some of the main theorems herein will be a part of a unified theory if one ever exists.

Now we state our main mathematical results. The group $\mathscr{R}U(n,n)$ acts via the linear fractional map

$$\mathscr{G}_g(m) = (\alpha m + \beta)(\varkappa m + \gamma)^{-1}$$

on $\mathscr{R}M_n$, the $n \times n$ matrices with entries from $\mathscr{R}$; here $g = \begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix} \in \mathscr{R}U(n, n)$ and $m \in \mathscr{R}M_n$. The first undertaking in this article is to determine some basics about the orbits of this transformation group. In particular we determine the orbits of the constant elements in $\mathscr{R}M_n$ and find that they are just what one would hope.

To describe these orbits define $\mathscr{R}M(j, k, l)$ to be the matrix functions in $\mathscr{R}M_{j+k+l}$ whose values at each point on the circle are matrices with j singular values less than 1, k singular values equal to 1, and l singular values bigger than one, that is, matrices in a set we denote by $M(j, k, l)$. Recall the singular values of a matrix m are the eigenvalues of $|m| = (mm^*)^{\frac{1}{2}}$. We shall prove (§ 3)

THEOREM I.1. *The group $\mathscr{R}U(n, n)$ acting as transformations on $\mathscr{R}M_n$ has $\frac{(n+1)(n+2)}{2}$ orbits which intersect the constants. They are precisely the sets $\mathscr{R}M(i, j, k)$ where $i+j+k=n$.*

That $\mathscr{R}M(n, 0, 0)$ is an orbit of $\mathscr{R}U(n, n)$ is already known since it is the main content of an embedding theorem due to engineers Darlington, Belevitch, Ono--Yasuro, and to Potapov; for greater generality see [3], [12]. The analogue of Theorem 1.1 for matrices with complex entries (a trivial consequence of Witt's Theorem, cf. [24]) is a nice example of a general classification of orbits which is worked out in J. Wolf's article [43]. Theorem I.1 is decidedly nontrivial primarily because the field is not algebraically closed and this forces us to extend existing Wiener-Hopf factorization theory. In operator theoretic terms Theorem I.1 amounts to a study of when a given rational matrix function can be embedded as the upper diagonal entry of a 2×2 block rational matrix function with $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$-unitary values for some signature matrix J (i.e., J is self-adjoint and unitary).

The next part of the paper, §4, introduces and develops properties of a certain class of matrix-valued functions. The most typical such class is a generalization of $H^\infty(M_n)$, which one might call *symplectic* $H^\infty(M_n)$, defined as follows.

Let $[\,,]$ be a Hermitian bilinear form on $\mathbf{C}^n$ and call M in M_n a $[\,,]$-bounded matrix provided there is an $r > 0$ such that

$$[My, Mx] \leqslant r[x, y]$$

for all x, y in $\mathbf{C}^n$. The $[\,,]$ analogue of $H^\infty(M_n)$ is just the class of analytic and uniformly $[\,,]$-bounded functions on the disk. *There is an inner-outer factorization, a Wiener-Hopf factorization, a maximum principle, and a Nevanlinna-Pick type interpolation theorem for this class.* The classical theory where $[\,,]$ is positive definite applies to passive circuits or strictly amplifying devices [18], but for a mixed active-passive device (such as a transistor) one requires a signed bilinear form.

The interpolation theory for $[\,,]$-H^∞ functions is best described in a Grassmannian setting. Instead of studying $[\,,]$-contractive matrices in M_n one studies n-dimensional subspaces in $\mathbf{C}^{2n}$ (by considering the graphs of matrices in M_n) which are positive in a certain bilinear form $\langle\, ,\rangle$ on $\mathbf{C}^{2n}$ built from $[\,,]$. One identifies $[\,,]-H^\infty$ with these subspace-valued functions on the unit disk and considers such functions $\mathbf{F}$ for which the subspaces $\mathbf{F}(z_j)$ contain given subspaces S_j, that is $\mathbf{F}$ which "interpolate" S_j at z_j. The Nevanlinna-Pick Theorem in this context says

THEOREM I.2. *Suppose $n = 1$. An $\mathbf{F}$ which is analytic on the disk with $\langle\, ,\rangle$-positive values and which interpolates the one dimensional spaces S_j at z_j for $j = 1, \ldots, N$ exists if and only if the matrix*

$$\Lambda = \left\{ \frac{\langle s_j, s_k\rangle}{1 - z_j\bar{z}_k} \right\}_{j,\, k=1,\, \ldots,\, N}$$

is positive definite. Here $s_j \neq 0$ is any basis vector for S_j.

This is a "coordinate independent" statement of the Nevanlinna-Pick Theorem and we shall prove a generalization of it in § 4b. The coordinate-free setting turns out to be very important in our study, since the $[\,,]$-$H^\infty(M_n)$ functions themselves appear to be badly singular (unless the form $[\,,]$ is positive definite) only because of the "coordinates" implicit in their definition.

The ultimate concern of this paper is the optimization of certain classes of functions on the orbits of $\mathscr{R}U(n, n)$ and on particular subsets of these orbits. The functions to be optimized all arise from the cross ratio of two matrices. Define the cross ratio $\mathscr{E}(S, K)$ of two matrices S, K to be

$$\mathscr{E}(S, K) = (1 - SS^*)^{-\frac{1}{2}}(S - K)(S - K^*)^{-1}(S^* - K^*)(S^* - K)^{-1}(1 - SS^*)^{\frac{1}{2}}.$$

This is a slight modification [16] of C. L. Siegel's definition. From $\mathscr{E}$ one can construct infinitely many metrics on $M(n, 0, 0)$ which are invariant under the action of $U(n, n)$, for example $\operatorname{arctanh}\|\mathscr{E}(S, K)\|$ is an analogue of the classical Poincaré metric on the unit disk called the Carathéodory metric. The main goal of this article and of [18] is

PROBLEM I.3. Given S in $\mathscr{R}M(j, 0, l)$ with $j + l = n$, find H in $\mathscr{R}\mathscr{B}H^\infty(M_n)$ so that the k^{th} singular value of $\mathscr{E}(S, H)(e^{i\theta})$ is a prescribed function of θ.

Here $\mathscr{R}\mathscr{B}H^\infty(M_n)$ denotes the collection of those functions in $\mathscr{R}M_n$ strictly bounded in norm by 1 on the closed unit disk $\{|z| \leqslant 1\}$. While the problem is stated in terms of achieving a prescribed function it turns out to be equivalent to an optimization problem.

The article [16] concentrated on the case where S is in $\mathscr{R}M(n, 0, 0)$ and among other things explicitly on

$$\min_{H\in\mathscr{R}\mathscr{B}H^\infty} \|\mathscr{E}(S, H)\|_{L^\infty(M_n)},$$

that is, the "Poincaré distance" of S to $\mathscr{R}\mathscr{B}H^\infty(M_n)$. This article attacks the general problem for S in $\mathscr{R}M(j, 0, l)$ rather than S in $\mathscr{R}M(n, 0, 0)$.

To put these results in perspective, one can think of [16] as consisting of two parts. The first part converts Problem I.3 for $l = 0$ into an interpolation problem. The second applies the theory of interpolation developed over 60 years to solve the problem. This paper successfully generalizes part one of [16] to the full $l \neq 0$ problem. It develops a flexible machine which can be used to reduce many problems of type I.3 to interpolation problems. Furthermore, just as in the classical case, we are able to give a general matrix positive-definiteness test as to when certain problems of this type have a solution. This is a fairly conclusive step in a long campaign to perform such a reduction on a broad class of problems. We now have a collection of matrix interpolation problems; the easiest can be solved but many are still open. Thus the focus of efforts can shift to them and to making the matrix tests truly practical.

Application of our results to design of an amplifier which consists only of a transistor and energy conserving components is described in § 7. Also, in § 6 we explore the mathematical consequences of some important physical constraints, namely, that the amplifier be stable over all frequencies, and that real-valued inputs give rise to real (as opposed to complex)-valued outputs. Section 8 treats an amplifier consisting of a transistor and passive components.

OUTLINE

§ 1. Gives only results over the field $\mathbf{C}$ rather than over $\mathscr{R}$. It sets the stage for Theorem I.1 and then goes on to solve one central optimization problem over $\mathbf{C}$ in a manner which illustrates the strategy for the general case.

§ 2. Describes signed Wiener-Hopf factorization.

§ 3. Proves Theorem I.1.

§ 4. Treats symplectic H^∞.

§ 5. Treats the basic optimization problem.

§ 6. Adds physical constraints to § 5.

§ 7. Applies all of the above to amplifiers.

§ 8. Extends some results of §§ 1–7 to a certain subsemigroup of $\mathscr{R}\mathrm{GL}(n)$.

1. RESULTS OVER THE FIELD C

In this section we derive some basic properties about $U(n, n)$ and its orbits in M_n over the field $\mathbf{C}$ rather than $\mathscr{R}$. Recall that $U(n, n)$ is the group of $2n \times 2n$ matrices g which satisfy

$$g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};$$

any such $g =: \begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix} \in M_{2n}$ acts on an $n \times n$ matrix m ($m \in M_n$) according to the formula

$$\mathscr{G}_g(m) = (\alpha m + \beta)(\varkappa m + \gamma)^{-1},$$

with $\mathscr{G}_{g_1} \circ \mathscr{G}_{g_2} = \mathscr{G}_{g_1 g_2}$. A more general class of matrices which will be useful consists of those g which satisfy

$$g^* \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix} g = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix}$$

where each J_l is a signature matrix, i.e. $J_l^2 = I_l$ and $J_l^* = J_l$. Henceforth the letter J or J_l will denote a signature matrix and $U(J_1, J_2)$ will denote the class of g's above. We will say $g \in U(J_1, J_2)$ is nondegenerate if in addition γ is invertible. Note $U(J_1, J_2)$ is a group if and only if $J_1 = J_2$; also $U(n, n) = U(I, I)$. The fact that $U(n, n)$ maps the set of strict contraction matrices $M(n, 0, 0)$ into itself generalizes simply. Define a (J_1, J_2)-contraction to be an operator $T\colon \mathbf{C}^n \to \mathbf{C}^n$ which satisfies

$$T^* J_2 T \leqslant J_1 .$$

Call T a (J_1, J_2)-strict contraction or unitary whenever the inequality is strict or an equality. It is known that if T is (J_1, J_2)-contractive (or (J_1, J_2)-unitary), then T^* is (J_2, J_1)-contractive (respectively (J_2, J_1)-unitary) (see [38]).

PROPOSITION 1.1. *A nondegenerate linear fractional map $\mathscr{G}_g$ with coefficient matrix $g =: \begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}$ in $U(J_1, J_2)$ maps a dense set of (J_1, J_1)-contractions (strict contractions) (unitaries) onto a dense set of (J_2, J_2)-contractions (strict contractions) (unitaries). If g satisfies only*

$$g^* \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix} g \leqslant \begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix} \tag{1.1}$$

then we still have that $\mathscr{G}_g$ takes a dense set of (J_1, J_1)-contractions (strict contractions) into (J_2, J_2)-contractions (strict contractions).

The proof will be postponed to the end of § 1 after we introduce an equivalent alternative form of writing $\mathscr{G}_g$. In view of Proposition 1.1, it is natural to refer to maps $\mathscr{G}_g$ with g in $U(J_1, J_2)$ as (J_1, J_2)-cascade maps. If the reader is only interested in Theorem I.1 he can skip to the proof of Proposition 1.1.

We next introduce, for any signature matrix J, a "metric" $\mathscr{E}_J$ on M_n, the values of which will be similarity equivalence classes of M_n, such that the class of maps $\{\mathscr{G}_g \mid g \in U(J_1, J_2)\}$ is precisely the set of isometries from $\mathscr{E}_{J_1}$ to $\mathscr{E}_{J_2}$, that is

$$\mathscr{E}_{J_2}(\mathscr{G}_g(s), \mathscr{G}_g(h)) \cong \mathscr{E}_{J_1}(s, h) \quad \text{for } g \in U(J_1, J_2),\ s, h \in M_n,$$

where "$\cong$" means "is similar to". For the case where all J's equal I, the "cross ratio" of two matrices introduced by C. L. Siegel [40] accomplishes this. More generally, recall that there is a "cross-ratio" of four matrices

$$\mathscr{C}(z_1, z_2, z_3, z_4) = (z_1 - z_2)(z_1 - z_3)^{-1}(z_4 - z_3)(z_4 - z_2)^{-1}$$

(whenever the necessary inverses exist) which is invariant up to similarity under any linear fractional map [16]. One can use this to generate finer invariants for maps $\mathscr{G}_g$ with g from $U(J_1, J_2)$ as follows. For J a signature and s a matrix, define the J-Schwarz reflection by

$$s^{\sim J} = Js^{*-1}J$$

whenever it exists. It is a generalization of Lemma 2.2 in [16] that

LEMMA 1.2. *If g in M_{2n} is invertible, then*

$$\mathscr{G}_{g_0}(s)^{\sim J_2} = \mathscr{G}_g(s^{\sim J_1})$$

where $g_0 = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix} g^{*-1} \begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix}$. *In particular if g is in $U(J_1, J_2)$, then $g_0 = g$, and $\mathscr{G}_g$ intertwines $\sim J_1$ with $\sim J_2$.*

Proof. Set $h = \mathscr{G}_{g_0}(s)$. Then $s = \mathscr{G}_{g_0^{-1}}(h)$ where

$$g_0^{-1} = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix} g^* \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix} = \begin{pmatrix} J_1\alpha^*J_2 & -J_1\varkappa^*J_2 \\ -J_1\beta^*J_2 & J_1\gamma^*J_2 \end{pmatrix}.$$

If we solve the equation $s = \mathscr{G}_{g_0^{-1}}(h)$ for h, we obtain

$$\begin{aligned} h &= (sJ_1\beta^*J_2 + J_1\alpha^*J_2)^{-1}(sJ_1\gamma^*J_2 + J_1\varkappa^*J_2) = \\ &= J_2(J_1sJ_1\beta^* + \alpha^*)^{-1}(J_1sJ_1\gamma^* + \varkappa^*)J_2. \end{aligned}$$

Hence

$$h^{\sim J_2} = (\alpha + \beta J_1 s^* J_1)(\varkappa + \gamma J_1 s^* J_1)^{-1} = \mathscr{G}_g(s^{\sim J_1})$$

as claimed.

Now use the symmetry $\sim$ to "reduce" the cross ratio, namely define

$$\mathscr{E}_J(s, h) = \mathscr{C}(h, s, s^{\sim J}, h^{\sim J})$$

and observe (in the spirit of [20]) that for g in $U(J_1, J_2)$

$$\mathscr{E}_{J_2}(\mathscr{G}_g(s), \mathscr{G}_g(h)) = \mathscr{C}(\mathscr{G}_g(h), \mathscr{G}_g(s), \mathscr{G}_g(s^{\sim J_1}), \mathscr{G}_g(h^{\sim J_1}))$$

is similar to

$$\mathscr{C}(h, s, s^{\sim J_1}, h^{\sim J_1}) = \mathscr{E}_{J_1}(s, h).$$

We have obtained

PROPOSITION 1.3. *If g is in $U(J_1, J_2)$, then $\mathscr{E}_{J_2}(\mathscr{G}_g(s), \mathscr{G}_g(h))$ is similar to $\mathscr{E}_{J_1}(s, h)$ whenever all relevant inverses exist.*

The invariant $\mathscr{E}_J$ bears a special relation to 0, namely,

$$\mathscr{E}_J(s, 0) = sJs^*J = ss^+$$

where $s^+ = Js^*J$ is the adjoint of the matrix s with respect to the bilinear form (Jx, y). We refer to the eigenvalues of the matrix ss^+, in analogy with the case where $J = I$, as the squares of the J-singular values for s. The analogy is especially good when s is a J-contraction (see [38] and § 6).

Recall (from the Introduction) that $M(j, k, l)$ denotes the class of all $n \times n$ matrices $(n = j + k + l)$ m with j singular values < 1, k singular values $= 1$, and l singular values > 1. In the classical case where we let $U(I, I)$ act on $M(n, 0, 0)$, for many problems it is convenient to be able to map a given $m \in M(n, 0, 0)$ to the 0-matrix via an isometry $\mathscr{G}_g$ $(g \in U(I, I))$. The fact that this is always possible is essentially Darlington's Theorem for the frequency independent case. In working with the more general class $M(j, 0, n-j)$, this is still possible, as long as we are willing to take g in $U(I, J)$. The result might be termed a "J-Darlington Theorem" for the frequency independent case.

LEMMA 1.4. *Let s be a matrix in $M(j, 0, n-j)$ and set $J = \begin{pmatrix} I_j & 0 \\ 0 & I_{n-j} \end{pmatrix}$. Then there is a g in $U(J, I)$ (respectively h in $U(I, J)$) so that $\mathscr{G}_g(0) = s$ (respectively $\mathscr{G}_h(s) = 0$).*

Proof. Since $s \in M(j, 0, l)$ there is an invertible matrix q such that $I - s^*s = q^*Jq$. (Indeed, choose $q = |I - s^*s|u$ where u is a unitary which diagonalizes $I - s^*s$ and $|I - s^*s|$ is the nonnegative square root of $(I - s^*s)^2$.) Similarly there is an invertible matrix r such that $I - ss^* = r^*Jr$. Set

$$g_s = \begin{pmatrix} r^{-1}J & sq^{-1}J \\ s^*r^{-1}J & q^{-1}J \end{pmatrix}. \tag{1.2}$$

Then it is easily verified that g_s is in $U(J, I)$, and clearly $\mathscr{G}_{g_s}(0) = sq^{*-1}J(q^{*-1}J)^{-1} = s$.

To build h, observe that $\mathscr{G}_h(s) = 0$ is the same as $\mathscr{G}_h^{-1}(0) = \mathscr{G}_{h^{-1}}(0) = s$. Hence it suffices to take

$$h = g_s^{-1} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} g_s^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} r^{-1*} & -r^{-1*}s \\ q^{-1*}s^* & q^{-1*} \end{pmatrix}. \tag{1.3}$$

The lemma follows.

Given a matrix s in $M(j, 0, l)$, we let g_s denote any matrix in $U(J, I)$ with $\mathscr{G}_{g_s}(0) = s$, and g^s any matrix in $U(I, J)$ with $\mathscr{G}_{g^s}(s) = 0$.

The main problem to be addressed in this paper is stated in the introduction as Problem I.2. It is an optimization problem involving $U(n, n)$ over $\mathscr{R}$. Naturally there is a corresponding (much easier) problem over **C** which we shall now solve. The solution is highly instructive, since it gives the thread of the more general argument. Physically it corresponds to designing a certain type of transistor amplifier to have large gain at one fixed frequency and the result we obtain is surely no surprise to many engineers; our (systematic) approach is certainly new.

The problem is: Given $A \in M(1, 0, 1)$ find

$$\delta := \max_{B \in M(2,0,0)} \text{e.v}_1[\mathscr{E}_I(B, A)]$$

where e.v_1 is the largest eigenvalue of the matrix. Find all B which give the maximum value. The answer is

THEOREM 1.5. *The maximum is $\delta = \infty$. The set of all matrices B achieving this maximum is*

$$\left\{ \lim_{r \uparrow \infty} \mathscr{G}_{g_A} \left(u \begin{bmatrix} c & 0 \\ 0 & r \end{bmatrix} v \right) : \textit{where } 0 \leqslant c < 1 \textit{ and } u, v \textit{ are } J\textit{-unitary matrices} \right\}$$

where ∞ is formally permitted as an entry. Here $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. First convert the problem to a simpler form by performing a sequence of manipulations on $\mathscr{E}_I(B, A)$. By Lemma 1.4 the matrix g^A in $U(I, J)$ induces a map $\mathscr{G}_{g^A}$ with $\mathscr{G}_{g^A}(A) = 0$. By Proposition 1.3 $\mathscr{E}_I(B, A)$ and $\mathscr{E}_J(\mathscr{G}_{g^A}(B), 0) = \mathscr{G}_{g^A}(B)[\mathscr{G}_{g^A}(B)]^+$ have the same eigenvalues. By Proposition 1.1

$$\mathscr{G}_{g^A}: M(2, 0, 0) \to \{\text{all strict } J\text{-contractions}\}.$$

Thus the basic problem is converted to finding

$$\max_{M \text{ a strict } J\text{-contraction}} \text{e.v}_1(MM^+).$$

This is easy to compute because after all $K_r := \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}$ is a J-contraction whenever $r > 1$. Its J-singular values, namely the square roots of the eigenvalues of $K_r K_r^+$, are $\{0, r\}$; thus the maximum is ∞. Furthermore, an arbitrary J-contraction with a J-singular value equal to $r > 1$ has the form $u \begin{bmatrix} c & 0 \\ 0 & r \end{bmatrix} v$ where u, v are arbitrary J-unitary matrices and $0 \leqslant c < 1$ (see [38]). Thus the set of all matrices $B \in M(2, 0, 0)$ for which the optimum ∞ is achieved is given by the formula in the theorem.

Having stated the main results of the section we turn next to the proof of Proposition 1.1. We begin by introducing another way of writing linear fractional maps which is commonplace in engineering. If $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with matrix entries a, b, c, d in M_n define a map F_u on M_n by

$$F_u(s) = a + bs(1 - ds)^{-1}c. \tag{1.4}$$

If c is invertible one can write $F_u(s) = \mathscr{G}_g(s)$ where

$$g = \begin{pmatrix} b - ac^{-1}d & ac^{-1} \\ -c^{-1}d & c^{-1} \end{pmatrix}. \tag{1.5}$$

Conversely, any map $\mathscr{G}_g$ with $g = \begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}$ with γ invertible can be written as F_u with

$$u = \begin{pmatrix} \beta\gamma^{-1} & \alpha - \beta\gamma^{-1}\varkappa \\ \gamma^{-1} & -\gamma^{-1}\varkappa \end{pmatrix}. \tag{1.6}$$

Thus F_u is *nondegenerate* if and only if c is invertible. For example, the matrix g_s of Lemma 1.4 transforms via (1.6) to

$$\Delta_s := \begin{pmatrix} s & r^* \\ Jq & -Jqs^*r^{-1}J \end{pmatrix}. \tag{1.7}$$

Thus $F_{\Delta_s}(0) = s$. Likewise g^s transforms to a matrix we shall denote ∇_s, i.e. $F_{\nabla_s}(s) = 0$. One can show that if g is in $U(J_1, J_2)$ (resp. g satisfies (1.1)) and γ is invertible then the corresponding u is $\left(\begin{pmatrix} J_2 & 0 \\ 0 & J_1 \end{pmatrix}, \begin{pmatrix} J_2 & 0 \\ 0 & J_1 \end{pmatrix}\right)$-unitary (resp. contractive) with invertible c. Also the converse holds.

In the rest of this paper most proofs (including the one immediately following) will be given in terms of F_u rather than $\mathscr{G}_g$ since the F_u conventions behave well over $\mathscr{C}$ when analyticity properties are an issue.

Proof of Proposition 1.1. The proof is an adaptation of that of [19] for the case where all signatures are the identity. We do it more generally than is required for the proposition. Suppose u is $\left(\begin{pmatrix} J_2 & 0 \\ 0 & J_{1*} \end{pmatrix}, \begin{pmatrix} J_{2*} & 0 \\ 0 & J_1 \end{pmatrix}\right)$-contractive and s is a (J_1, J_{1*})-contraction for which $1 - ds$ is invertible. (In Proposition 1.1 $J_{1*} = J_1$ and $J_{2*} = J_2$.) By an easy perturbation argument such s's are dense in the set of all (J_1, J_{1*})-contractions. Let x_1 be any vector in $\mathbf{C}^n$ and set $y_1 = F_u(s)x_1$. Motivated by the physical meaning of $F_u(s)$ as a cascade loading of u on s, we rewrite the equation

$$y_1 = F_u(s)x_1$$

as the system of equations

$$(1.8)\qquad \begin{aligned} y_1 &= ax_1 + bx_2 \\ y_2 &= cx_1 + dx_2 \\ x_2 &= y = sx \\ y_2 &= x. \end{aligned}$$

Hence

$$(1.9)\qquad \begin{aligned} (J_2x_1, x_1) - (J_{2*}y_1, y_1) &= (J_2x_1, x_1) + \{(J_{1*}x_2, x_2) - (J_{1*}y, y)\} - \\ &\quad - (J_{2*}y_1, y_1) + \{(J_1x, x) - (J_1y_2, y_2)\} = \\ &= [(J_2x_1, x_1) + (J_{1*}x_2, x_2) - (J_{2*}y_1, y_1) - (J_1y_2, y_2)] + \\ &\quad + [(J_1x, x) - (J_{1*}y, y)] \geqslant 0 \end{aligned}$$

with equality if both u is $\left(\begin{pmatrix} J_2 & 0 \\ 0 & J_{1*} \end{pmatrix}, \begin{pmatrix} J_{2*} & 0 \\ 0 & J_1 \end{pmatrix}\right)$-unitary and s is (J_1, J_{1*})-unitary. Furthermore, using (1.8) to express (1.9) only in terms of u, s and the arbitrary x_1, we obtain the general relation

$$(1.10)\qquad \begin{aligned} &J_2 - F_u(s)^*J_{2*}F_u(s) = \\ &=(c^*(1-s^*d^*)^{-1}s^*) \left\{\begin{pmatrix} J_2 & 0 \\ 0 & J_{1*} \end{pmatrix} - u^*\begin{pmatrix} J_{2*} & 0 \\ 0 & J_1 \end{pmatrix} u\right\} \begin{pmatrix} 1 \\ s(1-ds)^{-1}c \end{pmatrix} + \\ &+ c^*(1-s^*d^*)^{-1}\{J_1 - s^*J_{1*}s\}(1-ds)^{-1}c. \end{aligned}$$

From this equation it follows immediately that if c is invertible and s is a strict (J_1, J_{1*})-contraction, then $F_u(s)$ is a strict (J_2, J_{2*})-contraction, and conversely, if $F_u(s)$ is a strict contraction, then necessarily c is invertible.

It remains only to show that when u is unitary and c is invertible, then the image of a dense set of strict contractions is a dense set of strict contractions, and similarly for unitaries. For this we switch to the $\begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}$-convention, where composition of maps corresponds to multiplication of matrices. In particular, if $\begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}$

is $\left(\begin{pmatrix} J_{1*} & 0 \\ 0 & -J_1 \end{pmatrix}, \begin{pmatrix} J_{2*} & 0 \\ 0 & -J_2 \end{pmatrix}\right)$-unitary with γ invertible, then $\begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}^{-1} =$ $= \begin{pmatrix} J_{1*} & 0 \\ 0 & -J_1 \end{pmatrix} \begin{pmatrix} \alpha^* & \varkappa^* \\ \beta^* & \gamma^* \end{pmatrix} \begin{pmatrix} J_{2*} & 0 \\ 0 & -J_2 \end{pmatrix}$ is $\left(\begin{pmatrix} J_{2*} & 0 \\ 0 & -J_2 \end{pmatrix}, \begin{pmatrix} J_{1*} & 0 \\ 0 & -J_1 \end{pmatrix}\right)$-unitary with its lower diagonal coefficient invertible, and hence $(F_u)^{-1}$ maps a dense set of strict (J_2, J_{2*})-contractions into strict (J_2, J_{2*})-contractions, and a dense set of (J_2, J_{2*})-unitaries into (J_1, J_{1*})-unitaries. The proposition follows.

We conclude the section by sketching a proof of

THEOREM 1.6. *The orbits of $U(n, n)$ acting on M_n are the sets $M(j, k, l)$ with $j + k + l = n$.*

This is well known [43], but our proof is different from the usual one based on Witt's theorem and, more importantly, serves as an outline for our proof of Theorem I.1. In particular it shows the role of the (I, J)-cascade maps in our approach.

Proof. In the $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ notation we are concerned with the orbits of (I_{2n}, I_{2n})-unitaries $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with c invertible acting according to the formula (1.4). It is clear from formula (1.10) (with all J's equal I) that the classes $M(j, k, l)$ are left invariant by all such F_u, since any given s belongs to $M(j, k, l)$ if and only if $1 - s^*s$ has j positive eigenvalues, k eigenvalues equal to 0, and l negative eigenvalues.

Conversely, suppose s is in $M(j, k, l)$. We will produce a unitary $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with c invertible such that

$$F_u(s) = \begin{pmatrix} 0_j & & 0 \\ & \sqrt{2} I_l & \\ 0 & & I_k \end{pmatrix}$$

as follows. First, we can choose u_1 of the form $\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}$ (with U and V unitary) so that

$$F_{u_1}(s) = \begin{pmatrix} s_1 & 0 \\ 0 & I_k \end{pmatrix}$$

where $s_1 \in M(j, 0, l)$. Thus if we can produce a $2(j+l) \times 2(j+l)$ unitary v_1 such that $F_{v_1}(s_1) = \begin{pmatrix} 0_j & 0 \\ 0 & \sqrt{2} I_l \end{pmatrix}$, and set

$$u_2 = \begin{pmatrix} v_1 & 0 \\ 0 & \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \end{pmatrix},$$

then $F = F_{u_2} \circ F_{u_1}$ has the desired properties.

Let V_{s_1} be the $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary matrix given by Lemma 1.3, so $F_{V_{s_1}}(s_1) = 0$, and $F_{V_{s_1}}$ maps strict contractions into strict J-contractions. If we then let $\tau = \Delta_{\begin{pmatrix} 0_j & 0 \\ 0 & \sqrt{2}I_l \end{pmatrix}}$, then $F_\tau(0) = \begin{pmatrix} 0_j & 0 \\ 0 & \sqrt{2}I_l \end{pmatrix}$ and, since τ is $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$-unitary, F_τ maps strict J-contractions into strict contractions. Thus the composition $F_\tau \circ F_{V_{s_1}}$ maps strict contractions into strict contractions, and if we choose v_1 so that $F_{v_1} = F_\tau \circ F_{V_{s_1}}$, v_1 has all the properties required.

2. SIGNED SPECTRAL FACTORIZATIONS

Since the field $\mathscr{R}$ of rational functions is not algebraically closed (in particular the square root of a rational function is not rational), the proofs of the previous section do not extend immediately to the case where the underlying field **C** is replaced by $\mathscr{R}$. An adequate substitute is the notion of spectral factorization (both signed and unsigned) for a self-adjoint valued rational function.

Let $L^\infty(M_n)$ be the Banach space of M_n-valued functions on the circle $\{|z| = 1\}$ which are uniformly bounded in norm and $H^\infty(M_n)$ the closed subspace of those functions arising as the a.e.-existing radial limits of functions analytic on the disk $\{|z| < 1\}$. The letter $\mathscr{R}$ which stands for rational functions will be used frequently as a prefix, e.g. $\mathscr{R}L^\infty$ means rational functions uniformly bounded on the circle. A function H in $L^\infty(M_n)$ is said to be *uniformly invertible* if its inverse H^{-1} is in $L^\infty(M_n)$. For a self-adjoint valued function H in $L^\infty(M_n)$, we say that H has a *spectral factorization* if $H(e^{i\theta}) = A(e^{i\theta})^* A(e^{i\theta})$ and an *(analytic) signed spectral factorization* if $H(e^{i\theta}) = A(e^{i\theta})^* JA(e^{i\theta})$ where A is in $H^\infty(M_n)$ and J is a (constant) signature matrix. It is convenient to note here that the existence of a *rational signed spectral factorization*

$$H(e^{i\theta}) = A(e^{i\theta})^* JA(e^{i\theta}), \quad A \in \mathscr{R}L^\infty(M_n)$$

is equivalent to the existence of an *analytic* signed spectral factorization, as one can always cancel any poles of A inside the disk by multiplying by a scalar inner function (an H^∞ function with values of modulus 1 on the unit circle) without affecting the values of A^*JA on the unit circle. Thus Theorem 2.1, below, establishes not only the existence of a rational signed spectral factorization but also of an analytic one.

THEOREM 2.1. *Let H in $\mathscr{R}L^\infty(M_n)$ be uniformly invertible and self-adjoint. Then the signature $(l, 0, n-l)$ of the matrix $H(e^{i\theta})$ is independent of θ and there exists an A in $\mathscr{R}H^\infty(M_n)$ such that*

$$H(e^{i\theta}) = A(e^{i\theta})^* JA(e^{i\theta}).$$

Here $J = \begin{pmatrix} I_l & 0 \\ 0 & -I_{n-l} \end{pmatrix}$.

Theorem 2.1 is due to Nikolaichuk and Spitkovskii [33—34] (see also [11]). For another proof see [8]. The authors are very grateful to the referee and to K. Clancey for pointing out [33—34], since we had overlooked it and rederived their results.

We refer to the factorization $H = A^*JA$ as called outer whenever A is outer. It can be shown that the set of H as in Theorem 2.1 with outer signed spectral factorizations is L^∞ norm dense. As is well known (see [15]), any invertible function $H \in \mathscr{R}M_n$ has a so called *canonical factorization*

$$H(e^{i\theta}) = B(e^{i\theta})^* D(e^{i\theta}) A(e^{i\theta})$$

where A and B are outer (i.e. $A \cdot H^2(\mathbf{C}^n)$ and $B \cdot H^2(\mathbf{C}^n)$ are dense in $H^2(\mathbf{C}^n)$) and D is a diagonal matrix of the form

$$D(e^{i\theta}) = \operatorname{diag}\{e^{ik_1\theta}, \ldots, e^{ik_n\theta}\}$$

for some integers $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n$. The integers $\{k_1, \ldots, k_n\}$ are uniquely determined by H and are called the *(right) partial factorization indices* of H. If H is also self-adjoint valued on the unit circle, then H has an outer signed spectral factorization as above if and only if all partial indices are zero. If the values of H are positive-definite, then all partial indices of H are zero, and hence H has an outer factorization as above with $J = I$; this is classical spectral factorization.

Finally, to give perspective we remark that the set of partial factorization indices for a self-adjoint uniformly invertible matrix function $H(e^{i\theta})$ is not stable under small (self-adjoint) perturbations of H, except for the case where all partial indices are 0; in fact, the case where all partial indices are zero is the "*generic*" case. Analogous results were given by Gohberg-Kreĭn [15], who worked with general invertible matrix functions, and thus did not restrict themselves to self-adjoint perturbations; for the self-adjoint case see [33—34] also. A precise statement of what one can readily prove is

THEOREM 2.2. (i) *If $H(e^{i\theta}) \in \mathscr{R}M_n$ is self-adjoint and uniformly invertible with all partial indices equal to* 0, *then there is an $\varepsilon > 0$ such that, for any self-adjoint and invertible $K(e^{i\theta})$ with $\|H - K\|_\infty < \varepsilon$, $K(e^{i\theta})$ also has all its partial indices equal to* 0.

(ii) *Given any self-adjoint, invertible $H(e^{i\theta}) \in \mathscr{R}M_n$ and $\varepsilon > 0$, there exists a self-adjoint invertible $K(e^{i\theta}) \in \mathscr{R}M_n$ with $\|H - K\|_\infty < \varepsilon$ such that all partial indices for K are* 0.

3. ORBITS OF $\mathscr{R}U(n, n)$

The main accomplishment of this section is to prove Theorem I.1 which characterizes the orbits of the constants in $\mathscr{R}M_n$ under $\mathscr{R}U(n, n)$. The proof is easy. For an alternative proof see [8]. Besides the formulation of the theorem in terms of g

in $U(n, n)$ one could state it in terms of the "cascade" maps F_u given by (1.4). This second formulation is in the spirit of what has been done for the well-understood case of $\mathscr{R}M(n, 0, 0)$.

So we begin with the second formulation of the $k = 0$ case of Theorem 1.1.

THEOREM 3.1. *Let* $S(z) \in \mathscr{R}M(j, 0, l)$. *Then there exist* $B(z)$, $C(z)$, $D(z) \in \mathscr{R}L^\infty(M_n)$, *with* $C(z)$ *invertible for* $|z| = 1$, *such that*

$$\Delta_S(z) = \begin{pmatrix} S(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$*-unitary for* $|z| = 1$, *where* $J = \begin{pmatrix} I_j & 0 \\ 0 & -I_l \end{pmatrix}$.

Proof. By Theorem 2.1, there exist Q and R in $\mathscr{R}L^\infty(M^n)$ such that

$$I - S^*(z)S(z) = Q^*(z)JQ(z)$$

and

$$I - S(z)S^*(z) = R^*(z)JR(z).$$

Then

$$\Delta_S(z) = \begin{pmatrix} S(z) & R^*(z) \\ JQ(z) & -JQ(z)S^*(z)R(z)^{-1}J \end{pmatrix}$$

is $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$-unitary for $|z| = 1$, as desired.

COROLLARY 3.2. *For* $S(z)$ *in* $\mathscr{R}M(j, 0, l)$, *there exist* $A(z)$, $B(z)$, $C(z) \in \mathscr{R}L^\infty(M^n)$ $(n = j + l)$ *with* $C(z)$ *invertible for* $|z| = 1$, *such that*

$$\nabla_S(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & S^*(z) \end{pmatrix}$$

is $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$*-unitary for* $|z| = 1$.

As in § 1 for the constant case, we have $F_{\Delta_S}(0) = S$ and $F_{\nabla_S}(S) = 0$.

We shall need the following.

LEMMA 3.3. *If* H *in* $\mathscr{R}L^\infty(M_n)$ *is self-adjoint valued and* $\operatorname{rank} H(e^{i\theta}) = k$ *is a constant, then there is an inner function* U *in* $\mathscr{R}H^\infty(M_n)$ *such that*

$$H(e^{i\theta}) = U(e^{i\theta})^* \begin{pmatrix} N(e^{i\theta}) & 0 \\ 0 & 0_{n-k} \end{pmatrix} U(e^{i\theta})$$

with N *in* $\mathscr{R}L^\infty(M_k)$ *self-adjoint and uniformly invertible.*

Proof. It is well known (cf. [15], [32], [39]) that an outer rational spectral factorization always exists for a positive-definite rational matrix function; thus there is a rational outer function φ (with range in $\mathbf{C}^k$) so that $H^2 = \varphi^*\varphi$. Then $\varphi\varphi^*$ is uniformly invertible and positive definite, and hence has an outer factorization $\varphi\varphi^* = \psi^*\psi$ where $\psi^{\pm 1} \in \mathcal{R}H^\infty(M_k)$. Then it is easily checked that $v_1 = \varphi\varphi^{-1} \in \mathcal{R}H^\infty(M_{n\times k})$ is inner, with range equal to the range of H. By a similar construction starting with $I - v_1 v_1^*$ instead of H^2, we can construct an inner $v_2 \in \mathcal{R}H^\infty(M_{n\times(n-k)})$ with range equal to the kernel of H. Then $U = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ is an $(n \times n)$-inner function which satisfies all the requirements of the lemma.

Proof of Theorem I.1. Theorem I.1 amounts to the statement that if S is in $\mathcal{R}M(j, k, l)$ then there is a g in $\mathcal{R}U(n, n)$ so that $\mathcal{G}_g(S)$ is a constant matrix.

The first step is to prove that there exist rational phase functions U and V so that

$$V(z)S(z)U(z) = \begin{pmatrix} S_1(z) & 0 \\ 0 & I_k \end{pmatrix} \tag{3.1}$$

where S_1 is in $\mathcal{R}M(j, 0, l)$. From Lemma 3.3 we know that U, V exist so that

$$U^*(I - S^*S)U = \begin{pmatrix} N_1 & 0 \\ 0 & 0_k \end{pmatrix} \quad \text{and} \quad V(I - SS^*)V^* = \begin{pmatrix} N_2 & 0 \\ 0 & 0_k \end{pmatrix}$$

where N_1 and N_2 are uniformly invertible. Set $M = VSU$. Then

$$M^*M = \begin{pmatrix} I - N_1 & 0 \\ 0 & I_k \end{pmatrix} \quad \text{and} \quad MM^* = \begin{pmatrix} 1 - N_2 & 0 \\ 0 & I_k \end{pmatrix}$$

which implies that M has the form (3.1). This via a direct sum argument reduces the problem to that of mapping S_1 to a constant with a g in $\mathcal{R}U(j + l, j + l)$.

To accomplish this observe that V_{S_1} obtained in Corollary 3.2 is $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary and $F_{V_{S_1}}(S_1) = 0$. $\left(\text{Here } J = \begin{pmatrix} I_j & 0 \\ 0 & -I_l \end{pmatrix}.\right)$ By Proposition 1.1 $F_{V_{S_1}}$ maps contractions to J-contractions. Let $\tau = \Delta_{\begin{pmatrix} 0_j & 0 \\ 0 & \sqrt{2}I_l \end{pmatrix}}$; then $F_\tau(0) = \begin{pmatrix} 0_j & 0 \\ 0 & \sqrt{2}I_l \end{pmatrix}$ and since τ is $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$-unitary F_τ maps J-contractions to contractions. So $F_\tau \cdot F_{V_{S_1}}$ maps contractions to contractions and has the form F_w for w a unitary operator.

Furthermore, the coefficient c in the matrix representation $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible since the map F_w is the composition of maps whose matrices of coefficients enjoy this property.

REMARK. Proposition 3.1 is really only a partial generalization of the classical Darlington theorem since a key conclusion of it is that when $S \in \mathscr{R}M(n, 0, 0) \cap H^\infty$ one can find a unitary Δ_S in H^∞. In our Theorem I.1 no H^∞ condition plays a role; however, the full classical Darlington theorem follows trivially from Theorem I.1 by using inner functions to cancel unwanted poles inside the disk.

4. SYMPLECTIC $\mathscr{R}L^\infty$ AND $\mathscr{R}H^\infty$

Classical $L^\infty(M_n)$ consists of all uniformly bounded matrix valued functions on the circle. A natural class of matrices on **C** which arise surprisingly often are the J bounded ones, namely, $M \in M_n$ satisfying

$$(JMx, Mx) \leqslant r(Jx, x) \tag{4.1}$$

for all $x \in \mathbf{C}^n$ for some $r \geqslant 0$. An equivalent more geometric notion is that of a *plus-operator* (with respect to the $(-J)$-inner product rather than the J-inner product), for which an extensive theory has been developed (see [27–28] and [10]). In this section we shall study functions whose values are J-bounded, in particular, we shall concentrate on a space H_J^∞ of functions analytic on the disk with uniformly J-bounded values. Our goal is to give the basic properties of these and related sets of "symplectic bounded" functions which generalize standard L^∞ and H^∞ theory. This machinery is needed in the study of amplifiers; it certainly plays a major role in ours. The reason that H^∞ arises is that physical circuits correspond not to arbitrary rational functions but to ones which satisfy some analyticity properties. Amplifiers can have some modes which are energy dissipating and some which amplify (e.g. a transistor has one passive and one active mode); this forces one to use a signed bilinear form rather than a definite one. We saw in Section 1 when operators with J-bound equal to one (J-contractions) were introduced to solve the gain optimization problem (at fixed frequency). Thus it should be no surprise that variable frequency problems force us to study such spaces over the rational functions.

We begin by discussing the basics of J-bounded matrices and defining precisely several classes of functions. Any r satisfying (4.1) will be called a J-bound for M. In the familiar case $J = I$, the set of all J-bounds for M is the semibounded interval $\{\|M\| \leqslant r\}$. For the general case, we have the following.

PROPOSITION 4.1. *For a given $n \times n$ matrix M, the set of J-bounds for M is a (possibly empty) interval. When nonempty we denote its left endpoint as $|M|_J^-$ (the lower J bound for M) and its right endpoint as $|M|_J^+$ (the upper J bound for M).*

Proof. Without loss of generality, we may suppose that 1 is a J-bound for M, (so that M is a J-contraction), and $J = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then, if M is a strict J-contraction, by the theory of Potapov [38], M has a J-singular value decomposition

$$M = U \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix} V$$

where U and V are J-unitary, D_- is diagonal with eigenvalues between 0 and 1, and D_+ is diagonal with eigenvalues larger than 1. It is then easy to check that if we choose $|M|_J^-$ to be the largest eigenvalue of D_- and $|M|_J^+$ to be the smallest eigenvalue of D_+, then the set of J-bounds for M is the interval $|M|_J^- \leqslant r \leqslant |M|_J^+$. Since any J-contraction M can be approximated by one of the form above, the general result follows.

We now define symplectic analogues of such classical function spaces as L^∞, H^∞ and H_I^∞. Namely, we define $L_J^\infty(M_n)$ to be the set of measurable uniformly J-bounded matrix functions, and $\mathscr{R}H_J^\infty(M_n)$ to be all M in $\mathscr{R}L_J^\infty(M_n)$, the rational $L_J^\infty(M_n)$ functions, for which $M(z)$ is uniformly J-bounded on the unit disk $\{|z| \leqslant 1\}$.

For $J \neq I$, none of these sets is linear or normable in the usual sense; by definition they *are* closed under scalar multiplication. We shall refer to them only as symplectic *function sets* rather than as function spaces. As in the definite case, we let $\mathscr{B}L_J^\infty(M_n)$ be the set of all $L_J^\infty(M_n)$-functions with uniform J-bound of 1, and similarly for $\mathscr{R}\mathscr{B}H_J^\infty(M_n)$. As distinct from the definite case, these "balls" are not convex.

This section is divided into three parts. In Part (a) we shall show among other things that there is a Wiener-Hopf factorization, an inner-outer factorization, and a symplectic H_I^∞. In Part (b) we give an effective theory of interpolation extending that of Nevanlinna-Pick and others. In Part (c) we give a further extension of Nevanlinna-Pick theory.

(a) Factorization

We begin with some definitions. A function φ is said to be *J-inner* if φ is in $\mathscr{R}\mathscr{B}H_J^\infty$ and has J-unitary values on the boundary of the disk. A function F in $\mathscr{R}L_J^\infty$ is said to be *J-outer* if F is in $\mathscr{R}H_J^\infty$ and is analytic and invertible on the unit disk $\{|z| < 1\}$. The following has been proved by Potapov [38], although not quite in this language.

THEOREM 4.2. *Any A in $\mathscr{R}H_J^\infty$ can be factored as $A = \varphi F$ where φ is J-inner and F is J-outer. If $A = \varphi_1 F_1$ is another such factorization, then there is a constant J-unitary u such that $\varphi_1 = \varphi u$, $F_1 = u^\dagger F$. Here $u^\dagger = Ju^*J$.*

Proof. Potapov's factorization result is stated for J-contractions. Since any A in $\mathcal{R}H_J^\infty$ is a scalar multiple of a J-contraction, Theorem 4.2 easily follows from Potapov's result.

Also there is an outer spectral factorization result for $\mathcal{R}L_J^\infty$.

THEOREM 4.3. *A rational matrix function H in $\mathcal{R}L_J^\infty$ is J-self-adjoint valued with positive eigenvalues on the unit circle if and only if $H = A^\dagger A$ for some uniformly invertible (on the unit circle) J-outer function A ($A^\dagger = JA^*J$).*

Proof. By the results of § 2, the self-adjoint valued function JH has a factorization $JH = B^*JB$ for some rational B. Now by Theorem 4.7 to come, there is a J-inner φ so that $C = \varphi B \in \mathcal{R}H_J^\infty$. Then by the result of Potapov (Theorem 4.2 above), C in turn has a factorization $C = \psi A$ where ψ is J-inner and A is J-outer. Then clearly $A = \psi^\dagger \varphi B$ satisfies $A^\dagger A = H$.

Conversely, if $H = A^\dagger A$ for a uniformly invertible J-outer A, then clearly $H \in \mathcal{R}L_J^\infty$ and is J-self-adjoint valued. The fact that all the eigenvalues of $H(e^{i\theta})$ are positive follows from the discussion in Part (c) below.

The set $\mathcal{R}H_l^\infty(M_n)$ consists of those functions in $\mathcal{R}L^\infty(M_n)$ which have at most l poles in the disk. The symplectic generalization of this is essential to our paper. Actually we study only the unit ball $\mathcal{R}\mathcal{B}H_{J,l}^\infty(M_n)$ and just define $\mathcal{R}H_{J,l}^\infty(M_n)$ to consist of scalar multiples of $\mathcal{R}\mathcal{B}H_{J,l}^\infty(M_n)$.

Let E be any J-expansive matrix ($E^*JE - J$ is strictly positive definite). For each integer $j \geqslant 0$, complex number z in D and matrix function K in $\mathcal{R}M_n$ which is analytic at z_0, define an integer

$$N_{K,E,j}(z_0) = \dim \left\{ x \in \mathbf{C}^n : \frac{d^\alpha}{dz^\alpha}\{(K(z) - E)x\}\Big|_{z=z_0} = 0 \quad \text{for} \quad 0 \leqslant \alpha \leqslant i \right\}.$$

If K has a pole at z_0, apply the above definition to the restriction of K to the largest subspace on which K is analytic. In any case, we define the *order of contact between K and E at z_0* by

$$N(K, E)(z_0) = \sum_j N_{K,E,j}(z_0).$$

We now define $\mathcal{R}\mathcal{B}H_{J,l}^\infty$ by

$$\mathcal{R}\mathcal{B}H_{J,l}^\infty = \{K \in \mathcal{R}\mathcal{B}L_J^\infty : \sum_{z \in D} N(K, E)(z) \leqslant l\}.$$

As the notation suggests it will turn out that the definition is independent of the particular J-expansive matrix E. In particular, if we formally take $E = \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix}$

$\left(\text{where } J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\right)$, the assertion is roughly *that K has at most a total of l poles on a positive subspace or zeroes on a negative subspace inside the disk* (the two worst ways to violate being a J-contraction). Thus if we refer to a pole on a positive subspace or a zero on a negative subspace as a "J-pole", we can phrase the definition as saying that *K is constrained to have at most l J-poles*. This then is a canonical J-analogue of the space $\mathscr{RB}H_l^\infty(M_n)$. The fundamental fact concerning $\mathscr{RB}H_{J,l}^\infty$ is the following.

THEOREM 4.4. (i) *The definition of $\mathscr{RB}H_{J,l}^\infty$ is independent of the J-expansive matrix E.*

(ii) *For f any nondegenerate (I, J)-cascade map with constant coefficients, the set $f(\mathscr{RB}H_l^\infty(M_n))$ is independent of f, and is equal to $\mathscr{RB}H_{J,l}^\infty$.*

(iii) *$\mathscr{RB}H_{J,0}^\infty = \mathscr{RB}H_J^\infty$, that is, $\mathscr{RB}H_{J,0}^\infty$ can alternatively be characterized as the set of functions K in $\mathscr{R}M_n$ such that $K(z)$ is a J-contraction for $|z| \leqslant 1$.*

To prove Theorem 4.4, it will be convenient to use a special assumption which also came up in [16]. We say that a function K in $\mathscr{R}L^\infty(M_n)$ satisfies (N) if

(N) for z_0 any pole of K inside the disk, $K(z)^{-1}$ is a uniformly bounded analytic function of z for z in a neighborhood of z_0.

We shall need an easy consequence of (N).

LEMMA 4.5. *Assume $K \in \mathscr{R}L^\infty(M_n)$ satisfies assumption* (N). *Then, for a point z_0 inside the disk, the following are equivalent:*

(i) *z_0 is a pole for K,*

(ii) *z_0 is a pole for* $\det K$,

(iii) *z_0 is a zero for* $\det K^{-1}$,

(iv) *K^{-1} has analytic continuation to a neighborhood of z_0 and z_0 is a zero for K^{-1}.*

Proof. Let z_0 be a pole for K. Since K satisfies (N), $K^{-1}(z)$ has an analytic continuation to z_0. If the analytic continuation were invertible at z_0, then $K(z) = [K^{-1}(z)]^{-1}$ would be analytic at z_0, a contradiction. Thus (i) $\Rightarrow$ (iv). If K^{-1} is analytic at z_0 and $K^{-1}(z_0)$ is not invertible, then necessarily $\det K^{-1}(z)$ has a zero at z_0 and (iv) $\Rightarrow$ (iii). Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and the lemma follows.

Proof of Theorem 4.4. A dense subset of $\mathscr{RB}L^\infty(M_n)$ is the subset

$\mathscr{D} = \{H \in \mathscr{RB}L^\infty(M_n) : H$ satisfies (N) and all poles of H inside the disk are simple$\}$.

Then, by Lemma 4.5, it follows that for $H \in \mathscr{D}$, $\dim\ker H^{-1}(z) \leqslant 1$, and such an H is in $\mathscr{RB}H_l^\infty(M_n)$ if $\dim\ker H^{-1}(z_j) = 1$ for at most l points $\{z_j\}$; for each j we choose a basis vector y_j for the one-dimensional space $\ker H^{-1}(z_j)$:

$$H^{-1}(z_j)y_j = 0.$$

Now let f be a constant coefficient (I, J)-cascade map of the form $f = F_v$ where $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary matrix; it is convenient to assume that c and d are invertible. Also let $K(z) = f(H(z)) = a + bH(z)(I - dH(z))^{-1}c$ where H is in $\mathscr{D} \cap \mathscr{R}\mathscr{B}H_I^\infty(M_n)$ as above. Compute that

$$H^{-1}(z_j)y_j = 0 \Leftrightarrow -dy_j = (H^{-1}(z_j) - d)y_j \Leftrightarrow$$

$$\Leftrightarrow (H^{-1}(z_j) - d)^{-1}dy_j = -y_j \Leftrightarrow$$

$$\Leftrightarrow H(I - dH)^{-1}dy_j|_{z=z_j} = -y_j \Leftrightarrow$$

$$\Leftrightarrow b^{-1}(K(z_j) - a)c^{-1}dy_j = -y_j \Leftrightarrow$$

$$\Leftrightarrow K(z_j)c^{-1}dy_j = (-b + ac^{-1}d)y_j \Leftrightarrow$$

$$\Leftrightarrow K(z_j)x_j = (a - bd^{-1}c)x_j = Ex_j,$$

where $x_j = c^{-1}dy_j$ and $E = (a - bd^{-1}c)$. The only assumption used in the above calculation not already explicitly stated is that $H^{-1}(z_j) - d$ be invertible. However, one can approximate H with a function which does satisfy this and then the final conclusion still follows by taking a limit. Since f maps a dense subset of $\mathscr{R}\mathscr{B}H_I^\infty(M_n)$ onto a dense subset of $f(\mathscr{R}\mathscr{B}H_I^\infty(M_n))$, it remains only to show that $E = a - bd^{-1}c$ is a general J-expansive matrix. Thus we need the following.

LEMMA 4.6. *A matrix r is J-expansive if and only if it can be written in the form*

$$r = a - bd^{-1}c$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is* $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$*-unitary.*

Proof. Suppose $r = a - bd^{-1}c$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary, we have

$$a^*Ja + c^*c = J \qquad a^*Jb + c^*d = 0$$

$$b^*Ja + d^*c = 0 \qquad b^*Jb + d^*d = I,$$

so

$$r^*Jr = a^*Ja - c^*d^{-1*}b^*Ja - a^*Jbd^{-1}c + c^*d^{-1}b^*Jbd^{-1}c =$$

$$= (J - c^*c) + c^*c + c^*c + c^*d^{-1*}(1 - d^*d)d^{-1}c =$$

$$= J + c^*d^{-1*}d^{-1}c > J,$$

and hence r is J-expansive.

Conversely, if r is J-expansive, r is $(-J)$-contractive, and by the results of § 1 there is an $(I, -J)$-cascade map F_w which sends 0 to r. Hence w is a $\begin{pmatrix} -J & 0 \\ 0 & I \end{pmatrix}$-unitary matrix of the form $w = \begin{pmatrix} r & x \\ y & z \end{pmatrix}$. Then $w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & r \\ z & y \end{pmatrix}$ is $\left(\begin{pmatrix} I & 0 \\ 0 & -J \end{pmatrix}, \begin{pmatrix} -J & 0 \\ 0 & I \end{pmatrix} \right)$-unitary. By the results of § 1 this forces r to have the form $r = \alpha - \beta\gamma^{-1}\varkappa$ where $\begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix}$ is $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary. This concludes the lemma.

To conclude the proof of (ii), we now need only show that $F_v(\mathscr{B}\mathscr{R}H_I^\infty(M_n))$ is independent of the choice of v. If v_1 and v_2 are two such matrices then $F_u = F_{v_1}^{-1} \circ \circ F_{v_2}$ is a (constant coefficient) (I, I)-cascade map. By the argument above, the image of $\mathscr{R}\mathscr{B}H_I^\infty(M_n)$ under F_u is the closure of the set

$$\{H \in \mathscr{R}\mathscr{B}L^\infty(M_n) \colon \sum_{z \in \mathscr{D}} N_{H,L}(z) \leqslant l\}$$

where L is *any* matrix with $\|L^{-1}\| < 1$. In particular, if we formally take L equal to $\operatorname{diag}\{\infty, \ldots, \infty\}$, it is clear that the above set is identical to $\mathscr{R}\mathscr{B}H_I^\infty(M_n)$, and thus $F_{v_1}(\mathscr{R}\mathscr{B}H_I^\infty(M_n)) = F_{v_2}(\mathscr{R}\mathscr{B}H_I^\infty(M_n))$ as desired.

Finally consider the case $l = 0$. By (ii), $\mathscr{R}\mathscr{B}H_{J,0}^\infty(M_n)$ is equal to $f(\mathscr{R}\mathscr{B}H^\infty(M_n))$ where f is any nondegenerate (I, J)-cascade map. Since any F in $\mathscr{R}\mathscr{B}H^\infty(M_n)$ is a contraction on the closed unit disk $\{|z| \leqslant 1\}$, and f maps contractions into J-contractions, it follows that any $K = f(F)$ in $\mathscr{R}\mathscr{B}H_{J,0}^\infty(M_n)$ has J-contractive values for $|z| \leqslant 1$, that is, $K \in \mathscr{R}\mathscr{B}H_J^\infty(M_n)$. One can reverse the argument by applying f^{-1} instead of f. Thus (iii) follows as well.

The following result says that the relationships between $\mathscr{R}L_J^\infty$ and $\mathscr{R}H_{J,l}^\infty$ and $\mathscr{R}H_J^\infty$ are much like that between their counterparts with $J = I$.

THEOREM 4.7. (i) The maximum principle: *If $A \in \mathscr{R}H_J^\infty(M_n)$ and if r is a J-bound for all matrices $A(e^{i\theta})$, then r is a J-bound for all $A(z)$ with $|z| < 1$.*

(ii) $\mathscr{R}L_J^\infty = \bigcup_{l=0}^{\infty} \mathscr{R}H_{J,l}^\infty$, *that is, any A in $\mathscr{R}L_J^\infty$ is in $\mathscr{R}H_{J,l}^\infty$ for some l.*

(iii) *If A is in $\mathscr{R}L_J^\infty$, then φA is in $\mathscr{R}H_J^\infty$ for some J-inner φ.*

Proof. (i) is clear from the argument before.

Assume A is in $\mathscr{R}\mathscr{B}L_J^\infty$, and let E be any J-expansive matrix. Then, since A is rational,

$$l = \sum_{z \in D} N_{A,E}(z) < \infty.$$

Then by Theorem 4.4, A is in $\mathscr{R}\mathscr{B}H_{J,l}^\infty$. This proves (ii).

To prove (iii), the idea is to multiply A on the left by a J-inner Blaschke factor to cancel off the J-poles of A one at a time. This is simply the reverse of the technique which Potapov [38] used to produce a symplectic inner-outer factorization. We omit the details.

Finally, we mention that one could define symplectic function sets $L^{\infty}_{J_1, J_2}(M_n)$ (M_n-valued functions F on the unit circle such that $\langle J_2 F(e^{i\theta})x, F(e^{i\theta})x\rangle \leqslant r\langle J_1 x, x\rangle$ for some $r > 0$ for all x in $\mathbf{C}_n$) and similarly $H^{\infty}_{J_1, J_2}(M_n)$ and $H^{\infty}_{J_1, J_2; l}(M_n)$. Since it is simply a matter of bookkeeping to adapt the theorems of this section to this more general situation, and since we do not have specific need for these sets here, we do not pursue this level of generality.

(b) Symplectic interpolation problems: simplest type

We next consider extensions of the Nevanlinna-Pick theory of interpolation to the classes $\mathscr{R}H^{\infty}_J$ and $\mathscr{R}H^{\infty}_{J,l}$. For $\mathbf{z} = (z_1, \ldots, z_N)$ an N-tuple of complex numbers in the disk, and N-tuples $\mathbf{p} = (p_1, \ldots, p_N)$ and $\mathbf{x} = (x_1, \ldots, x_N)$ of operators $p_j : \mathbf{C}^{k_j} \to \mathbf{C}^n$ and $x_j : \mathbf{C}^{k_j} \to \mathbf{C}^n$ for $1 \leqslant k_j \leqslant N$, we define the interpolating sets of functions (multiplicity one case, i.e. no constraints are prescribed on derivatives)

$$\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x}) = \{H \in \mathscr{R}M_n : H(z_j)p_j = x_j \text{ for } j = 1, \ldots, N\}$$

and

$$\mathscr{I}^* = \mathscr{I}^*(\mathbf{z}, \mathbf{p}, \mathbf{x}) = \{H \in \mathscr{R}M_n : p_j^* H(z_j) = x_j^* \text{ for } j = 1, \ldots, N\}.$$

Then $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ consists of functions satisfying right interpolation conditions, and $\mathscr{I}^*(\mathbf{z}, \mathbf{p}, \mathbf{x})$ corresponds to left interpolation. Such sets arise in the description of the range of a rational cascade map acting on $\mathscr{R}H^{\infty}(M_n)$. As a natural generalization of Nevanlinna-Pick-Takagi interpolation, we consider the problem

(4.2 a) Given a scalar r, determine if there is an interpolating function F in $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ which is also in $r\mathscr{R}\mathscr{B}H^{\infty}_{J,l}(M_n)$.

In particular one has the problem

(4.2b) What is the minimum r for which such an F exists?

and for the case $J \neq I$ it makes equal mathematical sense to ask

(4.2b′) What is the maximum r for which such an F exists?

When $\mathscr{I}^*$ replaces $\mathscr{I}$ in the above, we refer to these problems as (4.2 a*), (4.2b*) and (4.2 b′*) respectively.

To formulate the solution of (4.2a), (4.2b) and (4.2b′), we define the Pick matrices

$$\Lambda_{\mathscr{I}}(r) = \left[\frac{r p_j^* J p_k - x_j^* J x_k}{1 - \bar{z}_j z_k} \right]_{j,k=1,\ldots,N} \tag{4.3}$$

and

$$\Lambda_{\mathscr{I}^*}(r) = \left[\frac{r p_j^* J p_k - x_j^* J x_k}{1 - z_j \bar{z}_k} \right]_{j,k=1,\ldots,N} .$$

Then, "generically",

Problem (4.2 *a*) *has a solution if and only if* $\Lambda_{\mathscr{I}}(1)$ *has at most* l *negative eigenvalues; furthermore, when this is the case, there exists a solution* F *which is* J*-unitary on the unit circle. The solution to* (4.2 *b*) *is equal to*

$$\min\{r\colon \Lambda_{\mathscr{I}}(r) \text{ has at most } l \text{ negative eigenvalues}\}$$

and the solution to (4.2 *b*′) *is equal to*

$$\max\{r\colon \Lambda_{\mathscr{I}}(r) \text{ has at most } l \text{ negative eigenvalues}\}.$$

To obtain solutions to (4.2a*), (4.2b*) and (4.2b′*), simply replace $\Lambda_{\mathscr{I}}(r)$ with $\Lambda_{\mathscr{I}^*}(r)$ in the above.

For the scalar case with $l = 0$, the above is classical Nevanlinna-Pick interpolation [31], and for $l > 0$ is of the type considered by Takagi [41] and Adamjan, Arov and Kreĭn [1]. For the matrix case with $J = I$, with $l = 0$ the result is equivalent to Nehari's theorem for block Hankel matrices ([30], [1]) and is also closely related to the commutant lifting theorem of Sz-Nagy and Foiaş [32] (see [18]). With $l > 0$ the general result has been worked out by Nudelman [36] and one of the authors [4]. Also [21], [22], [23] and [35] treat the $J \neq I$ case when $l = 0$. Papers of I. V. Kovalishyn and V. P. Potapov [26] and of I. L. Fedchin [13], [14] describe the connection between matrix functions in the class $\mathscr{R}U^+(n, n)$ and the Nevanlinna-Pick-Schur problem in the indefinite case. A difficulty for the symplectic case not present in the definite case ($J = I$) is that the set of J-contractions is not compact and we need to be able to deal with matrix functions which have a pole at the interpolating point, or which equal ∞ identically on a subspace. To give this a sound mathematical underpinning, we embed the rational matrix-valued functions into a natural compactification, a "rational Grassmannian manifold", and solve an interpolation problem in that setting.

Let M_n^{2n} be the set of all n-dimensional subspaces of $\mathbf{C}^{2n}$; it is called a Grassmannian manifold. Each $n \times n$ matrix m can be identified with a subspace $\mathrm{G}(m)$ by the following procedure. Write $\mathbf{C}^{2n}$ as $\mathbf{C}^n \oplus \mathbf{C}^n$, then define a subspace $\mathrm{G}(m)$ of $\mathbf{C}^{2n}$, the graph of m by

$$\mathrm{G}(m) = \{x \oplus mx\colon x \in \mathbf{C}^n\} .$$

While to each m we get a subspace the reverse is not true since some subspaces correspond to an m which formally maps some vectors to ∞. The matrix m is called the "angle operator" for $\mathrm{G}(m)$ with respect to the "basis" $\mathbf{C}^n \oplus 0$ and $0 \oplus \mathbf{C}^n$.

We next define $\mathscr{R}M_n^{2n}$ to be n-dimensional (over the field $\mathscr{R}$) subspaces of the rational vector space $\mathscr{R}\mathbf{C}^{2n}$ ($2n$-tuples of rational functions). Thus any element $\mathbf{F}$ in $\mathscr{R}M_n^{2n}$ can be written as the linear span (with scalar rational coefficients) of n basis vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ in $\mathscr{R}\mathbf{C}^{2n}$. As above, corresponding to any rational matrix function $F \in \mathscr{R}M_n$ is an element of $\mathscr{R}M_n^{2n}$, namely, its graph

$$\mathrm{G}(F) = \{r \oplus Fr : r \in \mathscr{R}\mathbf{C}^n\} \subset \mathscr{R}\mathbf{C}^{2n}.$$

We next introduce the operation of "point evaluation at z_0" for any complex number z_0, a well-defined map of $\mathscr{R}M_n^{2n}$ into M_n^{2n}. For this we use the fact that any $\mathbf{F} \in \mathscr{R}M_n^{2n}$ has a basis $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ such that no $\mathbf{r}_j$ has a pole at the prescribed point z_0 and $\{\mathbf{r}_1(z_0), \ldots, \mathbf{r}_n(z_0)\}$ is linearly independent as a set of vectors in $\mathbf{C}^{2n}$. We then define

$$\mathbf{F}(z_0) = \bigvee \{\mathbf{r}_1(z_0), \ldots, \mathbf{r}_n(z_0)\} \subset \mathbf{C}^{2n}.$$

We leave it to the reader to check that $F(z_0)$ is independent of the particular basis $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ chosen for $\mathbf{F}$. One can also think of $\mathscr{R}M_n^{2n}$ as being the set of all n-dimensional subspace valued functions (of $\mathbf{C}^{2n}$) defined on $\mathbf{C}$ (or the Riemannian sphere) which are rational in a simple sense. Note that if F is in $\mathscr{R}M_n$ and is analytic at z_0, then

$$\mathrm{G}(F)(z_0) = \mathrm{G}(F(z_0)).$$

We are now ready to introduce our more general interpolation problem. To do this define a bilinear form $\langle\ ,\ \rangle_r$ on $\mathbf{C}^{2n}$ by

$$\langle x, y\rangle_r = r(Jx_1, y_1) - (Jx_2, y_2)$$

where $x = x_1 \oplus x_2$ and $y = y_1 \oplus y_2$ are in $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^n$. Now we shall set definitions and prove an interpolation theorem about Grassmannian valued functions which respect an arbitrary nondegenerate bilinear form $\langle\ ,\ \rangle$ on $\mathbf{C}^{2n}$ in a certain way.

Let $\langle\ ,\ \rangle$ be such a form. For convenience we shall assume that $\langle\ ,\ \rangle$ has signature (n, n) (i.e. $\langle x, y\rangle = (Hx, y)$ where H has n positive eigenvalues and n negative eigenvalues).

A subspace $\mathscr{M}$ in M_n^{2n} is said to be *positive* if

$$\langle x, x\rangle \geqslant 0$$

and *negative* if

$$\langle x, x\rangle \leqslant 0$$

for all x in $\mathscr{M}$. The space $\mathscr{M}$ is called *strictly positive* or *strictly negative* if strict inequality holds for $x \neq 0$. Note that the graph $\mathrm{G}(m)$ of an m in M_n is a $\langle\ ,\ \rangle_r$-positive subspace of $\mathbf{C}^{2n}$ if an only if r is a J-bound for m. For a given $\langle\ ,\ \rangle$ define

$$\mathscr{R}\mathscr{P}M_n^{2n} = \{\mathbf{F} \in \mathscr{R}M_n^{2n} : F(\mathrm{e}^{\mathrm{i}\theta}) \text{ is } \langle\ ,\ \rangle \text{ positive for } 0 \leqslant \theta \leqslant 2\pi\}$$

and

$$\mathcal{RP}^{+}M_n^{2n} = \{\mathbf{F} \in \mathcal{R}M_n^{2n} : F(z) \text{ is } \langle\,,\rangle \text{ positive for } |z| < 1\}.$$

For $\langle\,,\rangle_r$ one can think of $\mathcal{RP}M_n^{2n}$ as the Grassmannian analogue of $r\mathcal{RB}L_J^{\infty}(M_n)$ because of the angle operator-graph correspondence. Naturally, in this case

$$\mathrm{G}(r\mathcal{RB}L_J^{\infty}(M_n)) \subset \mathcal{RP}M_n^{2n}$$

and one actually has equality if $J = I$ since the graph of any n-dimensional $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ positive space is the graph of some K in $\mathcal{B}M_n$. For $J \neq I$ the inequality is strict. Also for this case, the set $\mathcal{RP}^{+}M_n^{2n}$ contains

$$\mathrm{G}(r\mathcal{RB}H_J^{\infty}(M_n)).$$

To describe the generalized analogue $\mathcal{RP}_l^{+}M_n^{2n}$ of $r\mathcal{RB}H_{J,l}^{\infty}(M_n)$, let E be any fixed strictly $\langle\,,\rangle$ negative subspace of $\mathbf{C}_n^{2n}$. Roughly speaking $\mathcal{RP}_l^{+}M_n^{2n}$ is

$$\{\mathbf{F} \in \mathcal{R}M_n^{2n} : \sum_{z \in D} \dim(F(z) \cap E) \leqslant l\}$$

but this set is a little too big because F at some z_j could have "higher order contact" with E, and this should be counted[1]. The content of Theorem 4.4 in this slightly more general context is that the definition of $\mathcal{RP}_l^{+}M_n^{2n}$ is independent of the particular choice of strictly negative subspace $E \subset \mathbf{C}^{2n}$. Naturally, when $\langle\,,\rangle = \langle\,,\rangle_r$,

$$\mathrm{G}(r\mathcal{RB}H_{J,l}^{\infty}(M_n)) \subset \mathcal{RP}_l^{+}M_n^{2n}$$

as before and the remarks (concerning when equality holds) apply.

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[1] The order of contact $N(\mathbf{F}, \mathbf{G})(z_0)$ between $\mathbf{F}$ and $\mathbf{G}$ in $\mathcal{R}M_n^{2n}$ at z_0 is defined to be

$$N(\mathbf{F}, \mathbf{G})(z_0) = \sum_{j \geqslant 0} N(\mathbf{F}, \mathbf{G}, j)(z_0)$$

where

$$N(\mathbf{F}, \mathbf{G}, j)(z_0) = \dim\{x \in F(z_0) \cap G(z_0) \,|\, \text{there are rational } \mathbf{C}^{2n}\text{-valued functions } f \in \mathbf{F}$$

$$\text{and } g \in \mathbf{G} \text{ with } x = f(z_0) = g(z_0) \text{ and } \frac{\mathrm{d}^{\alpha}}{\mathrm{d}z^{\alpha}}(f(z) - g(z))\Big|_{z=z_0} = 0 \text{ for } 0 \leqslant \alpha \leqslant j\}.$$

We then define the order of contact between a function $\mathbf{F}$ and a subspace E as just the order o contact between $\mathbf{F}$ and the constant function $\mathbf{E}$ whose only values is E. The precise definition of $\mathcal{RP}_l^{+}M_n^{2n}$ is

$$\{\mathbf{F} \in \mathcal{RP}M_n^{2n} : \sum_{z \in D} N(\mathbf{F}, \mathbf{E})(z) \leqslant l\}.$$

We are now ready to pose our more general interpolation problem. Let $\mathbf{z} = \{z_1, \ldots, z_N\}$ be an N-tuple of complex numbers in the unit disk, and let $\mathbf{S} = \{S_1, \ldots, S_N\}$ be an N-tuple of subspaces of $\mathbf{C}^{2n}$. We define interpolating classes of rational subspace-valued functions by

$$\mathscr{I}(\mathbf{z}, \mathbf{S}) = \{\mathbf{F} \in \mathscr{R}M_n^{2n} : \mathbf{F}(z_j) \supset S_j \text{ for all } j = 1, \ldots, N\}$$

and

$$\mathscr{I}^*(\mathbf{z}, \mathbf{S}) = \{\mathbf{F} \in \mathscr{R}M_n^{2n} : \mathbf{F}(z_j) \subset S_j \text{ for all } j = 1, \ldots, N\}.$$

One can check that the elements of $\mathscr{I}(\mathbf{z}, \mathbf{S})$ which are graphs are the graphs of the interpolating class of rational functions $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$, where $S_j = \operatorname{Ran}\begin{bmatrix} p_j \\ x_j \end{bmatrix}$, and that the graph elements of $\mathscr{I}^*(\mathbf{z}, \mathbf{S})$ are precisely the graphs of functions in $\mathscr{I}^*(\mathbf{z}, \mathbf{p}, \mathbf{x})$ where this time $S_j = \operatorname{Ran}\begin{bmatrix} -x_j \\ p_j \end{bmatrix}^{\perp}$ (where $\perp$ is the orthogonal complement in $\mathbf{C}^{2n}$). Thus these classes are the natural Grassmannian analogues of the interpolating function sets considered previously. The interpolation problem we wish to solve is the following:

(4.4) *Determine if there is an* $\mathbf{F}$ *in* $\mathscr{I}(\mathbf{z}, \mathbf{S})$ *which is also in* $\mathscr{R}\mathscr{P}_l^+ M_n^{2n}$.

For example, when $l = 0$ we wish to find a positive subspace valued function $\mathbf{F}$ on $|z| < 1$ such that $\mathbf{F}(z_j) \supset S_j$ for each j. When $\mathscr{I}^*(\mathbf{z}, \mathbf{S})$ replaces $\mathscr{I}(\mathbf{z}, \mathbf{S})$ in (4.4), we refer to the problem as (4.4*).

If we identify functions in $\mathscr{R}H_{J,l}^\infty(M_n)$ with their graphs in $\mathscr{R}M_n^{2n}$, we see that the constraints in problem (4.4) with $\langle\, ,\rangle_r$ are the same as those in (4.2); the only difference is that in (4.4) the set of possible solutions is enlarged to include rational subspaces in $\mathscr{R}M_n^{2n}$ which are not the graphs of rational functions. By the remarks above, when $J = I$, and $\langle\, ,\rangle = \langle\, ,\rangle_r$ every subspace in $\mathscr{R}\mathscr{P}_l^+ M_n^{2n}$ is a graph, and thus problems (4.2(a)) and (4.4) are equivalent for this special case.

Now we state the solution to problem (4.3(a)) and thereby give a test to determine if r lies between the maximum and minimum possible in problems (4.2(b)) and (4.2(c)). The referee informed us that similar results were obtained by T. S. Ivanchenko [21], [22], [23] when $l = 0$, and by A. A. Nudelman when $l = 0$ in [35] and also when $l > 0$ but $J = I$ in [36].

To define an appropriate Pick matrix for these problems let s_j^u denote *any* basis for the space S_j and for fixed j, k let $\langle s_j^u, s_k^v\rangle$ denote the $\dim S_j \times \dim S_k$ matrix of $\langle\, ,\rangle$ inner products. Define a Pick matrix to be the matrix-entried matrix

$$\Lambda_{\mathscr{I}(\mathbf{z}, \mathbf{S})} := \left[\frac{\langle s_j^u, s_k^v\rangle}{1 - \bar{z}_j z_k}\right]_{j,k=1,2,\ldots,N}.$$

Also define

$$\Lambda_{\mathcal{I}^*(\mathbf{z},\,\mathbf{S})} := \left[\frac{-\langle t_j^u, t_k^v \rangle}{1 - \bar{z}_j z_k} \right]_{j,k=1,2,\dots,N}$$

where the t_j^u are *any* basis for the $\langle\,,\rangle$ orthogonal complement of S_j. Note that when $\langle\,,\rangle = \langle\,,\rangle_r$ and $S_j = \left(\operatorname{Ran}\begin{bmatrix} -x_j \\ p_j \end{bmatrix}\right)^{\perp}$, then the $\langle\,,\rangle$-orthogonal complement of $S_j = \operatorname{Ran}\begin{bmatrix} x_j \\ p_j \end{bmatrix}$, and, with appropiate choice of basis,

$$\Lambda_{\mathcal{I}^*(\mathbf{z},\,\mathbf{p},\,\mathbf{x})} = \Lambda_{\mathcal{I}^*(\mathbf{z},\,\mathbf{S})}\,.$$

It is also easy to see that

$$\Lambda_{\mathcal{I}(\mathbf{z},\,\mathbf{p},\,\mathbf{x})} = \Lambda_{\mathcal{I}(\mathbf{z},\,\mathbf{S})}\,.$$

THEOREM 4.8. *Problem* (4.3) *has a solution if and only if the matrix* $\Lambda_{\mathcal{I}(\mathbf{z},\,\mathbf{S})}$ *has at most* l *negative eigenvalues. Furthermore when this is the case there exists a solution* **F** *such that* $\mathrm{F}(z)$ *is a* $\langle\,,\rangle$ *null space for any* z *on the unit circle.*

To obtain the solution to problem (4.3*), replace $\Lambda_{\mathcal{I}(\mathbf{z},\,\mathbf{S})}$ by $\Lambda_{\mathcal{I}^*(\mathbf{z},\,\mathbf{S})}$ in the above.

Proof. We consider only problem (4.3); (4.3*) is similar. The idea of the proof is to reduce the problem to the classical Nevanlinna-Pick interpolation theorem. The reduction goes like this. We have a non-degenerate Hermitean bilinear form $\langle\,,\rangle$ of signature (n, n) on $\mathbf{C}^{2n}$. Let $\mathcal{M}$ be an n-dimensional strictly positive subspace of $\mathbf{C}^{2n}$. Let $\mathcal{M}'$ be its $\langle\,,\rangle$-orthogonal complement, so $\mathcal{M}$ is a strictly negative subspace. With an n-dimensional positive subspace $\mathcal{N}$ one associates an angle operator $K : \mathcal{M} \to \mathcal{M}'$ satisfying $\mathcal{N} = \{x + Kx : x \in \mathcal{M}\}$ with the property $-\langle Kx, Kx\rangle \leqslant \langle x, x\rangle$ for all $x \in \mathcal{M}$. Let us identify $\mathcal{M}$ with $\mathbf{C}^n$ in such a way that $\langle\,,\rangle$ restricted to $\mathcal{M}$ becomes the usual inner product on $\mathbf{C}^n$. Do the same with $\mathcal{M}'$ and $-\langle\,,\rangle$. Then with $\mathcal{N}$ we associate a conventional contraction K on $\mathbf{C}^n$. Thus $\mathcal{RP}_l^+ M_n^{2n}$ corresponds *exactly* to $\mathcal{RB}H_l^\infty(M_n)$ and similarly the interpolation problem $\mathcal{I}(\mathbf{z}, \mathbf{S})$ becomes $\mathcal{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ for any pairs $\mathbf{p}, \mathbf{x}$ with the property that $\dot{S}_j = \begin{bmatrix} p_j \\ x_j \end{bmatrix}$ identifies with S_j. The strongest existing theorem on classical interpolation [4] tells us that an interpolating function $M \in \mathcal{RB}H_l^\infty(M_n)$ exists if and only if the Pick matrix $\Lambda := \Lambda^{(1)}_{\mathcal{I}(\mathbf{z},\,\mathbf{p},\,\mathbf{x})}$ defined in (4.3) has at most l negative eigenvalues where the J which appears in Λ equals I. To convert Λ to the Pick matrix in the theorem note that $\langle\,,\rangle$ on $\mathbf{C}^{2n}$ has been identified with $(\mathcal{J}x, y)$ on $\mathbf{C}^{2n}$ where $\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $(,)$ is the usual inner product on $\mathbf{C}^{2n}$, and note that the business part of Λ is

$$p_k^* p_j - x_k^* x_j$$

which is just $[p_k^*, x_k^*]\mathscr{J}\begin{bmatrix} p_j \\ x_j \end{bmatrix}$. Now if $\dot{s}_j^v$ is a basis for $\dot{S}_j$ this matrix equals

$$Q_k^*(\mathscr{J}\dot{s}_k^u, \dot{s}_j^v)Q_j$$

for some invertible $\dim S_k \times \dim S_k$ matrix Q_k. Thus $\Lambda_{\mathscr{I}(z, p, x)}$ can be re-expressed as

$$\left[\frac{Q_k^*\langle s_k^u, s_j^v\rangle Q_j}{1 - z_k\bar{z}_j}\right]_{j,k=1,\ldots,N}.$$

This is $Q^*\Lambda_{\mathscr{I}(z, S)}Q$ where $Q = \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_N \end{pmatrix}$ and so has the same signature as the matrix in the theorem.

The reader may be wondering why we have spent this time bothering to restate the old fashion interpolation theorem in Grassmannian setting. The reason is that the Grassmannian version is in a sense coordinate independent. We had only an M_n^{2n}-valued function and a bilinear form $\langle\, ,\rangle$. The proof depended on selecting good coordinates for representing $\langle\, ,\rangle$. However, in our application the bilinear form $\langle\, ,\rangle$ will be presented in badly behaved coordinates and the corresponding angle operators will frequently be singular. For example, $\langle\, ,\rangle_r$ was defined that way. The Grassmannian Theorem 4.8 seems to be the easiest vehicle for changing coordinates.

Problem (4.3) suggests another natural question. Suppose we work not with positive subspace valued functions $\mathscr{R}\mathscr{P}^+M_n^{2n}$ but with the functions $\mathscr{R}\mathscr{P}_{n-k}^kM_n^{2n}$, each of whose values at a point z in the disk is a subspace on which $\langle\, ,\rangle$ has a fixed signature $(k, n-k)$. A natural question is (4.4'): Determine if there is an $\mathbf{F}$ in $\mathscr{I}(\mathbf{z}, \mathbf{S})$ which is also in $\mathscr{R}\mathscr{P}_{n-k}^kM_n^{2n}$. We shall not go through the construction here but the answer is essentially yes.

THEOREM 4.9. *If* $\dim S_j \leqslant$ *the index* k *of negativity for all* j, *then one can find an interpolating* $\mathbf{F}$ *in* $\mathscr{I}(\mathbf{z}, \mathbf{S})$ *which is also in* $\mathscr{R}\mathscr{P}_{n-k}^kM_n^{2n}$.

(c) SYMPLECTIC INTERPOLATION PROBLEMS INVOLVING J-SINGULAR VALUES.

In [16] the second author considered the interpolation problem, more general than the matrix Nevanlinna-Pick problem discussed above, of finding a matrix function F in $\mathscr{R}\mathscr{B}H_1^\infty(M_n)$ whose singular values on the unit circle are prescribed functions of θ $(0 < \theta \leqslant 2\pi)$. (Recall that the singular values of a matrix m are the square roots of the eigenvalues of m^*m.) In this subsection we formulate and give a partial solution of a symplectic analogue of this problem. The symplectic interpolation problems discussed so far are still not general enough to handle our application to an optimization problem for transistor amplifiers to be discussed in § 7; the generalized symplectic interpolation problem we now discuss is.

We first need to develop the symplectic generalization of singular values. Let J be the signature matrix $\begin{pmatrix} I_j & 0 \\ 0 & -I_{n-j} \end{pmatrix}$ acting on $\mathbf{C}^n$. A result of [38] is that any strict J-contraction $T \in M_n$ has a *J-singular decomposition*

$$T = U \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix} V$$

where U and V are J-unitary matrices, D_- and D_+ are positive diagonal matrices, and the diagonal entries of D_- are less than 1 (listed in order of decreasing magnitude) while those of D_+ are greater than 1 (listed in order of increasing magnitude). Also, if $T^\dagger$ denotes the adjoint JT^*J of T in the J-inner product, then $T^\dagger T = V^\dagger \begin{pmatrix} D_-^2 & 0 \\ 0 & D_+^2 \end{pmatrix} V$ is a J-self-adjoint J-contraction, and has a unique J-self-adjoint J-contractive square root

$$|T|_J = V^\dagger \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix} V,$$

called the *J-modulus of T*, with eigenvalues coinciding with those of D_- and D_+. Then T has an essentially unique J-polar decomposition

$$T = W|T|_J$$

where $W = UV$ is J-unitary, and $|T|_J$ is J-self-adjoint with nonnegative eigenvalues (coinciding with the diagonal entries of D_+ and D_- above). In analogy with the positive definite case, we refer to the eigenvalues of $|T|_J$ as the *J-singular values* of T and denote them as

$$s^+_{J,n-j}(T) \geqslant \ldots \geqslant s^+_{J,1}(T) > 1 > s^-_{J,1}(T) \geqslant \ldots \geqslant s^-_{J,j}(T). \tag{4.5}$$

Thus $D_+ = \operatorname{diag}\{s^+_{J,1}(T), \ldots, s^+_{J,n-j}(T)\}$ and $D_- = \operatorname{diag}\{s^-_{J,1}(T), \ldots, s^-_{J,j}(T)\}$.

A minimax characterization of the eigenvalues of a nonsingular J-positive matrix H has been obtained by Langer [29] and Phillips [37]. Since $I - T^\dagger T$ is invertible and J-positive if T is a strict J-contraction and the eigenvalues of $I - T^\dagger T$ are related to the J-singular values of T via the simple transformation $x \to \sqrt{1-x}$, one can adapt Phillips' techniques (which simplify considerably in the finite dimensional case) to obtain the following characterization of the J-singular values of a J-contraction. When there is no ambiguity, we refer to a J-positive subspace of $\mathbf{C}^n$ as a *positive subspace.* A positive subspace $\mathscr{P} \subset \mathbf{C}^n$ is said to be *maximal positive* if $\mathscr{P}$ is not properly contained in any other positive subspace. We similarly define *maximal negative* subspaces.

THEOREM 4.10. *Let T be a strict J-contraction with J-singular values given by* (4.5) *above. Then*

$$[s_{J,l}^{-}(T)]^2 = \sup_{\mathcal{P}} \inf_{V \subset \mathcal{P}} \sup_{x \in V} \frac{(JTx, Tx)}{(Jx, x)}$$

where $\mathcal{P}$ runs through all maximal positive subspaces of K and $V \subset \mathcal{P}$ has codimension $l - 1$ as a subspace of $\mathcal{P}$. Similarly

$$[s_{J,l}^{+}(T)]^2 = \inf_{\mathcal{N}} \sup_{V \subset \mathcal{N}} \inf_{x \in V} \frac{(JTx, Tx)}{(Jx, x)}$$

where $\mathcal{N}$ runs through all maximal negative subspaces of K, and $V \subset \mathcal{N}$ has codimension $l - 1$ as a subspace of $\mathcal{N}$.

For T a strict J-contraction represented as $T = U\begin{bmatrix} D_- & 0 \\ 0 & D_+ \end{bmatrix} V$ as above, let $\operatorname{diag}_+|T|_J$ denote the diagonal matrix D_+ and $\operatorname{diag}_-|T|_J$ denote the diagonal matrix D_- associated with T in this way. For T a not necessarily strict J-contraction we still associate matrices $\operatorname{diag}_+|T|_J$ and $\operatorname{diag}_-|T|_J$ by approximating T by strict contractions. Then the eigenvalues of $\operatorname{diag}_+|T|_J$ and $\operatorname{diag}_-|T|_J$ are still the square roots of the eigenvalues of $T^\dagger T$, but the eigenvalue 1 may be associated with a Jordan chain on an isotropic subspace (see [38]). The more general problem we consider here is that of estimating all the eigenvalues of $\operatorname{diag}_+|T|_J$ from above simultaneously, or of estimating all of the eigenvalues of $\operatorname{diag}_-|T|_J$ from below simultaneously; that is, for a diagonal matrix w_+ of length $n - j$ ($w_+ \geqslant I$) with eigenvalues listed in order of increasing magnitude, and a diagonal matrix w_- of length j ($0 \leqslant w_- \leqslant I$) with eigenvalues listed in order of decreasing magnitude, we characterize when both $\operatorname{diag}_+|T|_J \geqslant w_+$ and $\operatorname{diag}_-|T|_J \leqslant w_-$. The proof is a direct application of the minimax principle above.

THEOREM 4.11. *Let T be a J-contraction and w_+, w_- diagonal matrices as above. Let w be the J-contraction $\begin{pmatrix} w_- & 0 \\ 0 & w_+ \end{pmatrix}$ on $\mathbf{C}^n$. Then*

$$\operatorname{diag}_+|T|_J \geqslant w_+ \qquad \textit{and} \qquad \operatorname{diag}_-|T|_J \leqslant w_-$$

if and only if Tuw^{-1} is a J-contraction for some J-unitary u.

Proof. If $\operatorname{diag}_+|T|_J \geqslant w_+$ and $\operatorname{diag}_-|T|_J \leqslant w_-$ and we choose $u = V^\dagger$ where $T = U\begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix} V$ is the J-singular decomposition for T, then

$$Tuw^{-1} = U\begin{pmatrix} D_- w_-^{-1} & 0 \\ 0 & D_+ w_+^{-1} \end{pmatrix}$$

is a J-contraction.

Conversely, suppose Tuw^{-1} is a J-contraction for some u, so that

$$w^{*-1}u^*T^*JTuw^{-1} \leqslant J$$

or

$$u^*T^*JTu \leqslant w^*Jw.$$

If T and w are strict J-contractions, then by the minimax principle above (Theorem 4.9)

$$s^-_{J,l}(T) = s^-_{J,l}(Tu) \leqslant s^-_{J,1}(w)$$

and similarly

$$s^+_{J,l}(T) \geqslant s^+_{J,l}(w)$$

and hence

$$\operatorname{diag}_-|T|_J \leqslant w_- \quad \text{and} \quad \operatorname{diag}_+|T|_J \geqslant w_+.$$

The theorem now follows by approximation.

We now return to the interpolation problem. It is convenient to view the problems as being concerned with matrix functions in $\mathscr{R}H^\infty_{J,l}(M_n)$ rather than with rational subspace-valued functions in $\mathscr{R}\mathscr{P}^+_l M^{2n}_n$. Then the theorems hold only generically, but we understand that they hold exactly if we adjoin ideal elements to $\mathscr{R}H^\infty_{J,l}(M_n)$ corresponding to elements of $\mathscr{R}\mathscr{P}^+_l M^{2n}_n$ which are not graphs. The problems discussed in § 4 (b) are concerned with performing an interpolation with a function F in $\mathscr{R}H^\infty_{J,l}(M^n)$ while controlling the J-bound of F on the circle. Since the maximum J-bound for F is $|F|^+_J = s^+_{J,1}(F)$ and the minimum J-bound for F is $|F|^-_J = s^-_{J,1}(F)$, this amounts to controlling the particular J-singular values $s^+_{J,1}(F)$ and $s^-_{J,1}(F)$ in a nonindependent way. In this subsection, we consider the problem of performing the interpolation while controlling all the J-singular values simultaneously and independently. This is a canonical generalization of work done previously by the second author [16] for the case $J = I$.

Let $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ be the class of all rational matrix functions satisfying a set of interpolating conditions as in § 4(b) and let $w = \begin{pmatrix} w_- & 0 \\ 0 & w_+ \end{pmatrix}$ be an invertible diagonal matrix function in $\mathscr{R}L^\infty_J$ with $w_+ = \operatorname{diag}_+|w|_J$ and $w_- = \operatorname{diag}_-|w|_J$. The problem which we wish to consider is:

(4.6) Determine if there is an F in $\mathscr{R}H^\infty_{J,l}$ which is also in $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ such that

$$\text{(4.6 a)} \qquad \begin{cases} \operatorname{diag}_+|F|_J(e^{i\theta}) \geqslant w_+(e^{i\theta}) \\ \operatorname{diag}_-|F|_J(e^{i\theta}) \leqslant w_-(e^{i\theta}). \end{cases}$$

For example, in the transistor amplifier problem, we are interested in performing the interpolation with an F in $\mathscr{R}\mathscr{B}H^\infty_{J,l}$ with a prescribed lowest J-singular value

$s_{J,l}^{-}(F)(e^{i\theta}) = \eta(e^{i\theta}) < 1$. We should then choose $w_+ = I$ and $w_- = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & 0 \\ & 0 & & \ddots \\ & & & & \eta \end{pmatrix}$ in the above.

To analyze (4.6), we first note that by Theorem 4.10 applied pointwise, (4.6 a) holds for some F in $\mathscr{R}L_J^\infty(M_n)$ if and only if

> $F(e^{i\theta})u(e^{i\theta})w(e^{i\theta})^{-1}$ is a J-contraction for all θ, for some $u \in \mathscr{R}L_J^\infty(M_n)$ with J-unitary values on the unit circle.

Let us say that any μ of the form $\mu = u^\dagger w^2 u$, where $u \in \mathscr{R}L_J^\infty(M_n)$ is J-unitary on the unit circle, is *J-equivalent to* w^2. By Theorem 4.3, any such μ has a J-outer spectral factorization

$$\mu = \alpha_\mu^\dagger \alpha_\mu$$

where $\alpha_\mu \in \mathscr{R}\mathscr{B}H_J^\infty(M_n)$. Then $\alpha_\mu^{-1}\alpha_\mu^{\dagger-1} = u^{-1}w^{-2}u^{\dagger-1}$ and Fuw^{-1} is a J-contraction on the unit circle if and only if $F\alpha_\mu^{-1}$ is. Thus our problem is to determine if

> There exists an F in $\mathscr{R}H_{J,l}^\infty(M_n) \cap \mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ and μ which is J-equivalent to w, so that $F(e^{i\theta})\alpha_\mu(e^{i\theta})^{-1}$ is a J-contraction for all θ.

Since α_μ is J-outer, $K = F\alpha_\mu^{-1}$ is in $\mathscr{R}H_{J,l}^\infty(M_n)$ if and only if F is. Then $F\alpha_\mu^{-1}$ is in addition J-contractive on the circle if and only if K is in $\mathscr{R}\mathscr{B}H_{J,l}^\infty(M_n)$. We also note that F is in $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ if and only if K is in $\mathscr{I}(\mathbf{z}, \alpha_\mu(\mathbf{z})\mathbf{p}, \mathbf{x})$ where

$$\alpha_\mu(\mathbf{z})\mathbf{p} = \{\alpha_\mu(z_1)p_1, \ldots, \alpha_\mu(z_N)p_N\}.$$

Combining this analysis with Theorem 4.8 applied to $\langle\,,\rangle$ given by $\langle\,,\rangle = (\mathscr{J}\cdot, \cdot)$ where $\mathscr{J} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$ (in the generic form for rational matrix functions rather than for rational subspace valued functions), we obtain

THEOREM 4.12. *Problem* (4.6) *has a solution if and only if the* $N \times N$ *block matrix*

$$A = \frac{1}{1 - \bar{z}_j z_k}(p_j^*\alpha_\mu(z_j)^* J\alpha_\mu(z_k)p_k - x_j^* J x_k) \quad j, k = 1, \ldots, N$$

has at most l negative eigenvalues for some μ which is J-equivalent to w^2. Furthermore, whenever this is the case, one can choose a solution of (4.6) *such that* (4.6 a) *holds with equality on the unit circle.*

REMARK 1. The solution is reasonably satisfactory from a computational point of view except for the presence of the α_μ. Even in the definite case, there appears to be no useful characterization of the possible values of $\alpha_\mu(z_i)$ which one must put in the above matrix. In the definite case, one can show that if one does not demand control on the norm of the interpolating function L, and the interpolating conditions are never of full rank, one can arrange that $\|\text{s.s.v.}(L)\|_\infty = 0$ (that is, if the interpolating conditions are never of full rank, and $w = \begin{pmatrix} \infty & & 0 \\ & \ddots & \\ & & \infty \\ 0 & & 0 \end{pmatrix}$, the problem always has a solution.) This is the case where the largest singular value (= the norm) is as far as possible from the smallest singular value. This is essentially a restatement of Theorem 4.9. On the other hand, if one demands that all singular values be equal, the interpolating function (if it exists) is uniquely determined up to a multiplicative unitary constant.

5. "POINCARÉ DISTANCE" PROBLEM

This section treats the basic Problem I.3 about finding functions with a specified cross ratio which in the study of amplifiers corresponds to constructing an amplifier with prescribed gain.

The first step is to convert problems involving the cross ratio to ones involving the range of a fixed map $\mathscr{F}$. This conversion is a part of what was called the duality principle in [16] and we state it very formally as a lemma.

LEMMA 5.1. *If $S \in \mathscr{R}M(j, 0, l)$ and $\mathscr{F}$ is any rational (I, J)-linear fractional map with $\mathscr{F}(S) = 0$, then the family of matrix functions*

$$\{\mathscr{E}(S, H) : H \in \mathscr{R}\mathscr{B}H^\infty(M_n)\}$$

and the family of J-selfadjoint matrix functions

$$|\mathscr{F}(\mathscr{R}\mathscr{B}H^\infty(M^n))|_J^2 = \{\mathscr{F}(H)J\mathscr{F}(H)^*J \mid H \in \mathscr{R}\mathscr{B}H^\infty(M_n)\}$$

are identical up to pointwise similarity (on the circle).

Proof. Since $\mathscr{F}(S) = 0$ we have $\mathscr{E}(S, H)$ is similar to $\mathscr{E}_J(0, \mathscr{F}(H)) = \mathscr{F}(H)J\mathscr{F}(H)^*J$ by Proposition 1.3 applied pointwise.

The central problem now is to describe the range of a given rational (I, J)-linear fractional map. A major result of [16] was to describe $\mathscr{K}(\mathscr{R}\mathscr{B}H(M_n))$ for any rational (I, I)-cascade map $\mathscr{K}$. If we consider only the (generic) multiplicity-one case, the result is that *there is a non-negative integer l, and N-tuples $z = (z_1, \ldots, z_N)$ con-*

tained in the disk, N-tuples $p = (p_1, \ldots, p_N)$ *and* $x = (x_1, \ldots, x_N)$ *of operators* $p_j, x_j : \mathbf{C}^{k_j} \to \mathbf{C}^n$ *such that*

$$\mathscr{K}(\mathscr{R}\mathscr{B}H^\infty(M_n)) \cong \mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x}) \cap \mathscr{R}\mathscr{B}H_l^\infty(M_n)$$

when $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ is as in § 4. Here $A \cong B$ means that $A(e^{i\theta})^* A(e^{i\theta})$ and $B(e^{i\theta})^* B(e^{i\theta})$ are similar for each θ. For the general case, one must also insert interpolation conditions on derivatives. Using the machinery developed in § 4, we obtain

THEOREM 5.2. *In the generic (multiplicity-one) case, if* $\mathscr{G}$ *is a rational* (I, J)*-linear fractional map, then there exists an integer* $l \geqslant 0$ *and N-tuples* $\mathbf{z}$, $\mathbf{p}$, *and* $\mathbf{x}$ *such that "generically",*

$$\mathscr{G}(\mathscr{R}\mathscr{B}H^\infty(M_n)) \underset{J}{\cong} \mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x}) \cap \mathscr{R}\mathscr{B}H_{J,l}^\infty(M_n).$$

Here $A \underset{J}{\cong} B$ *means* $A(e^{i\theta})^\dagger A(e^{i\theta})$ *is similar to* $B(e^{i\theta})^\dagger B(e^{i\theta})$ *for each* θ, *where* $T^\dagger$ *is* JT^*J.

Proof. By adjusting the phase, we assume $\cong$ holds with $=$. To obtain a mathematically clean version of Theorem 5.2 (i.e., an *exact* description of $\mathscr{G}(\mathscr{R}\mathscr{B}H^\infty(M_n))$ rather than a "generic" description), we again must go to the Grassmannian. Observe that if $\mathscr{G}_G$ (where $G = \begin{pmatrix} \alpha & \beta \\ \varkappa & \gamma \end{pmatrix} \in \mathscr{R}M_{2n}$) is the rational linear fractional map

$$\mathscr{G}_G(f) = (\alpha f + \beta)(\varkappa f + \gamma)^{-1}$$

considered as acting on $\mathscr{R}M_n$ (whenever the necessary inverses exist), then the induced action U_G on the Grassmannian $\mathscr{R}M_n^{2n}$ (via the operator-graph correspondence) is

$$U_G : \{r \oplus fr \mid r \in \mathscr{R}\mathbf{C}^n\} \to$$

$$\to \{r \oplus (\alpha f + \beta)(\varkappa f + \gamma)^{-1} r \mid r \in \mathscr{R}\mathbf{C}^n\} =$$

$$= \{(\varkappa f + \gamma) q \oplus (\alpha f + \beta) q \mid q \in \mathscr{R}\mathbf{C}^n\} =$$

$$= \left\{ \begin{pmatrix} \gamma & \varkappa \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} q \\ fq \end{pmatrix} \middle| \, q \in \mathscr{R}\mathbf{C}^n \right\}$$

where we have set $q = (\varkappa f + \gamma)^{-1} r$. Thus the action U_G on subspace valued functions in $\mathscr{R}M_n^{2n}$ is simply multiplication by the matrix function $G' = PGP$ $\left(P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$.

We next observe that, for the special case where $G = g \in M_{2n}$ is a constant matrix, the maps U_g act very simply on interpolating sets $\mathscr{I}(\mathbf{z}, \mathbf{S})$ in $\mathscr{R}M_n^{2n}$:

$$U_g(\mathscr{I}(\mathbf{z}, \mathbf{S})) = \mathscr{I}(\mathbf{z}, g'\mathbf{S})$$

where $g'\mathbf{S} = \{g'S_1, \ldots, g'S_N\}$. Furthermore, *if* $\langle \cdot, \cdot \rangle_1$ *and* $\langle \cdot, \cdot \rangle_2$ *are two bilinear forms on* $\mathbf{C}^{2n}$ *and* $\mathscr{R}\mathscr{P}_{1,l}^{+}M_n^{2n}$ *and* $\mathscr{R}\mathscr{P}_{2,l}^{+}M_n^{2n}$ *are the respective subsets of* $\mathscr{R}M_n^{2n}$ *associated with these forms defined in* § 4b, *and if* g' *is a* $(\langle\, ,\rangle_1, \langle\, ,\rangle_2)$*-isometry* ($\langle g'x, g'x\rangle_2 = \langle x, x\rangle_1$ *for all* x *in* $\mathbf{C}^{2n}$), *then*

$$U_g(\mathscr{R}\mathscr{P}_{1,l}^{+}M_n^{2n}) = \mathscr{R}\mathscr{P}_{2,l}^{+}M_n^{2n}.$$

Now note that any $G \in \mathscr{R}U(I, J)$ can be written as $G = gK$ where g is a constant matrix in $U(J, I)$, and $K = g^{-1}G$ is in $\mathscr{R}U(I, I)$. By the result of [16] mentioned above, for the generic multiplicity one case

$$U_K(\mathscr{R}\mathscr{P}_1^{+}M_n^{2n}) = \mathscr{R}\mathscr{P}_{1,l}^{+}M_n^{2n} \cap \mathscr{I}(\mathbf{z}, \mathbf{S})$$

for some n-tuples $\mathbf{z}$ and $\mathbf{S}$, where $\langle x, x\rangle_1 = \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} x, x\right)$ for x in $\mathbf{C}^{2n}$. Now since g is in $U(J, I)$, $g' = PgP$ is as well, and thus, from the above remarks,

$$U_g(\mathscr{R}\mathscr{P}_{1,l}^{+}M_n^{2n}) \cap \mathscr{I}(\mathbf{z}, \mathbf{S}) = \mathscr{R}\mathscr{P}_{2,l}^{+}M_n^{2n} \cap \mathscr{I}(\mathbf{z}, g\mathbf{S})$$

where $\langle x, x\rangle_2 = \left(\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} x, x\right)$ $(x \in \mathbf{C}^{2n})$. Since $G = gK$, we have $\mathscr{G}_G = \mathscr{G}_g\mathscr{G}_K$ and $U_G = U_gU_K$ and thus

$$U_G(\mathscr{R}\mathscr{P}_1^{+}M_n^{2n}) = \mathscr{R}\mathscr{P}_{2,l}^{+}M_n^{2n} \cap \mathscr{I}(\mathbf{z}, g\mathbf{S}).$$

This is the exact Grassmannian version of the conclusion of Theorem 5.2.

We are now ready to deal with the "Poincaré distance problem" (Problem I.3 in the Introduction).

(I.3) Given S in $\mathscr{R}M(j, 0, k)$ $(j + k = n)$ find H in $\mathscr{R}\mathscr{B}H^{\infty}(M_n)$ such that

$$\text{s.s.v. } \mathscr{E}(S, H)(e^{i\theta}) \leqslant \eta(e^{i\theta}) > 0$$

where s.s.v.(m) = the smallest singular value of the matrix m and where η is a prescribed rational function. Now by Lemma 5.1

$$\mathscr{E}(S, H) \cong \mathscr{G}_{gS}(H)\mathscr{G}_{gS}(H)^{\dagger}$$

where $\mathscr{G}_{gS}$ is any rational (I, J)-linear fractional map which maps S to 0 $\left(\text{where } J = \begin{pmatrix} I_j & 0 \\ 0 & -I_k \end{pmatrix}\right)$. (Such exist by Corollary 3.2.) By Theorem 5.2, $\mathscr{G}_{gS}(\mathscr{R}\mathscr{B}H^\infty(M_n))$ generically is a set of the form $\mathscr{B}H^\infty_{J,l}(M_n) \cap \mathscr{I}(\mathbf{z}, \mathbf{x}, \mathbf{p})$, where l, $\mathbf{z}$, $\mathbf{x}$ and $\mathbf{p}$ in principle can be computed from S. Problem I.3 becomes

$$\text{(5.1)} \qquad \text{Find } K \text{ in } \mathscr{B}H^\infty_{J,l}(M_n) \cap \mathscr{I}(\mathbf{z}, \mathbf{x}, \mathbf{p}) \text{ with s.-}J\text{-s.v. } K(e^{i\theta}) \leqslant \eta(e^{i\theta})$$

where s.-J-s.v.(m) = the smallest J-singular value of m. If we introduce weight functions $w_+(e^{i\theta}) = I_k$ and $w_-(e^{i\theta}) = \operatorname{diag}\{1, \ldots, 1, \eta(e^{i\theta})\}$ $(j \times j)$, (5.1) is the same as Problem 4.6 above. Theorem 4.4 can therefore be used to solve the Poincaré distance problem (I.3). We state the result as

THEOREM 5.3. (i) *For a given S in $\mathscr{R}M(j, 0, k)$, there exist an integer $l \geqslant 0$ and N-tuples $\mathbf{z}$, $\mathbf{p}$, $\mathbf{x}$ as above such that a solution of the Poincaré distance problem* I.3 *exists if and only if the matrix*

$$A = \left[\frac{1}{1 - \bar{z}_j z_k}(p_j^* \alpha_\mu(z_j)^* J \alpha_\mu(z_k) p_k - x_j^* J x_k)\right]_{j,k=1,\ldots,N}$$

has at most l negative eigenvalues for some μ which is J-equivalent to $w^2 :=$
$$= \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \eta^2 \end{pmatrix}.$$

(ii) *Whenever this is the case, a solution H exists with*

$$\text{s.s.v. } H(e^{i\theta}) = \eta(e^{i\theta}).$$

This is a direct generalization of Theorem 5.2 [16], the main theorem of [16], to a generic S in $\mathscr{R}M(j, 0, k)$; the paper [16] treated the $k = 0$ case. Recall α_μ is the J-outer spectral factorization $\alpha_\mu^T \alpha_\mu = \mu$ of μ.

6. MATHEMATICAL CONSEQUENCES OF SOME PHYSICAL CONSTRAINTS

In this section we wish to discuss the mathematical implications which so far have been ignored of two types of constraints which are required for a realistic solution of the electric circuit problems to be discussed in the next section. The first is that an input signal consisting of real-valued frequencies should give rise to an output signal of real-valued frequencies (as opposed to complex-valued). The second is that the amplifier be *stable*. We discuss each in turn.

(R) REAL ON THE REAL LINE.

Mathematically, the first constraint means that the entries of any rational matrix function which is the frequency response function for a physically realizable circuit must be real on the real line. We can formalize this notion as follows. For S in $\mathcal{R}M_n$, define $\hat{S}$ to be

$$\hat{S}(z) := S(\bar{z})^{*T}$$

(where M^T is the transpose of a matrix M). Observe that $(ST)^\wedge = \hat{S}\hat{T}$. The condition then that real frequencies go to real frequencies is that

$$\hat{S} = S.$$

Let us call any such S *real*. The point of the next chain of results is that all the analysis done in the preceding sections can be done with real matrix functions. We have not included these results until now because mathematically the extra "reality" constraint tends to be only a nuisance.

It is a straight-forward matter of adapting the argument of Vekua ([42], pp. 45–49) and the arguments in §§ 2 and 3 to verify the following "real" Darlington theorem.

THEOREM 3.1R. *Let S in $\mathcal{R}M(j, 0, l)$ be real. Then there exist* **real** *A, B, C in $\mathcal{R}L^\infty(M_n)$ with $C(z)$ invertible for $|z| = 1$, such that*

$$V_S = \begin{pmatrix} A & B \\ C & S \end{pmatrix}$$

is $\begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$-unitary for $|z| = 1$, where $J = \begin{pmatrix} I_j & 0 \\ 0 & -I_l \end{pmatrix}$.

In the equivalent symplectic formalism, this means that g^s can be chosen to be real. This enables us to get a real version of the duality principle. Let us say a cascade map $\mathscr{F} = F_U$ is *real* if U is real. It is again an easy check that the range of a real (I, J)-cascade map, acting on real matrix functions in $\mathcal{R}\mathcal{B}H^\infty(M_n)$, is equivalent (in the generic multiplicity one case) to the set of all real functions in $\mathcal{R}\mathcal{B}H^\infty_{J,l}$ which are also in a *real* interpolation set $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$. The interpolation set $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ is said to be real if $\hat{K}(z) \equiv K(\bar{z})^{*T}$ is in $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ whenever $K(z)$ is. This constraint implies that the interpolation conditions $K(z_j)p_j = x_j$ $(j = 1, \ldots, N)$ occur symmetrically, that is they can be arranged to satisfy

(6.1) For some k_1 between 0 and $\left[\frac{N}{2}\right]$ and all $j \leqslant k_1$

$$\operatorname{Im} z_j > 0 \quad z_{j+k_1} = \bar{z}_j$$

$$p_{j+k_1} = p_j \quad x_{j+k_1} = \hat{x}_j$$

while for all $j > 2k_1$

$$\mathrm{Im} z_j = 0 \quad p_j = \hat{p}_j \quad x_j = \hat{x}_j.$$

For the real version of the Poincaré distance problem discussed in § 5, we assume that we are given a *real* matrix function S in $\mathscr{R}M(j, 0, k)$ and wish to decide when there is a *real* H in $\mathscr{R}\mathscr{B}H^\infty(M_n)$ such that

$$\text{s.e.v. } \mathscr{E}(S, H)(e^{i\theta}) \leqslant \eta(e^{i\theta})$$

for some given real rational $\eta(e^{i\theta})$. Using the real Lemma 5.1 and a real version of Lemma 4.3, we see that this problem is equivalent to:

(5.1R) Find a real L in $\mathscr{B}H^\infty_{J,l}(M_n) \cap \mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ with

$$\text{s.-}J\text{.-s.v. } L(e^{i\theta}) \leqslant \eta(e^{i\theta}).$$

where in addition, $\mathscr{I}(\mathbf{z}, \mathbf{p}, \mathbf{x})$ is a *real* interpolation set.

For the case $l = 0$, we can use a real constant coefficient (J, I)-cascade map to transform our real $\mathscr{B}H^\infty_J$-interpolation problem to a real $\mathscr{B}H^\infty$-interpolation problem, the set of all solutions of which form a convex set. Thus if H is any solution, then $\frac{1}{2}(H + \hat{H})$ is a real solution. For the case of arbitrary l, we again can map to a real $\mathscr{B}H^\infty_l$ (instead of our original $\mathscr{B}H^\infty_{J,l}$) interpolation problem. The above simple convexity argument fails, but nevertheless it can be shown when $n = 1$ the optimal solution f to a finite interpolation is unique. If the interpolation requirements are real then $\hat{f}$ is also a solution; so $f = \hat{f}$. Thus a real solution can be found if any can be found. When $n > 1$, the argument becomes more complicated and depends on a representation for all solutions to the interpolating problem which can be derived from [7]. All coefficients in the representation must be real on the real axis; this guarantees that a real solution exists, if any exists. Thus Theorem 5.3 provides the solution to Problem 5.3R.

(S) STABILITY.

A problem motivated from consideration of electric circuits (see § 7 and [5]) is the following:

(6.2S) Given a real function S in $\mathscr{R}M(j, 0, n - j) \cap H^\infty(M_n)$ we wish to decide if there is a real H in $\mathscr{R}\mathscr{B}H^\infty(M_n)$ such that

$$\text{s.e.v. } \mathscr{E}(S, H)(e^{i\theta}) \leqslant \eta(e^{i\theta}) > 0$$

for some given real rational function η, with the additional constraint that H be stable. The constraint that H be stable is that $(I - SH)^{-1}$ have no poles on the unit disk $\{|z| \leqslant 1\}$.

If we apply Lemma 5.1 and Theorem 5.2 to reduce this problem to an interpolation problem as was done in § 5, we see that it is equivalent to

Find an L of the form $\mathscr{G}_{g}s(H)$ satisfying (5.1R) with $(I - SH)^{-1} \in H^{\infty}(M_n)$.
Following [5], we say that

$$H \text{ and } S^{-1} \textit{ never agree}$$

if $H - S^{-1}$ takes invertible values in $\{|z| \leqslant 1\}$. It is straightforward to verify that this is equivalent to our stability constraint. The argument in [5] shows (with extra care to include the *real* constraint) that the map $\mathscr{G} = \mathscr{G}_{g}s_{-1}$ can be chosen so that

$$H \textit{ and } S^{-1} \textit{ never agree} \Leftrightarrow L = \mathscr{G}(H) \textit{ and } 0 = \mathscr{G}(S^{-1}) \textit{ never agree.}$$

Thus we have shown that Problem (6.2S) is equivalent (for the multiplicity free case) to the following:

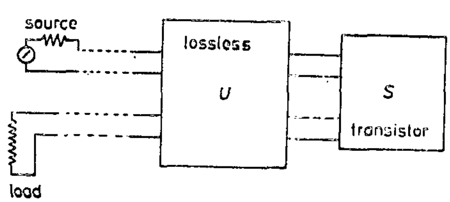
(6.3RS) Determine if there is a *real* L satisfying (5.1R) such that L and 0 never agree.

Here, of course, we assume that the interpolation constraint parameters **z**, **p**, **x** satisfy (6.1).

Problem (6.3 RS) is wide open, even for the case $J = I$, $l = 0$, for the matrix case. A solution for the scalar case (with $l = 0$) was offered in [5] but with the "real on the real line" constraint ignored. It is very straightforward to include this constraint in the analysis given in [5].

7. AMPLIFIERS

A fairly general configuration for an amplifier is given in the figure.



The transistor typically is given and the problem is to find a lossless circuit so that the power which a signal traveling from source to load gains is large and independent of frequency.

The transistor is specified by a 2×2 matrix

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

called its scattering matrix, the coupling circuit by an inner function $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrix entries A, B, C, D in $H^\infty(M_2)$, and the frequency response function for the whole amplifier is $\mathscr{F}_U(S)$. Motivation for this can be found in the exposition § 1 [17], or to a lesser extent in § 6 [16]. The gain of the amplifier at "frequency" θ is proportional to

$$|[F_U(S)(e^{i\theta})]_{12}|^2$$

where $[M]_{12}$ stands for the 12 entry of the 2×2 matrix M. Typically a transistor at frequencies in its operating range has a $|S_{12}|$ which is very large, $|S_{21}|$ is small (at audio frequencies almost 0), and $|S_{11}|$ and $|S_{22}|$ are both smaller than 1. So S has one singular value much bigger than 1 and one singular value less than one. Various mathematical problems which arise directly are

I. Narrow band amplifier. Fix θ_0, find

$$g^{\theta_0} = \max_{U \text{ inner}} |[F_U(S)(e^{i\theta_0})]_{12}|^2$$

and find a U which achieves this maximum.

II. Broad band amplifier. Find

$$g_S = \max_{U \text{ inner}} \inf_{\theta} |[F_U(S)(e^{i\theta})]_{12}|^2 \tag{7.1}$$

and the maximizing U.

Stability restriction. Do the optimization above subject to the constraint that the amplifier is stable. The stability constraint is that $F_U(S)$ and all small perturbations of $F_U(S)$ obtained by small perturbations of U and S be in $H^\infty(M_n)$. Physically, this means that a finite energy input should have a finite energy output. In other words the amplifier is not called on to do something so strenuous that it will (in practice) burn out. If $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so $F_U(S) = A + BS(I - DS)^{-1}C$, then stability means that $(I - DS)^{-1} \in H^\infty(M_n)$, or that D *never agrees with* S^{-1} at a z inside the unit disk (see [5] for more details).

Results. The first step in studying these problems is to convert them to a more canonical form.

Proposition 7.1.

(i)
$$g_S = \max_{\substack{U \text{ inner} \\ \text{in } H^\infty(M_4)}} \inf_{\theta} \text{ b.e.v. } |F_U(S)(e^{i\theta})| =$$

$$= \max_{H \in \mathscr{B}H^\infty(M_2)} \inf_{\theta} \text{ b.e.v. } \mathscr{E}(H(e^{i\theta}), S(e^{i\theta})^*)$$

and

$$g_S^{-1} = \min_{H \in \mathscr{B}H^\infty(M_2)} \|\text{s.e.v. } \mathscr{E}(H, S^{-1})\|_{L^\infty(\mathbf{C}^1)} .$$

Here b.e.v. = "*biggest eigenvalue*" *and* s.e.v. = "*smallest eigenvalue*".

(ii) *The matrix function* $F_U(S)(e^{i\theta})$ *which (approximately) achieves the maximum gain* g_S *is (approximately) off-diagonal.*

(iii) *The stable circuits in Figure* 1 *correspond to* $H \in \mathscr{B}H^\infty(M_n)$ *which never agree with* S^{-1} *at a z in the unit disk.*

Proof. We prove the first expression for g_S and the assertions concerning f_S (statement (ii)) only for the narrowband (constant frequency) case, since it is easy and especially informative, and refer the reader to [16] for the broadband case. Then the set of transformations $\{F_U : U\ 4 \times 4 \text{ unitary}\}$ is a group, and, for V and W unitary 2×2 matrices, the transformation $x \to VxW$ is an element of this group. Thus the maximum gain g_S, with the superfluous $e^{i\theta}$ suppressed, can be written as

$$g_S = \max_{\substack{U \text{ unitary } 4\times 4 \\ V, W \text{ unitary } 2\times 2}} |[VF_U(S)W]_{12}|^2.$$

Fixing U for the moment and using the freedom from V and W, we may suppose that

$$F_U(S) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \qquad s_1 = \|F_U(S)\| > s_2 = \|F_U(S)^{-1}\|^{-1}$$

is diagonal (by properties of the singular value decomposition of a matrix). It is then easy to see that

$$\max_{V, W \text{ } 2\times 2 \text{ unitaries}} \left| \left[V \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} W \right]_{12} \right|^2 = \left| \begin{bmatrix} 0 & s_1 \\ s_2 & 0 \end{bmatrix}_{12} \right|^2 = s_1^2 .$$

Letting U vary now, we deduce statement (ii) for constant matrices.

Now let us return to the broadband (variable frequency) case. Since

$$\text{b.e.v. } |F_U(S)|^2 = \text{b.e.v. } \mathscr{E}(0, F_U(S)) =$$
$$= \text{b.e.v. } \mathscr{E}(F_U^{-1}(0)^*, S^*) = \text{b.e.v. } \mathscr{E}(H, S^*)$$

where $H = F_U^{-1}(0)^*$ by the duality principle (Lemma 5.1 with $J = I$), the second expression for g_S follows from the first. By Lemma 1.2 (adapted for the cascade formalism),

$$g_S^{-1} = \min_U \text{s.e.v. } |F_U(S)|^2 =$$
$$= \min_U \text{s.e.v. } \mathscr{E}(0, F_U(S)) = \min_H \text{s.e.v. } \mathscr{E}(H, S^{-1})$$

which proves the last expression for g_S.

Finally, if $H = F_U^{-1}(0)^*$ where $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then $H = D$ and thus stability of the circuit means that H never agrees with S^{-1} at a z in the unit disk.

In the proposition g_S^{-1} is given as exactly the type of minimum this paper is devoted to computing. Thus the results of this paper apply directly. (Note that the S in § 6 S should be taken to be S^{-1} in the notation of this section.)

The solution to the narrowband problem is easy and surely no surprise to engineers. It is the content of Theorem 1.5. The answer is that an infinite gain is possible.

It should be clear how our main results Theorem 5.3i and Theorem 5.3R (with the *real* constraints discussed in § 6R added) are statements about the broadband equalization problem. Statement (ii) in the theorems together with the adaptation in § 6R amounts to the following physical principle for possibly unstable amplifiers.

PRINCIPLE. Suppose S in $\mathscr{R}M(1, 0, 1)$ is a "real" function. Then there is an optimum amplifier of the type in Figure 1 (with maximum gain g_S). The gain of this amplifier does not depend on the frequency at which the amplifier is operated. This amplifier has 0 front and back reflection. Also if it is possible to build an amplifier whose gain is greater than $g(e^{i\theta})$ for all θ, then there is an amplifier whose gain exactly equals $g(e^{i\theta})$ for all θ.

The Theorem 5.3i gives one something of a method for determining when a given gain is obtainable from an amplifier of the type in Figure 1. It says that for a given S, there are points $\mathbf{z}, \mathbf{p}, \mathbf{x}$ which after the theory is refined might well be computable, so that if one can find a certain type of α_μ making the matrix A positive then the gain is realizable. The solution is unsatisfactory because of the freedom present in the α_μ. On the other hand this is the exact generalization of what happens in the gain specification problem for passive circuits. That theory is now refined to the point of being potentially practical (see [17]), so maybe additional work on interpolation theory will someday handle the existing theory.

Other types of amplifiers are possible. It would be good to have a study of amplifiers where the "equalizing circuit" U is not required to be lossless but merely to be passive. We treat this in the next section. Another type of amplifier is the reflection type; much of [16], [17], [18] is devoted to a study of it. The mathematical issue is that of finding the Poincaré distance from a rational function F to $\mathscr{B}H^\infty$, that is, find $\inf_{H \in \mathscr{B}H^\infty(M_N)}$ b.e.v. $\delta(H, F)$. This problem is solved (see [16]) but without the crucial stability constraint [5].

8. THE PASSIVE SEMIGROUP IN GL(2n)

The paper so far has concerned $U(n, n)$, $\mathscr{R}U(n, n)$, $\mathscr{R}U^+(n, n)$ and an optimization problem on orbits. Now we look at a much bigger semigroup than $\mathscr{R}U^+(n, n)$ and show that optimizing over it gives the same answer as we already obtained.

This shows that any gain obtainable in Figure 1 through adroit selection of a passive (energy dissipating) circuit U can also be obtained with proper selection of a lossless (energy conserving) U. In practice a passive circuit U may be easier to build.

We begin by defining the semigroup $C(n, n)$ of $\mathrm{GL}(2n)$ to be all $J := \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ contraction operators. It contains $U(n, n)$ and has the property that if $c \in C(n, n)$ then $\mathscr{G}_c : \mathscr{B}M_n \to \mathscr{B}M_n$ by Lemma 1.1. Conversely if $\mathscr{G}_c : \mathscr{B}M_n \to \mathscr{B}M_n$, then c is a scalar multiple of a matrix in $C(n, n)$. A key fact for our purposes is the main result of [6].

THEOREM [6]. *Suppose $c \in C(n, n)$. If $S \in M(j, 0, n - j)$ and $H \in M(n, 0, 0)$, then*

$$\mathrm{e.v}_k \mathscr{E}(S, H) \geqslant \mathrm{e.v}_k \mathscr{E}(\mathscr{G}_c(S), \mathscr{G}_c(H))$$

for $k \geqslant n - j$.

We define $\mathscr{R}C(n, n)$ to be $C(n, n)$ over the field of rational functions, and $\mathscr{R}C^+(n, n)$ to be those c in $\mathscr{R}C(n, n)$ for which $\mathscr{G}_c : \mathscr{B}H^\infty(M_n) \to \mathscr{B}H^\infty(M_n)$. Recall that a map $\mathscr{G}_c$ can be written in the cascade formalism as a map F_U; the c in $\mathscr{R}C^+(n, n)$ correspond precisely to those U in $\mathscr{B}H^\infty(M_{2n})$.

Now we turn to the broadband gain optimization problem (7.1) but allow one to use any U in Figure 1 which is passive; that is U is in $\mathscr{B}H^\infty(M_4)$. The main result of this section is

THEOREM 8.1. *Suppose $S \in \mathscr{R}H^\infty M(1, 0, 1)$. The optimum gain*

$$\delta_{\mathscr{L}} = \max_{U \in \mathscr{L}} \inf_\theta |[F_U(S)(e^{i\theta})]_{21}|^2$$

over coupling circuits U in the class $\mathscr{L}$

(a) *is the same for $\mathscr{L}_1 = \mathscr{R}\mathscr{B}H^\infty(M_2)$ and for $\mathscr{L}_2 = \{U : U \text{ inner}\}$.*

(b) *is the same for $\mathscr{L}_3 =$ those "real" $\mathscr{R}\mathscr{B}H^\infty(M_2)$ for which the Fig. 1 circuit is stable and for $\mathscr{L}_4 =$ those "real" inner functions for which the Fig. 1 circuit is stable.*

Actually this is a very special case of the most general theorem which follows immediately from our techniques. Namely, a similar theorem holds for an arbitrary class of matrices $M(j, 0, n - j)$ and the problem of whether or not any given function in this class is a physically obtainable gain.

Proof. The proof amounts to generalizing parts of the argument behind Proposition 7.1. The first step, that

$$\delta_{\mathscr{L}} = \max_U \inf_\theta \| F_U(S) \|_{M_2}$$

holds for all of our classes $\mathscr{L}$, is the same as before (see § 6 [16]). Next we note that

$$\delta_{\mathscr{L}}^{-1} = \min_{U \in \mathscr{L}} \|\text{s.e.v.}\ |F_U(S)|\|_{L^\infty(\mathbf{C})}$$

and hence, by Lemma 1.2 (adapted to cascade transformation),

$$\delta_{\mathscr{L}}^{-1} = \min_{U \in \mathscr{L}} \|\text{s.e.v.}\ |F_{\tilde{U}}(\tilde{S})|\|_{L^\infty(\mathbf{C})}.$$

By Corollary 1.4 in [6], if U is in any of our classes $\mathscr{L}$,

$$\text{s.e.v.}\ |F_{\tilde{U}}(\tilde{S})| = \text{s.e.v.}\ \mathscr{E}(0,\ F_{\tilde{U}}(\tilde{S})) \geqslant \text{s.e.v.}\ \mathscr{E}(H,\ \tilde{S}^*)$$

where $H = F_{\tilde{U}}^{-1}(0)^*$, with equality if $U = \tilde{U}^{-1}$ is inner (e.g. U in $\mathscr{L}_2$ or $\mathscr{L}_4$). Next we must see which H's arise from U's in a given class $\mathscr{L}$.

Since $F_U(H^*) = 0$, a formal application of the version of Lemma 1.2 for cascade transforms gives $F_U(H^{-1})^{-1} = 0$. Express U as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$; then $F_U(H^{-1}) = A + B(H - D)^{-1}C$. Thus we see formally that $H = D$. To make this argument rigorous, we need only perturb slightly. Thus the class of H which arise from $\mathscr{L}_1 = \mathscr{R}\mathscr{B}H^\infty(M_2)$ is precisely $\mathscr{R}\mathscr{B}H^\infty(M_2)$, the same as for $\mathscr{L}_2$ by Darlington's theorem (see § 3 and Proposition 7.1). For $\mathscr{L}_3$ the H's are precisely those H in $\mathscr{R}\mathscr{B}H^\infty(M_2)$ which never agree with S^{-1}, the same as for $\mathscr{L}_4$ (see § 7).

Put this all together to obtain for part (a)

$$\delta_{\mathscr{L}_1}^{-1} \geqslant \min_{H \in \mathscr{B}H^\infty(M_2)} \|\text{s.e.v.}\ \mathscr{E}(H, S)^{-1})\|_{L^\infty} = \delta_{\mathscr{L}_2}^{-1},$$

and similarly for part (b)

$$\delta_{\mathscr{L}_3}^{-1} \geqslant \min_{\substack{H \in \mathscr{B}H^\infty(M_2) \\ H \text{ never agrees} \\ \text{with } H^{-1}}} \|\text{s.e.v.}\ \mathscr{E}(H, S^{-1})\|_{L^\infty} = \delta_{\mathscr{L}_4}^{-1}.$$

On the other hand, $\mathscr{L}_1 \supset \mathscr{L}_2$ and thus $\delta_{\mathscr{L}_2}^{-1} \geqslant \delta_{\mathscr{L}_1}^{-1}$ and (a) follows. Similarly $\mathscr{L}_3 \supset \mathscr{L}_4$ and (b) follows as well.

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