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K-GROUPS OF REDUCED CROSSED PRODUCTS BY FREE GROUPS

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The starting point for the present paper was a problem of R. V. Kadison about projections in the reduced C^*-algebra of a free group (see [18]). As conjectured by R. V. Kadison, we prove that there are no non-trivial projections in the reduced C^*-algebra of a free group.

Our results are in fact more general. For a C^*-algebra A endowed with an automorphic action of the free group on n generators F_n, we obtain a six terms cyclic exact sequence which can be used for the computation of the K-groups of the reduced crossed product of A by F_n. In particular for $C^*_r(F_n)$, the reduced C^*-algebra of F_n, we have $K_0(C^*_r(F_n)) \simeq \mathbf{Z}$ and $K_1(C^*_r(F_n)) \simeq \mathbf{Z}^n$. Now, J. Cohen has proved in [5] (see also [4]) that there are no non-trivial projections in the full C^*-algebra of the free group and recently J. Cuntz in [8] has proved that K_0 and K_1 of the full C^*-algebra of F_n are $\mathbf{Z}$ and respectively $\mathbf{Z}^n$. Thus our results show that the full group C^*-algebra and the reduced C^*-algebra of F_n have the same K-theory.

The main result of the present paper generalizes the exact sequence for crossed products by $\mathbf{Z}$ which we obtained in [17]. A modified proof of the exact sequence for crossed products by $\mathbf{Z}$, discovered by J. Cuntz [7], has the important feature that it avoids a certain argument about spectral projections in the initial proof. This feature of the proof in [7] turned out to be extremely useful for the generalization to the case of actions of free groups.

Though it won't be explicit in our proofs, let us also mention, that the natural framework for some of the constructions we use in this paper is the general theory of extensions of $\mathcal{K} \otimes A$ of G. G. Kasparov [14].

The paper has three sections. Section 1 contains preliminaries about the Toeplitz extension. Section 2 contains the main technical facts on which the proof of the exact sequence is based. In Section 3 the main result is obtained by putting together the results of the preceding sections.

A paper with the same title as the present one has been circulated as INCREST preprint No. 59/1980. Unfortunately one of the key-lemmas in that paper turned out to be false as discovered by W. Paschke. We thank W. Paschke for pointing out this error to us, which determined the present new attempt of computing the K-groups of reduced crossed products by free groups. The present paper has been circulated as INCREST preprint No. 62/1981.

§ 1

In this section we introduce the Toeplitz extensions for reduced crossed product by an action of a free group and prove some of its properties.

By F_n we shall denote the free group on n generators $g_1, \ldots, g_n$. For $g \in F_n$ we shall denote by $|g|$ the length of the corresponding word. By $\Gamma_k \subset F_n$ we shall denote the subset of Γ_n consisting of elements $g_{i_1}^{m_1} \ldots g_{i_s}^{m_s}$ ($s \geqslant 0$, $i_1 \neq i_2$, $i_2 \neq i_3, \ldots$ $\ldots, i_{s-1} \neq i_s$, $m_1 \neq 0, \ldots, m_s \neq 0$) such that $m_s > 0$ if $i_s = k$. Remark that the neutral element e of F_n is in Γ_k and $g_j\Gamma_k = \Gamma_k$ if $j \neq k$ and $g_k\Gamma_k = \Gamma_k \setminus \{e\}$.

Consider a unital C^*-algebra A with an action $\alpha: F_n \to \mathrm{Aut}A$. We shall assume that A is a C^*-algebra of operators on a Hilbert space H_0 and that the action α is implemented by a unitary representation $g \mapsto v_g$ of F_n on H_0 (i.e. $v_g a v_g^* = \alpha(g)a$ for $a \in A$, $g \in F_n$).

The reduced crossed product of A by α, $A \times_{\alpha r} F_n$ will be identified with the C^*-algebra of operators on $\ell^2(F_n, H_0) \simeq \ell^2(F_n) \otimes H_0$ generated by the operators:

$$\pi(a) = 1 \otimes a,$$

$$(u_g k)(h) = v_g k(g^{-1}h)$$

where $a \in A$, $g, h \in F_n$ and $k \in \ell^2(F_n, H_0)$.

In case $g = g_i$ we shall write also u_i, v_i instead of u_{g_i}, v_{g_i}.

Consider also the representation $g \mapsto R_g$ of F_n on $\ell^2(F_n, H_0)$ given by

$$R_g k(h) = k(hg).$$

Then $\{R_g\}_{g \in F_n}$ is in the commutant of $A \times_{\alpha r} F_n$ in $B(\ell^2(F_n, H_0))$.

Each Γ_k ($1 \leqslant k \leqslant n$) determines an extension of $\mathscr{K} \otimes A$ ($\mathscr{K}$ denotes the compact operators on some separable Hilbert space) by $A \times_{\alpha r} F_n$, which will be called a *Toeplitz extension*. In order to fix notations we shall construct this extension only for $k = n$.

On $\ell^2(\Gamma_n, H_0) \subset \ell^2(F_n, H_0)$ let ρ, U_i ($1 \leqslant i < n$) and S_n denote the restrictions of π, u_i ($1 \leqslant i < n$) and u_n respectively. The *Toeplitz algebra* $\mathscr{T}_n$ is the C^*-algebra generated by $\rho(A)$ and $U_1, \ldots, U_{n-1}, S_n$.

We have

$$I - S_n S_n^* = P_e$$

where P_e is the orthogonal projection of $\ell^2(\Gamma_n, H_0)$ onto its subspace $\ell^2(\{e\}, H_0)$. Note also that $U_1, \ldots, U_{n-1}$ are unitaries and

$$U_i \rho(a) U_i^* = \rho(\alpha(g_i)a) \quad (1 \leqslant i \leqslant n-1)$$

$$S_n^* \rho(a) S_n = \rho(\alpha(g_n^{-1})a).$$

It is also easy to see that the closed two-sided ideal $\mathcal{J}$ in $\mathcal{T}_n$ generated by P_e is isomorphic to $\mathcal{K}(\ell^2(\Gamma_n)) \otimes A$. To this end one uses conjugation by the unitary operator

$$\Lambda : \ell^2(\Gamma_n, H_0) \to \ell^2(\Gamma_n, H_0)$$

given by

$$(\Lambda k)(g) = v_g^* k(g)$$

for $k \in \ell^2(\Gamma_n, H_0)$, $g \in \Gamma_n$. In fact denoting by $e(g', g)$ the natural matrix units for $\mathcal{K}(\ell^2(\Gamma_n))$ this isomorphism can be described by the correspondences

$$e(g', g) \otimes a \mapsto \omega(g') P_e \omega(g^{-1}) \rho(\alpha(g)a)$$

where $\omega(g)$ for $g = g_{i_1}^{k_1} \ldots g_{i_m}^{k_m}$ is obtained by replacing g_j with U_j for $1 \leqslant j \leqslant n-1$, g_n^k by S_n^k for $k > 0$ and g_n^{-k} by $S_n^{*k} (k > 0)$.

1.1. LEMMA. *There exists a homomorphism*

$$p : \mathcal{T}_n \to A \times_{\alpha r} F_n$$

such that $p(\rho(a)) = \pi(a)$, $p(U_i) = u_i$ $(1 \leqslant i \leqslant n-1)$, $p(S_n) = u_n$. *Moreover* Ker$p$ *is the closed two-sided ideal* $\mathcal{J}$ *of* $\mathcal{T}_n$ *generated by* P_e, *and* $\mathcal{J}$ *is isomorphic to* $\mathcal{K} \otimes A$.

Proof. Let $W = R_{g_n}$. Then it is easily seen that for $m \to \infty$ we have

$$W^m \tilde{S}_n W^{*m} \xrightarrow{s} u_n$$

$$W^m \tilde{S}_n^* W^{*m} \xrightarrow{s} u_n^*$$

$$W^m \tilde{U}_i W^{*m} \xrightarrow{s} u_i \quad (1 \leqslant i \leqslant n-1)$$

$$W^m \tilde{U}_i^* W^{*m} \xrightarrow{s} u_i^* \quad (1 \leqslant i \leqslant n-1)$$

$$W^m \widetilde{\rho(a)} W^{*m} \xrightarrow{s} \pi(a) \quad (a \in A)$$

$$W^m \tilde{P}_e W^{*m} \xrightarrow{s} 0$$

where for $T \in B(\ell^2(\Gamma_n, H_0))$ we denote by $\tilde{T}$ the operator $T \oplus 0$ on $\ell^2(\Gamma_n, H_0) \oplus \oplus \ell^2(F_n \setminus \Gamma_n, H_0) = \ell^2(F_n, H_0)$.

Since $\rho(A) \cup \{S_n, S_n^*\} \cup \{U_i, U_i^*\}_{1 \leq i \leq n-1}$ is a set of generators for the Banach algebra $\mathscr{T}_n$, we infer that

$$\text{s-lim}\, W^m \tilde{T} W^{*m}$$

exists for all $T \in \mathscr{T}_n$ and the map

$$T \mapsto \operatorname*{s-lim}_{m \to \infty} W^m \tilde{T} W^{*m}$$

is a unital $*$-homomorphism of $\mathscr{T}_n$ onto $A \times_{\alpha r} F_n$. This is the homomorphism p we were looking for and is clearly unique.

Since $p(P_e) = 0$ it follows that $\mathrm{Ker}\, p \supset \mathscr{J}$.

To see that $\mathrm{Ker}\, p \subset \mathscr{J}$ we proceed as follows. For $x \in A \times_{\alpha r} F_n$ we define

$$\Phi(x) = P_{\Gamma_n} x \mid \ell^2(\Gamma_n, H_0)$$

where P_{Γ_n} is the orthogonal projection of $\ell^2(F_n, H_0)$ on $\ell^2(\Gamma_n, H_0)$. It is easily seen that Φ is a completely positive map and $\Phi(A \times_{\alpha r} F_n) \subset \mathscr{T}_n$ (this must be checked only for elements of the form $\pi(a)u_g$ which form a total subset of $A \times_{\alpha r} F_n$). Moreover it is easy to check that $\Phi(xy) - \Phi(x)\Phi(y) \in \mathscr{J}$ and $p(\Phi(x)) = x$ (again the checking must be done only for elements of the form $\pi(a)u_g$). Using $\Phi(xy) - \Phi(x)\Phi(y) \in \mathscr{J}$ we infer that $q \circ \Phi$ is a unital $*$-homomorphism of $A \times_{\alpha r} F_n$ into $\mathscr{T}_n / \mathscr{J}$ where q is the canonical map $\mathscr{T}_n \to \mathscr{T}_n / \mathscr{J}$. Moreover $q \circ \Phi$ is onto, since $\Phi(A \times_{\alpha r} F_n) \supset \rho(A) \cup \{S_n, U_1, \ldots, U_{n-1}\}$.

Consider now $y \in \mathrm{Ker}\, p$. Then there is $z \in A \times_{\alpha r} F_n$ such that $(q \circ \Phi)(z) = q(y)$ or equivalently $\Phi(z) - y \in \mathscr{J}$. But then $z = p(\Phi(z) - y) \in p(\mathscr{J}) = \{0\}$, so that $q(y) = 0$, which means $y \in \mathscr{J}$. Q.E.D.

The preceding lemma can be summarized by the exact sequence:

$$0 \to A \otimes \mathscr{K} \xrightarrow{\psi} \mathscr{T}_n \xrightarrow{p} A \times_{\alpha r} F_n \to 0.$$

For later use it will be useful to prove a lemma concerning certain representations of $\mathscr{T}_n$.

Let $u_1, \ldots, u_{n-1}$ be unitaries on some Hilbert space H and let $s_n \in B(H)$ be an isometry. We shall define a map $\sigma : F_n \to B(H)$. For g_n we define

$$\sigma(g_n^m) = \begin{cases} s_n^m & \text{if } m > 0 \\ I & \text{if } m = 0 \\ s_n^{*|m|} & \text{if } m < 0 \end{cases}$$

and for $g \in F_{n-1}$ (where F_{n-1} is the subgroup of F_n generated by $g_1, \ldots, g_{n-1}$) we define $\sigma(g)$ $(g \in F_{n-1})$ as the representation of F_{n-1} such that $\sigma(g_i) = u_i$ $(1 \leqslant i \leqslant n-1)$.

A general element of F_n can be always written in a unique way

$$g = g_n^{k_m} h_m g_n^{k_{m-1}} h_{m-1} \ldots g_n^{k_1} h_1 g_n^{k_0}$$

where $m \geqslant 0$, $h_j \in F_{n-1} \setminus \{e\}$ $(1 \leqslant j \leqslant m)$ and $k_j \neq 0$ for $1 \leqslant j \leqslant m-1$. Accordingly we define

$$\sigma(g) = \sigma(g_n^{k_n})\sigma(h_m)\sigma(g_n^{k_{m-1}})\sigma(h_{m-1}) \ldots \sigma(g_n^{k_1})\sigma(h_1)\sigma(g_n^{k_0}).$$

1.2. LEMMA. *Let $u_1, \ldots, u_{n-1}$ be unitaries on H and let s_n be an isometry on H. Let further μ be a non-degenerate representation of A on H such that*

$$u_j \mu(a) u_j^* = \mu(\alpha(g_j)a) \quad 1 \leqslant j \leqslant n-1$$

$$s_n^* \mu(a) s_n = \mu(\alpha(g_n^{-1})a).$$

Consider $P = I - s_n s_n^*$ and assume $g \in \Gamma_n \setminus \{e\} \Rightarrow P\sigma(g)P = 0$. Then

$$H_1 = \bigvee_{g \in \Gamma_n} \sigma(g)PH$$

*is a reducing subspace for $\{u_1, \ldots, u_{n-1}, s_n\} \cup \mu(A)$ and there is a unique *-representation φ of $\mathscr{T}_n$ on H_1 such that*

$$\varphi(U_j) = u_j | H_1 \quad 1 \leqslant j \leqslant n-1$$

$$\varphi(S_n) = s_n | H_1$$

$$\varphi(\rho(a)) = \mu(a) | H_1.$$

Proof. Note first that the definition of σ implies that

$$u_j \sigma(g) = \sigma(g_j g) \quad 1 \leqslant j \leqslant n-1$$

$$u_j^{-1} \sigma(g) = \sigma(g_j^{-1} g) \quad 1 \leqslant j \leqslant n-1$$

$$s_n^* \sigma(g) = \sigma(g_n^{-1} g)$$

for every $g \in F_n$. Moreover, for $g \in F_n \setminus \{g_n^{-1}\}$ we have

$$s_n \sigma(g) P = \sigma(g_n g) P.$$

Indeed, if $g_n g \neq e$, then we have $P\sigma(g_n g)P = 0$ and hence:

$$s_n \sigma(g) P = s_n s_n^* \sigma(g_n g) P =$$

$$= (I - P)\sigma(g_n g) P = \sigma(g_n g) P.$$

Using the above identities recurrently one easily gets that

$$(*)\qquad \sigma(g)\sigma(h)P = \sigma(gh)P, \quad (\forall)\ g \in F_n,\ (\forall)\ h \in \Gamma_n.$$

In particular, we have

$$\sigma(g)H_1 = \bigvee_{g' \in g\Gamma_n} \sigma(g')PH.$$

This relation for $g = g_j^{\varepsilon}$, $1 \leqslant j \leqslant n$, $\varepsilon = \pm 1$ together with $\sigma(g_n^{-1})P = 0$ gives that H_1 is a reducing subspace for $\{u_1, \ldots, u_{n-1}, s_n\}$.

To see that H_1 is also reducing for $\mu(A)$ note first that

$$(I - P)\mu(a)P\mu(a^*)(I - P) =$$

$$= s_n s_n^* \mu(a)(I - s_n s_n^*)\mu(a^*)s_n s_n^* =$$

$$= s_n\mu(\alpha(g_n^{-1})(aa^*))s_n^* - s_n\mu(\alpha(g_n^{-1})a)\mu(\alpha(g_n^{-1})a^*)s_n^* = 0.$$

Since $A = A^*$, this implies $[P, \mu(A)] = 0$ and hence:

$$s_n\mu(a) = s_n s_n^* \mu(\alpha(g_n)a)s_n = (I - P)\mu(\alpha(g_n)a)s_n = \mu(\alpha(g_n)a)s_n.$$

This relation, together with

$$u_j\mu(a) = \mu(\alpha(g_j)a)u_j \quad (1 \leqslant j \leqslant n - 1),$$

shows that

$$\sigma(g)\mu(a) = \mu(\alpha(g)a)\sigma(g).$$

Since P is reducing for $\mu(A)$ it follows that H_1 is reducing for $\mu(A)$.

Using $(*)$ it is easily seen that $\sigma(g)P$ for $g \in \Gamma_n$ is a partial isometry from PH to $\sigma(g)PH$ and that $g \neq g'$, $g, g' \in \Gamma_n$ implies

$$\sigma(g)PH \perp \sigma(g')PH.$$

Defining for $g, g' \in \Gamma_n$ the partial isometry

$$E(g', g) = \sigma(g')P\sigma(g^{-1})$$

it follows that $E(g', g)$ is a system of matrix units for $\mathcal{K}(\ell^2(\Gamma_n))$ and

$$E(g', g)\mu(a) = \mu(\alpha(g'g^{-1})a)E(g', g).$$

Denoting by $e(g', g)$ the natural matrix units for $\mathcal{K}(\ell^2(\Gamma_n))$ we easily see that

$$e(g', g) \otimes a \mapsto E(g', g)\mu(\alpha(g)a)|H_1$$

defines a representation of $\mathcal{K}(\ell^2(\Gamma_n)) \otimes A$ on H_1. Since $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{T}_n$, this representation extends to a representation of $\mathcal{T}_n$ on H_1.

To conclude the proof one must check that this representation of $\mathscr{T}_n$ coincides with φ on the generators $U_1, \ldots, U_{n-1}, S_n$, which we leave to the reader. Q.E.D.

We return now to the exact sequence

$$0 \to A \otimes \mathscr{K} \xrightarrow{\psi} \mathscr{T}_n \xrightarrow{p} A \times_{\alpha r} F_n \to 0.$$

It is easily seen that $\rho(A) \cup \{U_1, \ldots, U_{n-1}\}$ determine an injective *-homomorphism

$$d : A \times_{\alpha' r} F_{n-1} \to \mathscr{T}_n$$

where α' is the restriction of α to F_{n-1}. We shall denote by

$$i : A \to A \times_{\alpha' r} F_{n-1}$$

the canonical inclusion, so that

$$d \circ i = \rho.$$

On the other hand let α_n denote the action

$$\alpha_n : \mathbf{Z} \to \mathrm{Aut} A$$

given by $\alpha_n(k) = \alpha(g_n^k)$, $\mathscr{T}(A, \alpha_n, \mathbf{Z})$ the Toeplitz algebra corresponding to this action of $\mathbf{Z} \simeq F_1$ which coincides with the Toeplitz algebra for a crossed product by $\mathbf{Z}$ defined in [17]. Using the preceding Lemma 1.2 or Lemma 3.1 of [17], we have that $\rho(A) \cup \{S_n\} \subset \mathscr{T}_n$ determine an injective *-homomorphism

$$t : \mathscr{T}(A, \alpha_n, \mathbf{Z}) \to \mathscr{T}_n.$$

It is obvious that we have a factorization of ρ

$$\begin{array}{ccc} A & \xrightarrow{\quad\rho\quad} & \mathscr{T}_n \\ & \searrow \qquad \nearrow_t & \\ & \mathscr{T}(A, \alpha_n, \mathbf{Z}) & \end{array}$$

1.3. Lemma. *The diagram*

$$\begin{array}{ccc} K_j(A \otimes \mathscr{K}) & \xrightarrow{\psi_*} & K_j(\mathscr{T}_n) \\ \Big| \wr & & \Big\uparrow \rho_* \\ K_j(A) & \xrightarrow{\mathrm{id}_* - (\alpha(g_n^{-1}))_*} & K_j(A) \end{array}$$

is commutative ($j = 0, 1$).

Proof. Using the factorization

$$\begin{array}{ccc} A & \xrightarrow{\quad\rho\quad} & \mathscr{T}_n \\ & \searrow \qquad \nearrow_t & \\ & \mathscr{T}(A, \alpha_n, \mathbf{Z}) & \end{array}$$

the lemma is an immediate consequence of the corresponding result for $n = 1$ ([17], Lemma 2.1 and Lemma 2.2). Q.E.D.

§ 2

In this section we will show that the homomorphism $d : A \times_{\alpha' r} F_{n-1} \to \mathcal{T}_n$ induces isomorphisms between the corresponding K-groups i.e. that $d_* : \mathrm{K}_j(A \times_{\alpha' r} F_{n-1}) \to \mathrm{K}_j(\mathcal{T}_n)$ is an isomorphism for $j = 0, 1$.

We shall denote by H_τ and H, the Hilbert spaces $\ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(F_{n-1})$ and respectively $\ell^2(F_n) \otimes H_0 \otimes \ell^2(F_{n-1})$. Viewing $\ell^2(\Gamma_n)$ as a subspace of $\ell^2(F_n)$ we shall also view H_τ as a subspace of H. The representation of $\mathcal{T}_n$ on H_τ defined by

$$B(\ell^2(\Gamma_n) \otimes H_0) \supset \mathcal{T}_n \ni x \mapsto x \otimes 1 \in B(\ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(F_{n-1}))$$

will be denoted by τ, whereas the representation of $A \times_{\alpha r} F_n$ on H defined by

$$B(\ell^2(F_n) \otimes H_0) \supset A \times_{\alpha r} F_n \ni x \mapsto x \otimes 1 \in B(\ell^2(F_n) \otimes H_0 \otimes \ell^2(F_{n-1}))$$

will be denoted by σ.

Throughout this section we shall moreover assume that the covariant representation of (A, α, F_n) on H_0 defined by $A \subset B(H_0)$ and $g \mapsto v_g$ determines a faithful representation of the reduced crossed product $A \times_{\alpha r} F_n$ on H_0.

For any subset $E \subset \Gamma_n$, $Q_E \in B(\ell^2(\Gamma_n))$ will denote the orthogonal projection of $\ell^2(\Gamma_n)$ onto the subspace $\ell^2(E)$. Denoting by λ the left regular representation it is easily seen that for $g \in \Gamma_n$ the map

$$\nu_g(a) = Q_{\{g\}} \otimes (\alpha(g)a) \otimes 1$$

$$\nu_g(u_h) = Q_{\{g\}} \otimes v_g v_h v_g^* \otimes \lambda_h, \quad h \in F_{n-1}$$

extends to a non-unital faithful representation of $A \times_{\alpha' r} F_{n-1}$ on H_τ which we shall also denote by ν_g.

Denoting by $e(g', g)$ $(g', g \in \Gamma_n)$ the natural matrix units of $\mathcal{K}(\ell^2(\Gamma_n))$ the map

$$\nu(e(g', g) \otimes x) = (e(g', g) \otimes v_{g'g^{-1}} \otimes 1)\nu_g(x) =$$

$$= e(g', g) \otimes ((v_{g'} \otimes 1)x(v_g \otimes 1)^*),$$

where $g, g' \in \Gamma_n$, $x \in A \times_{\alpha' r} F_{n-1}$ extends to a faithful representation of $\mathcal{K} \otimes (A \times_{\alpha' r} F_{n-1})$ on H_τ. This is easily seen using the unitary $\Lambda \otimes 1 = \sum_{g \in \Gamma_n} Q_{\{g\}} \otimes v_g^* \otimes 1$ which by conjugation gives

$$(\Lambda \otimes 1)\nu(e(g', g) \otimes a)(\Lambda \otimes 1)^* = e(g', g) \otimes a \otimes 1$$

$$(\Lambda \otimes 1)\nu(e(g', g) \otimes u_h)(\Lambda \otimes 1)^* = e(g', g) \otimes v_h \otimes \lambda_h .$$

In fact the representation ν is such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\ \tau\ } & B(H_\tau) \\ \wr\| & & \uparrow \nu \\ \mathcal{K} \otimes A & \xrightarrow{\ \mathrm{id} \otimes i\ } & \mathcal{K} \otimes (A \times_{\alpha' r} F_{n-1}) \end{array}$$

where the isomorphism $\mathcal{J} \simeq \mathcal{K} \otimes A$ is the isomorphism considered in § 1.

This implies that

$$\tau(\mathscr{T}_n)\nu(\mathscr{K}\otimes i(A))\subset\nu(\mathscr{K}\otimes(A\times_{\alpha'r}F_{n-1})).$$

Since $\mathscr{K}\otimes 1\subset\mathscr{K}\otimes i(A)$ contains an approximate identity for $\mathscr{K}\otimes(A\times_{\alpha' r}F_{n-1})$ this implies that

$$\tau(\mathscr{T}_n)\nu(\mathscr{K}\otimes(A\times_{\alpha'r}F_{n-1}))\subset\nu(\mathscr{K}\otimes(A\times_{\alpha'r}F_{n-1})).$$

Let now $\check{\Gamma}_j$ denote the set $F_n\setminus\Gamma_j$ and remark the following decomposition of Γ_n into disjoint subsets

$$\Gamma_n=\{e\}\cup\Gamma_n g_n\cup\bigcup_{1\leqslant j\leqslant n-1}(\Gamma_j g_j\cup\check{\Gamma}_j).$$

Correspondingly we define an isometric operator

$$V\colon H_\tau\oplus\underbrace{H\oplus\ldots\oplus H}_{(n-1)\text{-times}}\to H_\tau$$

by the formula

$$V(\xi\oplus\eta_1\oplus\ldots\oplus\eta_{n-1})=$$

$$=(r_{g_n^{-1}}\otimes 1\otimes 1)\xi+\sum_{1\leqslant j\leqslant n-1}((\tilde{Q}_{\check{\Gamma}_j}\otimes 1\otimes 1)\eta_j+(r_{g_j^{-1}}\tilde{Q}_{\Gamma_j}\otimes 1\otimes\lambda_{g_j^{-1}})\eta_j)$$

where $g\mapsto r_g$ denotes the right regular representation and $\tilde{Q}_E$ denotes the orthogonal projection of $\ell^2(F_n)$ onto $\ell^2(E)$. Remark that the range of V is $\ell^2(\Gamma_n\setminus\{e\})\otimes\otimes H_0\otimes\ell^2(F_{n-1})$.

Let τ' denote the representation (non-unital) of $\mathscr{T}_n$ on H_τ defined by

$$\tau'(x)=V(\tau(x)\oplus\underbrace{(\sigma\circ p)(x)\oplus\ldots\oplus(\sigma\circ p)(x)}_{(n-1)\text{-times}})V^*.$$

It is easily seen that

$$\tau'(\rho(a))=Q_{\Gamma_n\setminus\{e\}}\otimes a\otimes 1.$$

We also have

$$\tau'(U_k)=\tilde{\lambda}_{g_k}Q_{\Gamma_n\setminus\{g_k^{-1},e\}}\otimes v_k\otimes 1+\tilde{\lambda}_{g_k^2}Q_{\{g_k^{-1}\}}\otimes v_k\otimes\lambda_{g_k^{-1}}\quad(1\leqslant k\leqslant n-1)$$

$$\tau'(S_n)=\tilde{\lambda}_{g_n}Q_{\Gamma_n\setminus\{e\}}\otimes v_n\otimes 1$$

where $\tilde{\lambda}_g$ for $g\in F_n$ is the compression $P_{\ell^2(\Gamma_n)}\lambda_g|\ell^2(\Gamma_n)$ of λ_g to $\ell^2(\Gamma_n)$.

To check these formulae remark first that in order to prove that $\tau'(x)=y$ it is sufficient to show that

$$y(Q_{\Gamma_n\setminus\{e\}}\otimes 1\otimes 1)=(Q_{\Gamma_n\setminus\{e\}}\otimes 1\otimes 1)y=y$$

and

$$yV = V(x \oplus \underbrace{(\sigma \circ p)(x) \oplus \ldots \oplus (\sigma \circ p)(x)}_{(n-1)\text{-times}}).$$

The first of these relations is obviously satisfied so we shall concentrate on the second. We have

$$V(\tau(U_k)\xi \oplus \sigma(u_k)\eta_1 \oplus \ldots \oplus \sigma(u_k)\eta_{n-1}) =$$

$$= \tau(U_k)Q_{\Gamma_n \setminus (\Gamma_k g_k \cup \check{\Gamma}_k \cup \{e\})} V(\xi \oplus \eta_1 \oplus \ldots \oplus \eta_{n-1}) +$$

$$+ (\tilde{Q}_{\check{\Gamma}_k} \otimes 1 \otimes 1)\sigma(u_k)\eta_k + (r_{g_k^{-1}} \tilde{Q}_{\Gamma_k} \otimes 1 \otimes \lambda_{g_k^{-1}})\sigma(u_k)\eta_k$$

and

$$(\tilde{Q}_{\check{\Gamma}_k} \otimes 1 \otimes 1)\sigma(u_k)\eta_k + (r_{g_k^{-1}} \tilde{Q}_{\Gamma_k} \otimes 1 \otimes \lambda_{g_k^{-1}})\sigma(u_k)\eta_k = \tau(U_k)(\tilde{Q}_{\check{\Gamma}_k \setminus \{g_k^{-1}\}} \otimes 1 \otimes 1)\eta_k +$$

$$+ \tau(U_k)(r_{g_k^{-1}} \tilde{Q}_{\Gamma_k} \otimes 1 \otimes \lambda_{g_k^{-1}})\eta_k + \tau(U_k)(r_{g_k^{-1}} \tilde{Q}_{\{g_k^{-1}\}} \otimes 1 \otimes \lambda_{g_k^{-1}})\eta_k =$$

$$= \tau(U_k)Q_{(\check{\Gamma}_k \setminus \{g_k^{-1}\}) \cup \Gamma_k g_k} V(\xi \oplus \eta_1 \oplus \ldots \oplus \eta_{n-1}) +$$

$$+ (\tilde{\lambda}_{g_k^2} \otimes v_k \otimes \lambda_{g_k^{-1}})(\tilde{Q}_{\{g_k^{-1}\}} \otimes 1 \otimes 1)\eta_k =$$

$$= \tau(U_k)Q_{(\check{\Gamma}_k \setminus \{g_k^{-1}\}) \cup \Gamma_k g_k} V(\xi \oplus \eta_1 \oplus \ldots \oplus \eta_{n-1}) +$$

$$+ (\tilde{\lambda}_{g_k^2} \otimes v_k \otimes \lambda_{g_k^{-1}})(Q_{\{g_k^{-1}\}} \otimes 1 \otimes 1)V(\xi \oplus \eta_1 \oplus \ldots \oplus \eta_{n-1})$$

which in view of $\tau(U_k) = \tilde{\lambda}_{g_k} \otimes v_k \otimes 1$ yield the formula for $\tau'(U_k)$.

The formula for $\tau'(S_n)$ follows immediately from

$$V(\tau(S_n) \oplus \sigma(u_n) \oplus \ldots \oplus \sigma(u_n)) = \tau(S_n)V$$

and the fact that the range of V is $(Q_{\Gamma_n \setminus \{e\}} \otimes 1 \otimes 1)H_\tau$.

The formulae for $\tau'(\rho(a))$, $\tau'(U_k)$, $\tau'(S_n)$ give immediately

$$\tau(\rho(a)) - \tau'(\rho(a)) = \nu(e(e, e) \otimes i(a))$$

$$\tau(U_k) - \tau'(U_k) =$$

$$= \nu(e(e, g_k^{-1}) \otimes 1) + \nu(e(g_k, e) \otimes 1) - \nu(e(g_k, g_k^{-1}) \otimes u_k^{-1})$$

$$\tau(S_n) - \tau'(S_n) = \nu(e(g_n, e) \otimes 1).$$

This implies

$$\tau'(\mathscr{T}_n)\nu(\mathscr{K} \otimes (A \times_{\alpha'r} F_{n-1})) \subset \nu(\mathscr{K} \otimes (A \times_{\alpha'r} F_{n-1}))$$

and

$$\tau(x) - \tau'(x) \in v(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})) \quad \text{for } x \in \mathscr{T}_n.$$

The preceding facts show that both τ and τ' are $*$-homomorphism of the Toeplitz algebra into the multiplier algebra of $\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})$ and that these $*$-homomorphisms are equal modulo the ideal $\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})$. This will determine a map

$$K_*(\mathscr{T}_n) \to K_*(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})) \simeq K_*(A \times_{\alpha' r} F_{n-1})).$$

The rest of this section will be actually devoted to the proof that this map is the inverse of

$$d_* : K_*(A \times_{\alpha' r} F_{n-1}) \to K_*(\mathscr{T}_n).$$

To do this we need some preliminaries.

First, consider M, N two C^*-algebras $\mathscr{I} \subset N$ a closed two-sided ideal and $\varphi_1, \varphi_2 : M \to N$ $*$-homomorphisms such that $\varphi_1(m) - \varphi_2(m) \in \mathscr{I}$ for $m \in M$.

Then for u a unitary in some $\mathscr{M}_k(\tilde{M})$ ($\mathscr{M}_k$ denoting $k \times k$ matrices and ~ denoting the fact that a unit has been adjoined) consider $[\tilde{\varphi}_1(u)\tilde{\varphi}_2(u^*)]_1$ in $K_1(\mathscr{I})$. It is easily seen that $[\tilde{\varphi}_1(u)\tilde{\varphi}_2(u^*)]_1$ depends only on $[u]_1$ in $K_1(M)$ and that $[u]_1 \to [\tilde{\varphi}_1(u)\tilde{\varphi}_2(u^*)]_1$ is a homomorphism $K_1(M) \to K_1(\mathscr{I})$.

For K_0 a corresponding homomorphism can be defined via suspensions.

This is the construction we shall use for the map

$$K_*(\mathscr{T}_n) \to K_*(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})).$$

Further we need the construction of a certain curve of homomorphisms $\tau_t : A \times_{\alpha' r} F_{n-1} \to B(H_\tau)$ ($t \in [0, \pi/2]$).

For each $t \in [0, \pi/2]$ and $1 \leqslant k \leqslant n-1$ define

$$\begin{aligned} W_k(t) = {} & Q_{\Gamma_n \setminus \{e, g_k\}} \otimes 1 \otimes 1 + \\ & + \cos t\, Q_{\{e\}} \otimes 1 \otimes 1 + \sin t\, \tilde{\lambda}_{g_k} Q_{\{e\}} \otimes 1 \otimes \lambda_{g_k}^{-1} - \\ & - e^{2it} \sin t\, \tilde{\lambda}_{g_k^{-1}} Q_{\{g_k\}} \otimes 1 \otimes \lambda_{g_k} + e^{2it} \cos t\, Q_{\{g_k\}} \otimes 1 \otimes 1. \end{aligned}$$

It is easily seen that $W_k(t)$ can be also written

$$\begin{aligned} W_k(t) = {} & Q_{\Gamma_n \setminus \{e, g_k\}} \otimes 1 \otimes 1 + \\ & + \cos t\, v(e(e, e) \otimes 1) + \sin t\, v(e(g_k, e) \otimes u_k^{-1}) - \\ & - e^{2it} \sin t\, v(e(e, g_k) \otimes u_k) + e^{2it} \cos t\, v(e(g_k, g_k) \otimes 1) \end{aligned}$$

which using the unitarity of the scalar matrix

$$\begin{pmatrix} \cos t & -e^{2it}\sin t \\ \sin t & e^{2it}\cos t \end{pmatrix}$$

shows that $W_k(t)$ is unitary.

Moreover we clearly have

$$W_k(t) - I \in \nu(\mathcal{K} \otimes (A \times_{\alpha' r} F_{n-1}))$$

where I is the identity operator on H_τ.

Define now the unitary operators

$$u_k(t) = W_k(t)\tau(U_k)$$

and note that if we write $H_\tau = X \otimes H_0$ where $X = \ell^2(\Gamma_n) \otimes \ell^2(F_{n-1})$ then $u_k(t)$ may be written as $w_k(t) \otimes v_k$ where $w_k(t)$ is some unitary in $B(X)$. We thus get a unitary representation

$$g \mapsto u_g(t) = w_g(t) \otimes v_g$$

of F_{n-1} such that $u_{g_k}(t) = u_k(t)$.

Recall now our assumption concerning the covariant representation of (A, α, F_n) on H_0 defined by $A \subset B(H_0)$ and $g \mapsto v_g$. Since this covariant representation gives a faithful representation of the reduced crossed product $A \times_{\alpha r} F_n$ it follows that $A \subset B(H_0)$ and $F_{n-1} \ni g \mapsto v_g$ will yield a faithful representation of the reduced crossed product $A \times_{\alpha' r} F_{n-1}$. From this it follows that the map

$$\tau_t(i(a)) = 1 \otimes a$$

$$\tau_t(u_g) = w_g(t) \otimes v_g$$

extends to a faithful representation τ_t of the reduced crossed product $A \times_{\alpha' r} F_{n-1}$ on H_τ. (See for instance [16], 7.7.5).

Remark further that for every $x \in A \times_{\alpha' r} F_{n-1}$ the map

$$[0, \pi/2] \ni t \to \tau(x) \in B(H_\tau)$$

is norm-continuous. Moreover we have

$$\tau_0(x) = \tau(d(x))$$

$$\tau_{\pi/2}(x) = \tau'(d(x)) + Q_{\{e\}} \otimes x$$

for $x \in A \times_{\alpha' r} F_{n-1}$ as can be easily seen by looking at $x \in i(A) \cup \{u_1, \ldots, u_{n-1}\}$.

Also since each $W_k(t) \in I + \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$ and

$$\tau(U_k)\nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})) \subset \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})),$$

we infer that

$$\tau_t(u_k) - \tau(U_k) \in \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})),$$

and hence since also $\tau_t(a) = \tau((d \circ i)(a))$ we have that

$$\tau_t(A \times_{\alpha' r} F_{n-1})\nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})) \subset \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$$

and

$$\tau_t(x) - \tau(d(x)) \in \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$$

for all $x \in A \times_{\alpha' r} F_{n-1}$.

2.1. LEMMA. *The homomorphism*

$$d_* : \mathrm{K}_j(A \times_{\alpha' r} F_{n-1}) \to \mathrm{K}_j(\mathscr{T}_n)$$

is injective for $j = 0, 1$.

Proof. The homomorphisms τ and τ' of $\mathscr{T}_n$ into

$$\tau(\mathscr{T}_n) + \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$$

determine a homomorphism

$$\Delta : \mathrm{K}_1(\mathscr{T}_n) \to \mathrm{K}_1(\nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})).$$

We shall in fact prove that $\Delta \circ d_*$ is the identity on

$$\mathrm{K}_1(A \times_{\alpha' r} F_{n-1}) \simeq \mathrm{K}_1(\nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})).$$

Since passing to matrix-algebras over A has the same effect on the other algebras considered, it will be sufficient to prove this for classes of unitaries in $A \times_{\alpha' r} F_{n-1}$.

Thus for a unitary $u \in A \times_{\alpha' r} F_{n-1}$ we must prove that

$$(\Delta \circ d_*)[u] = [\nu(e(e,e) \otimes u) + Q_{\Gamma_n \setminus \{e\}} \otimes 1 \otimes 1].$$

Consider the unitaries

$$w_t = \tau_t(u)(\tau'(d(u)) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}.$$

We have that $w_t \in I + \nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$ and $t \to w_t$ being norm-continuous for $t \in [0, \pi/2]$ we infer that $[w_0] = [w_{\pi/2}]$ in $\mathrm{K}_1(\nu(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1})))$.

Now

$$[w_0] = [\tau_0(u)(\tau'(d(u)) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}] =$$

$$= [\tau(d(u))(\tau'(d(u)) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}] =$$

$$= (\Delta \circ d_*)[u]$$

and

$$[w_{\pi/2}] = [\tau_{\pi/2}(u)(\tau'(d(u)) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}] =$$
$$= [\nu(e(e, e) \otimes u) + Q_{\Gamma_n \setminus \{e\}} \otimes 1 \otimes 1],$$

which concludes the proof of the injectivity of d_* for $j = 1$.

To get the same result for $j = 0$ apply the preceding result to $(A \otimes C(\mathbf{T}), \alpha \otimes \mathrm{id}_{C(\mathbf{T})}, F_n)$ which gives that

$$(d \otimes \mathrm{id}_{C(\mathbf{T})})_* : \mathrm{K}_1((A \times_{\alpha' \mathrm{r}} F_{n-1}) \otimes C(\mathbf{T})) \to \mathrm{K}_1(\mathcal{T}_n \otimes C(\mathbf{T}))$$

is injective. Using the natural isomorphism

$$\mathrm{K}_1(M \otimes C(\mathbf{T})) \simeq \mathrm{K}_0(M) \oplus \mathrm{K}_1(M)$$

for M unital we get the desired result also for $j = 0$. Q.E.D.

In order to prove the surjectivity of d_* we shall use again the map Δ and some homotopies. We begin with the construction of the homotopies.

Corresponding to the unital embedding $d: A \times_{\alpha' \mathrm{r}} F_{n-1} \to \mathcal{T}_n$ there is an embedding $\tilde{d}$ of the multiplier algebra of $\mathcal{K} \otimes (A \times_{\alpha' \mathrm{r}} F_{n-1})$ into the multiplier algebra of $\mathcal{K} \otimes \mathcal{T}_n$. Viewing τ and τ' as homomorphisms of $\mathcal{T}_n$ into the multipliers of $\nu(\mathcal{K} \otimes (A \times_{\alpha' \mathrm{r}} F_{n-1})) \simeq \mathcal{K} \otimes (A \times_{\alpha' \mathrm{r}} F_{n-1})$, we get by composition with $\tilde{d}$ two homomorphisms of $\mathcal{T}_n$ into the multiplier algebra of $\mathcal{K} \otimes \mathcal{T}_n$. For our purposes it will be convenient to have these homomorphisms expressed in terms of representations.

Consider $\mathcal{T}_n$, as usual, as a subalgebra of $B(H_0 \otimes \ell^2(\Gamma_n))$ and define $\tilde{\nu}_g : \mathcal{T}_n \to B(\ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(\Gamma_n))$ defined by

$$\tilde{\nu}_g(x) = Q_{\{g\}} \otimes ((v_g \otimes 1)x(v_g^* \otimes 1)),$$

where $g \in \Gamma_n$ and $x \in \mathcal{T}_n$. We define the representation

$$\tilde{\nu} : \mathcal{K} \otimes \mathcal{T}_n \to B(\ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(\Gamma_n))$$

by

$$\tilde{\nu}(e(g', g) \otimes x) = (\tilde{\lambda}_{g'g^{-1}} \otimes v_{g'g^{-1}} \otimes 1)\tilde{\nu}_g(x) =$$
$$= e(g', g) \otimes ((v_{g'} \otimes 1)x(v_g \otimes 1)^*).$$

Using $\Delta \otimes 1$ as for ν it is easily seen that $\tilde{\nu}$ is a faithful representation of $\mathcal{K}(\ell^2(\Gamma_n)) \otimes \mathcal{T}_n$ on $H_{\tilde{\tau}} = \ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(\Gamma_n)$.

In fact let $\omega_n \subset \Gamma_n$ be such that $F_{n-1}\omega_n = \Gamma_n$ and $g\omega_n \cap \omega_n = \emptyset$ for $g \in F_{n-1} \setminus \{e\}$; then identifying $\ell^2(\Gamma_n)$ with $\ell^2(F_{n-1}) \otimes \ell^2(\omega_n)$ and $H_{\tilde{\tau}}$ with $H_\tau \otimes \ell^2(\omega_n)$ we have $(\tilde{\nu} \circ (\mathrm{id}_{\mathcal{K}} \otimes d))(x) = \nu(x) \otimes 1$ for $x \in \mathcal{K} \otimes (A \times_{\alpha' \mathrm{r}} F_{n-1})$. Corresponding to $H_{\tilde{\tau}} = H_\tau \otimes \ell^2(\omega_n)$ we define

$$\tilde{\tau}(x) = \tau(x) \otimes 1$$
$$\tilde{\tau}'(x) = \tau'(x) \otimes 1$$

for $x \in \mathcal{T}_n$.

Since $(\mathrm{id}_{\mathscr{K}} \otimes d)(\mathscr{K} \otimes (A \times_{\alpha' r} F_{n-1}))$ contains an approximate identity for $\mathscr{K} \otimes \mathscr{T}_n$ it follows that

$$\tilde{\tau}(\mathscr{T}_n)\tilde{v}(\mathscr{K} \otimes \mathscr{T}_n) \subset \tilde{v}(\mathscr{K} \otimes \mathscr{T}_n)$$

$$\tilde{\tau}'(\mathscr{T}_n)\tilde{v}(\mathscr{K} \otimes \mathscr{T}_n) \subset \tilde{v}(\mathscr{K} \otimes \mathscr{T}_n)$$

and the homomorphisms of $\mathscr{T}_n$ into the multipliers of $\tilde{v}(\mathscr{K} \otimes \mathscr{T}_n)$ defined by $\tilde{\tau}$ and $\tilde{\tau}'$ are just the ones we wanted to construct.

The homomorphisms $\tilde{\tau}$ and $\tilde{\tau}'$ can be also described by the formulae

$$\tilde{\tau}(x) = x \otimes 1 \in B((\ell^2(\Gamma_n) \otimes H_0) \otimes \ell^2(\Gamma_n))$$

for $x \in \mathscr{T}_n \subset B(\ell^2(\Gamma_n) \otimes H_0)$ and

$$\tilde{\tau}'(\rho(a)) = Q_{\Gamma_n \setminus \{e\}} \otimes a \otimes 1$$

$$\tilde{\tau}'(U_k) = \tilde{\lambda}_{g_k} Q_{\Gamma_n \setminus \{e, g_k^{-1}\}} \otimes v_k \otimes 1 + \tilde{\lambda}_{g_k^2} Q_{\{g_k^{-1}\}} \otimes v_k \otimes \tilde{\lambda}_{g_k^{-1}} \quad (1 \leqslant k \leqslant n-1)$$

$$\tilde{\tau}'(S_n) = \tilde{\lambda}_{g_n} Q_{\Gamma_n \setminus \{e\}} \otimes v_n \otimes 1$$

where $H_{\tilde{\tau}}$ is written as $\ell^2(\Gamma_n) \otimes H_0 \otimes \ell^2(\Gamma_n)$.

Consider now the unitaries $\tilde{u}_k \in B(H_{\tilde{\tau}})$, $1 \leqslant k \leqslant n-1$ defined by

$$\tilde{u}_k = \tilde{\lambda}_{g_k} \otimes v_k \otimes 1 = \tilde{\tau}(U_k)$$

and for each $t \in [0, \pi/2]$ the isometry

$$s_n(t) = \tilde{\lambda}_{g_n} Q_{\Gamma_n \setminus \{e\}} \otimes v_n \otimes 1 + \cos t\, \tilde{\lambda}_{g_n} Q_{\{e\}} \otimes v_n \otimes 1 + \sin t\, Q_{\{e\}} \otimes v_n \otimes \tilde{\lambda}_{g_n} =$$

$$= \tilde{\tau}(S_n) - \tilde{v}(e(g_n, e) \otimes 1) + \cos t\, \tilde{v}(e(g_n, e) \otimes 1) +$$

$$+ \sin t\, \tilde{v}(e(e, e) \otimes S_n) \in \tilde{\tau}(S_n) + \tilde{v}(\mathscr{K} \otimes \mathscr{T}_n).$$

Our next goal is to show that there exists a *-representation:

$$\tilde{\tau}_t : \mathscr{T}_n \to B(H_{\tilde{\tau}})$$

such that

$$\tilde{\tau}_t(\rho(a)) = 1 \otimes a \otimes 1$$

$$\tilde{\tau}_t(U_k) = \tilde{u}_k$$

$$\tilde{\tau}_t(S_n) = s_n(t).$$

This will be done by using Lemma 1.2.

Consider $P(t) = 1 - s_n(t)s_n^*(t)$ and σ_t the map corresponding to σ of Lemma 1.2. Note that for each $t \in [0, \pi/2)$ $P(t)$ is the orthogonal projection onto the subspace of $H_{\tilde{\tau}}$ spanned by the vectors

$$\cos t Q_{\{e\}}\xi \otimes \eta \otimes h - \sin t \tilde{\lambda}_{g_n} Q_{\{e\}}\xi \otimes \eta \otimes \tilde{\lambda}_{g_n^{-1}} h.$$

It is not difficult to see that

$$P(t)\sigma_t(g)P(t) = 0 \quad \text{for } g \in \Gamma_n \setminus \{e\}.$$

Indeed, if $g = g_n^k$, $k > 0$ this is clear because $P(t) = 1 - s_n(t)s_n^*(t)$. Moreover

$$\sigma_t(g_n^k)P(t)H_{\tilde{\tau}} \subset \sum_{j=0}^{k+1} (Q_{\{g_n^j\}} \otimes 1 \otimes 1)H_{\tilde{\tau}},$$

so that writing $g \in \Gamma_n \setminus (\bigcup_{k \geq 0} \{g_n^k\})$ in the form $g = g' g_n^k$ with $g' \in \bigcup_{1 \leq j \leq n-1} (\Gamma_j g_j \cup \check{\Gamma}_j)$ it is easily seen that

$$\sigma_t(g)P(t)H_{\tilde{\tau}} \subset \sum_{j=0}^{k+1} (Q_{\{g' g_n^j\}} \otimes 1 \otimes 1)H_{\tilde{\tau}},$$

which is clearly orthogonal to $P(t)H_{\tilde{\tau}}$.

To check that $\bigvee_{g \in \Gamma_n} \sigma_t(g)P(t)H_{\tilde{\tau}} = H_{\tilde{\tau}}$ remark first that

$$\lim_{m \to \infty} s_n^{*m}(t)(Q_{\{g_n^k\}}\xi \otimes \eta \otimes \mu) = 0$$

for each $k \geq 0$ and $t \in [0, \pi/2)$.

This implies

$$(Q_{\{g_n^k\}} \otimes 1 \otimes 1)H_{\tilde{\tau}} \subset \bigvee_{k \geq 0} s_n^k(t)\, P(t)H_{\tilde{\tau}} \subset \bigvee_{g \in \Gamma_n} \sigma_t(g)P(t)H_{\tau}.$$

For $g \in \Gamma_n \setminus \{g_n^k \mid k \geq 0\}$ write $g = g' g_n^k$ where $k \geq 0$ and $g' \in \bigcup_{1 \leq j \leq n-1} (\Gamma_j g_j \cup \check{\Gamma}_j)$ and note that

$$\sigma(g')(Q_{\{g_n^k\}} \otimes 1 \otimes 1)H_{\tilde{\tau}} = (Q_{\{g\}} \otimes 1 \otimes 1)H_{\tilde{\tau}}.$$

This gives then

$$H_{\tilde{\tau}} = \bigvee_{g \in \Gamma_n} \sigma_t(g)P(t)H_{\tilde{\tau}}.$$

The fact that the unitaries $\tilde{u}_k (1 \leq k \leq n-1)$ and the isometry $s_n(t)$ implement the action α of F_n on $1 \otimes A \otimes 1$ is quite easy to see from the presence of the corresponding v_j in the second factors of the tensor products appearing in the formulae defining these operators.

Thus from Lemma 1.2 we conclude that $\tilde{\tau}_t : \mathcal{T}_n \to B(H_{\tilde{\tau}})$ is a well-defined homomorphism for $t \in [0, \pi/2)$ and by continuity this is also true for $t = \pi/2$.

Now for $t \in [\pi/2, \pi]$ we shall construct another curve of homomorphisms $\tilde{\tau}_t : \mathcal{T}_n \to B(H_{\tilde{\tau}})$, keeping this time the isometry constant and varying the unitaries. Consider for $t \in [\pi/2, \pi]$ the unitaries

$$\tilde{u}_k(t) = \tilde{\lambda}_{g_k} Q_{\Gamma_n \setminus \{g_k^{-1}, e\}} \otimes v_k \otimes 1 + \sin t\, \tilde{\lambda}_{g_k} Q_{\{g_k^{-1}\}} \otimes v_k \otimes 1 -$$

$$- \cos t\, \tilde{\lambda}_{g_k^2} Q_{\{g_k^{-1}\}} \otimes v_k \otimes \tilde{\lambda}_{g_k^{-1}} - e^{2it} \cos t\, Q_{\{e\}} \otimes v_k \otimes \tilde{\lambda}_{g_k} -$$

$$- e^{2it} \sin t\, \tilde{\lambda}_{g_k} Q_{\{e\}} \otimes v_k \otimes 1 \quad (1 \leqslant k \leqslant n-1)$$

and the isometry

$$s_n = \tilde{\lambda}_{g_n} Q_{\Gamma_n \setminus \{e\}} \otimes v_n \otimes 1 + Q_{\{e\}} \otimes v_n \otimes \tilde{\lambda}_{g_n}.$$

Remark that for $t = \pi/2$ we have $\tilde{u}_k(\pi/2) = \tilde{u}_k$ and $s_n = s_n(\pi/2)$. Further, remark also that

$$\tilde{\tau}(U_k^*)\tilde{u}_k(t) = (\tilde{\lambda}_{g_k^{-1}} \otimes v_k^* \otimes 1)\, \tilde{u}_k(t) =$$

$$= Q_{\Gamma_n \setminus \{g_k^{-1}, e\}} \otimes 1 \otimes 1 + \sin t\, Q_{\{g_k^{-1}\}} \otimes 1 \otimes 1 -$$

$$- \cos t\, \tilde{\lambda}_{g_k} Q_{\{g_k^{-1}\}} \otimes 1 \otimes \tilde{\lambda}_{g_k^{-1}} - e^{2it} \cos t\, \tilde{\lambda}_{g_k^{-1}} Q_{\{e\}} \otimes 1 \otimes \tilde{\lambda}_{g_k} - e^{2it} \sin t\, Q_{\{e\}} \otimes 1 \otimes 1 =$$

$$= Q_{\Gamma_n \setminus \{g_k^{-1}, e\}} \otimes 1 \otimes 1 + \sin t\, \tilde{\nu}(e(g_k^{-1}, g_k^{-1}) \otimes 1) -$$

$$- \cos t\, \tilde{\nu}(e(e, g_k^{-1}) \otimes U_k^*) - e^{2it} \cos t\, \tilde{\nu}(e(g_k^{-1}, e) \otimes U_k) -$$

$$- e^{2it} \sin t\, \tilde{\nu}(e(e, e) \otimes 1) \in I + \tilde{\nu}(\mathcal{K} \otimes \mathcal{T}_n),$$

$$\tilde{\tau}(S_n) - s_n = \tilde{\lambda}_{g_n} Q_{\{e\}} \otimes v_n \otimes 1 - Q_{\{e\}} \otimes v_n \otimes \tilde{\lambda}_{g_n} =$$

$$= \tilde{\nu}(e(g_n, e) \otimes 1) - \tilde{\nu}(e(e, e) \otimes S_n) \in \tilde{\nu}(\mathcal{K} \otimes \mathcal{T}_n).$$

We shall prove the existence of $\tilde{\tau}_t : \mathcal{T}_n \to B(H_{\tilde{\tau}})$ such that

$$\tilde{\tau}_t(\rho(a)) = 1 \otimes a \otimes 1$$

$$\tilde{\tau}_t(U_k) = \tilde{u}_k(t)$$

$$\tilde{\tau}(S_n) = s_n$$

where $t \in [\pi/2, \pi]$, $a \in A$ and $1 \leqslant k \leqslant n-1$. This will be done using Lemma 1.2.

Thus, consider $P = I - s_n s_n^*$. It is easily seen that

$$P = Q_{\{g_n\}} \otimes 1 \otimes 1 + Q_{\{e\}} \otimes 1 \otimes Q_{\{e\}}.$$

Since the space $(Q_{\Gamma_n g_n} \otimes 1 \otimes 1)H_{\tilde{\tau}}$ is reducing for $\tilde{u}_k(t)$ $(1 \leqslant k \leqslant n-1)$ and s_n, and since moreover

$$\sigma_t(g)(Q_{\{g_n\}} \otimes 1 \otimes 1)H_{\tilde{\tau}} = (Q_{\{gg_n\}} \otimes 1 \otimes 1)H_{\tilde{\tau}}$$

for $g \in \Gamma_n$, we see that in order to prove that $P\sigma_t(g)P = 0$ for $g \in \Gamma_n \setminus \{e\}$ it will be sufficient to prove that

$$(Q_{\{e\}} \otimes 1 \otimes Q_{\{e\}})\sigma_t(g)(Q_{\{e\}} \otimes 1 \otimes Q_{\{e\}}) = 0$$

for $g \in \Gamma_n \setminus \{e\}$. This in turn will follow once we establish the following fact: if $g \in \Gamma_n \setminus \{e\}$ is written in the form $g = g_{i_m}^{\alpha_m} g_{i_{m-1}}^{\alpha_{m-1}} \ldots g_{i_1}^{\alpha_1}$, $(i_1 \neq i_2, \ldots, i_{m-1} \neq i_m, \alpha_j \neq 0)$ then

$$\sigma_t(g)(Q_{\{e\}} \otimes 1 \otimes Q_{\{e\}})H_{\tilde{\tau}} \subset (Q_{\{e\}} \otimes 1 \otimes Q_{\{g\}})H_{\tilde{\tau}} + (Q_{\Gamma_{i_m}^{\operatorname{sign} \alpha_m}} \otimes 1 \otimes 1)H_{\tau}$$

where

$$\Gamma_k^+ = (\check{\Gamma}_k)^{-1} \cap \Gamma_n$$

$$\Gamma_k^- = (\Gamma_k g_k)^{-1} \cap \Gamma_n.$$

We shall establish this by induction on the length of g. Before doing this, let us write down the formulae for $\tilde{u}_k^*(t)$, s_n^*:

$$\tilde{u}_k^*(t) = \tilde{\lambda}_{g_k^{-1}} Q_{\Gamma_n \setminus \{e, g_k\}} \otimes v_k^* \otimes 1 +$$

$$+ \sin t\, \tilde{\lambda}_{g_k^{-1}} Q_{\{e\}} \otimes v_k^* \otimes 1 - \cos t\, \tilde{\lambda}_{g_k^{-2}} Q_{\{g_k\}} \otimes v_k^* \otimes \tilde{\lambda}_{g_k} -$$

$$- e^{-2it} \cos t\, Q_{\{e\}} \otimes v_k^* \otimes \tilde{\lambda}_{g_k^{-1}} - e^{-2it} \sin t\, \tilde{\lambda}_{g_k^{-1}} Q_{\{g_k\}} \otimes v_k^* \otimes 1;$$

$$s_n^* = \tilde{\lambda}_{g_n^{-1}} Q_{\Gamma_n \setminus \{g_n, e\}} \otimes v_n^* \otimes 1 + Q_{\{e\}} \otimes v_n^* \otimes \tilde{\lambda}_{g_n^{-1}}.$$

Looking at the formulae for $\tilde{u}_k(t)$, $\tilde{u}_k^*(t)$, s_n, s_k^* the assertion is immediately established for $|g| = 1$.

Assume our assertion holds for $|g| = p$ and let us prove it for $g' \in \Gamma_n$, where $|g'| = p + 1$. Writing $g' = g_j^{\varepsilon} g$ where $\varepsilon \in \{1, -1\}$, $g = g_{i_m}^{\alpha_m} \ldots g_{i_1}^{\alpha_1}$, $|g| = p$ we have that either $j \neq i_m$ or if $j = i_m$ then $\varepsilon\alpha_m > 0$. First remark that looking at the formulae for $\sigma_t(h)$ with $|h| = 1$ it is easily seen that

$$\sigma_t(g_j^\varepsilon)(Q_{\{e\}} \otimes 1 \otimes Q_{\{g\}})H_{\tilde{\tau}} \subset$$

$$\subset (Q_{\{e\}} \otimes 1 \otimes Q_{\{g_j^\varepsilon g\}})H_{\tilde{\tau}} + (Q_{\Gamma_j^{\mathrm{sign}\,\varepsilon}} \otimes 1 \otimes 1)H_{\tilde{\tau}}.$$

Thus, it will be sufficient to prove that

$$\sigma_t(g_j^\varepsilon)(Q_{\Gamma_{i_m}^{\mathrm{sign}\,\alpha_m}} \otimes 1 \otimes 1)H_{\tilde{\tau}} \subset$$

$$\subset (Q_{\Gamma_j^{\mathrm{sign}\,\varepsilon}} \otimes 1 \otimes 1)H_{\tilde{\tau}}.$$

Again, the case $i_m \neq j$ follows immediately from the formulae for $\sigma_t(g_j^\varepsilon)$ and the case $i_m = j$, $\mathrm{sign}\alpha_m = \mathrm{sign}\varepsilon$ follows also by inspection of the same formulae.

The fact that $\tilde{u}_k(t)$ $(1 \leqslant k \leqslant n-1)$ and s_n implement the action α on $1 \otimes A \otimes 1$ is easily seen from the appearence of the corresponding v_j's in the second factors of the tensor products in the formulae defining these operators.

Thus by Lemma 1.2 we conclude that $\tilde{\tau}_t$ is well-defined on the subspace $H_1(t) = \bigvee_{g\in\Gamma_n} \sigma_t(g)PH_{\tilde{\tau}}$, i.e. there is a representation $\tilde{\tau}_t^{(1)}\colon \mathscr{T}_n \to B(H_1(t))$ such that

$$\tilde{\tau}_t^{(1)}(\rho(a)) = (1 \otimes a \otimes 1)|H_1(t)$$

$$\tilde{\tau}_t^{(1)}(U_k) = \tilde{u}_k(t)|H_1(t)$$

$$\tilde{\tau}_t^{(1)}(S_n) = s_n|H_1(t).$$

Writing $H_{\tilde{\tau}} = L \otimes H_0$ where $L = \ell^2(\Gamma_n) \otimes \ell^2(\Gamma_n)$ there is $L_1(t) \subset L$ such that $H_1(t) = L_1(t) \otimes H_0$. Indeed this follows from the fact that

$$\sigma_t(g) = \sigma_t'(g) \otimes v_g$$

$$P(t) = P'(t) \otimes 1$$

where $\sigma_t'(g) \in B(L)$, $P'(t) \in B(L)$ and that hence each $\sigma_t(g)P(t)H_{\tilde{\tau}}$ is of the form $X \otimes H_0$ with $X \subset L$.

Now, $H_2(t) = H_{\tilde{\tau}} \ominus H_1(t) = L_2(t) \otimes H_0$ where $L_2(t) = L \ominus L_1(t)$. Moreover $\tilde{u}_k(t)|H_2(t)$, $s_n|H_2(t)$ are unitaries and

$$\tilde{u}_k(t)|H_2(t) = \tilde{u}_k''(t) \otimes v_k$$

$$s_n|H_2(t) = \tilde{u}_n''(t) \otimes v_n$$

$$(1 \otimes a \otimes 1)H_2(t) = I_{L_2(t)} \otimes a$$

where $\tilde{u}_k''(t)$ are unitaries for $1 \leqslant k \leqslant n$.

Our assumption that $g \mapsto v_g$ and $A \subset B(H_0)$ generate a faithful representation of the reduced crossed product $A \times_{\alpha r} F_n$ implies (keeping in mind [16], 7.7.5) that there is a representation π_t of $A \times_{\sigma r} F_n$ on $H_2(t)$ such that

$$\pi_t(\pi(a)) = 1 \otimes a$$

$$\pi_t(u_k) = \tilde{u}_k''(t) \otimes v_k \quad (1 \leqslant k \leqslant n).$$

Thus $\tilde{\tau}_t^{(2)}: \mathscr{T}_n \to B(H_2(t))$ given by $\tilde{\tau}_t^{(2)} = \pi_t \circ p$ is well defined and

$$\tilde{\tau}_t^{(2)}(\rho(a)) = (1 \otimes a \otimes 1)|H_2(t)$$

$$\tilde{\tau}_t^{(2)}(U_k) = \tilde{u}_k(t)|H_2(t)$$

$$\tilde{\tau}_t^{(2)}(S_n) = s_n|H_2(t).$$

This concludes the proof of the existence of $\tilde{\tau}_t = \tilde{\tau}_t^{(1)} \oplus \tilde{\tau}_t^{(2)}$ for $t \in [\pi/2, \pi]$.

Putting together the two paths of representations we get a path of representations $\tilde{\tau}_t (t \in [0, \pi])$ of $\mathscr{T}_n$ on $H_{\tilde{\tau}}$. Clearly $t \to \tilde{\tau}_t(x)$ is norm-continuous for each $x \in \mathscr{T}_n$. Moreover inspecting the computations done when defining $\tilde{\tau}_t(x)$ it is easily seen that $\tilde{\tau}_t(\mathscr{T}_n)\tilde{v}(\mathscr{K} \otimes \mathscr{T}_n) \subset \tilde{v}(\mathscr{K} \otimes \mathscr{T}_n)$ and $\tilde{\tau}_t(x) - \tilde{\tau}(x) \in \tilde{v}(\mathscr{K} \otimes \mathscr{T}_n)$ for every $x \in \mathscr{T}_n$.

Further note that

$$\tilde{\tau}_0(U_k) = \tilde{u}_k = \tilde{\tau}(U_k)$$

$$\tilde{\tau}_0(S_n) = s_n(0) = \tilde{\lambda}_{g_n} Q_{\Gamma_n \setminus \{e\}} \otimes v_n \otimes 1 + \tilde{\lambda}_{g_n} Q_{\{e\}} \otimes v_n \otimes 1 = \tilde{\tau}(S_n)$$

so that $\tilde{\tau}_0 = \tilde{\tau}$. Similarly

$$\tilde{\tau}_\pi(U_k) = \tilde{\lambda}_{g_k} Q_{\Gamma_n \setminus \{g_k^{-1}, e\}} \otimes v_k \otimes 1 +$$

$$+ \tilde{\lambda}_{g_k^2} Q_{\{g_k^{-1}\}} \otimes v_k \otimes \tilde{\lambda}_{g_k^{-1}} + Q_{\{e\}} \otimes v_k \otimes \tilde{\lambda}_{g_k} = \tilde{\tau}'(U_k) + \tilde{v}_e(U_k);$$

$$\tilde{\tau}_\pi(S_n) = s_n = \tilde{\lambda}_{g_n} Q_{\Gamma_n \setminus \{e\}} \otimes v_n \otimes 1 + Q_{\{e\}} \otimes v_n \otimes \tilde{\lambda}_{g_n} = \tilde{\tau}'(S_n) + \tilde{v}(S_n);$$

$$\tilde{\tau}_\pi(\rho(a)) = 1 \otimes a \otimes 1 = \tilde{\tau}'(\rho(a)) + \tilde{v}_e(\rho(a))$$

so that $\tilde{\tau}_\pi = \tilde{\tau}' + \tilde{v}_e$.

2.2. LEMMA. *The map*

$$d_* : \mathrm{K}_j(A \times_{\alpha' r} F_{n-1}) \to \mathrm{K}_j(\mathscr{T}_n)$$

is surjective for $j = 0, 1$.

Proof. As in the proof of Lemma 2.1 we shall use the homomorphism

$$\Delta : K_1(\mathcal{T}_n) \to K_1(v(\mathcal{K} \otimes (A \times_{\alpha' r} F_{n-1}))) \simeq K_1(A \times_{\alpha' r} F_{n-1})$$

and prove that $d_* \circ \Delta$ is the identity on $K_1(\mathcal{T}_n)$. Again as in the proof of Lemma 2.1 it will be sufficient to prove this for classes of unitaries $[u]$ where $u \in \mathcal{T}_n$.

Via the isomorphism $K_1(\mathcal{T}_n) \simeq K_1(\mathcal{K} \tilde{\otimes} \mathcal{T}_n)$ the class $(d_* \circ \Delta)[u]$ is the class of $(\mathrm{id}_{\mathcal{K}} \tilde{\otimes} d)(\tau(u)(\tau'(u) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1})$ in $K_1(\mathcal{K} \tilde{\otimes} \mathcal{T}_n)$. This in turn via the isomorphism $\mathcal{K} \tilde{\otimes} \mathcal{T}_n \simeq I + \tilde{v}(\mathcal{K} \otimes \mathcal{T}_n)$ is the same as the class of $\tilde{\tau}(u)(\tilde{\tau}'(u) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}$ in $K_1(I + \tilde{v}(\mathcal{K} \otimes \mathcal{T}_n))$. Consider the continuous path of unitaries in $I + \tilde{v}(\mathcal{K} \otimes \mathcal{T}_n)$ given by

$$w_t = \tilde{\tau}_t(u)(\tilde{\tau}'(u) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}.$$

We have

$$w_0 = \tilde{\tau}(u)(\tilde{\tau}'(u) + Q_{\{e\}} \otimes 1 \otimes 1)^{-1}$$

$$w_\pi = Q_{\Gamma_n \setminus \{e\}} \otimes 1 \otimes 1 + \tilde{v}_e(u) =$$

$$= Q_{\Gamma_n \setminus \{e\}} \otimes 1 \otimes 1 + \tilde{v}(e(e, e) \otimes u).$$

Since $[w_0] = [w_\pi]$ and since via the considered isomorphisms $[w_\pi]$ corresponds to $[u]$, our assertion is proved for $j = 1$.

The corresponding result for $j = 0$ is obtained by applying the preceding result to $(A \otimes C(\mathbf{T}), \alpha \otimes \mathrm{id}_{C(\mathbf{T})}, F_n)$ and using the natural isomorphism $K_1(M \otimes C(\mathbf{T})) \simeq \simeq K_0(M) \oplus K_1(M)$ for M unital. Q.E.D.

§ 3

In this section putting together the results of the preceding sections we obtain the main results of this paper.

3.1. THEOREM. *The diagram*

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*)} & K_1(A \times_{\alpha' r} F_{n-1}) & \xrightarrow{k_*} & K_1(A \times_{\alpha r} F_n) \\ \uparrow & & & & \downarrow \\ K_0(A \times_{\alpha r} F_n) & \xleftarrow{k_*} & K_0(A \times_{\alpha' r} F_{n-1}) & \xleftarrow{i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*)} & K_0(A) \end{array}$$

where k is the natural inclusion of $A \times_{\alpha' r} F_{n-1}$ into $A \times_{\alpha r} F_n$ and the vertical arrows correspond to the connecting homomorphisms in the exact sequence for the Toeplitz extension (modulo the isomorphisms $K_j(\mathscr{K} \otimes A) \simeq K_j(A)$), is an exact sequence.

Proof. Consider the exact sequence of K-theory for the Toeplitz extension:

$$\begin{array}{ccccc} K_1(\mathscr{K} \otimes A) & \xrightarrow{\psi_*} & K_1(\mathscr{T}_n) & \xrightarrow{\rho_*} & K_1(A \times_{\alpha r} F_n) \\ \uparrow & & & & \downarrow \\ K_0(A \times_{\alpha r} F_n) & \xleftarrow{\rho_*} & K_0(\mathscr{T}_n) & \xleftarrow{\psi_*} & K_0(\mathscr{K} \otimes A). \end{array}$$

By Lemma 1.3 the diagram

$$\begin{array}{ccc} K_j(\mathscr{K} \otimes A) & \xrightarrow{\psi_*} & K_j(\mathscr{T}_n) \\ \| & & \uparrow \rho_* \\ K_j(A) & \xrightarrow{\mathrm{id}_* - \alpha(g_n^{-1})_*} & K_j(A) \end{array}$$

is commutative for $j = 0, 1$. Since $\rho = d \circ i$ the diagram

$$\begin{array}{ccc} K_j(\mathscr{K} \otimes A) & \xrightarrow{\psi_*} & K_j(\mathscr{T}_n) \\ \| & & \uparrow d_* \\ K_j(A) & \xrightarrow{i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*)} & K_j(A \times_{\alpha' r} F_{n-1}) \end{array}$$

is also commutative. Now by Lemma 2.1 and Lemma 2.2 we have that d_* is an isomorphism so that $K_j(\mathscr{K} \otimes A) \xrightarrow{\psi} K_j(\mathscr{T}_n)$ may be replaced in the exact sequence by $K_j(A) \xrightarrow{i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*)} K_j(A \times_{\alpha' r} F_{n-1})$. Q.E.D.

3.2. COROLLARY. *We have $K_0(C_r^*(F_n)) \simeq \mathbf{Z}$ the generator being* [1]. *Also, we have $K_1(C_r^*(F_n)) \simeq \mathbf{Z}^n$ the generators being $[u_1], \ldots, [u_n]$.*

Proof. Applying Theorem 3.1 for $A = \mathbf{C}$ and α trivial, we get

$$K_0(C_r^*(F_{n-1})) \to K_0(C_r^*(F_n))$$

is an isomorphism for $n \geqslant 1$ (where $C_r^*(F_0) = \mathbf{C}$). This shows that

$$K_0(\mathbf{C}) \to K_0(C_r^*(F_n))$$

is an isomorphism proving our assertion concerning K_0.

For K_1 we get exact sequences

$$0 \to K_1(C_r^*(F_{n-1})) \xrightarrow{k_*} K_1(C_r^*(F_n)) \to \mathbf{Z} \to 0$$

the map $K_1(C_r^*(F_n)) \to \mathbf{Z}$ being given by the index corresponding to the extension

$$0 \to \mathscr{K} \to \mathscr{T}_n \to C_r^*(F_n) \to 0.$$

Since $u_n \in C_r^*(F_n)$ has index 1 it follows that $C_r^*(F_n)$ is isomorphic to $k_*(K_1(C_r^*(F_{n-1}))) \oplus \mathbf{Z}\cdot[u_n]$ where k_* is injective. Applying this fact several times our assertion concerning K_1 follows. Q.E.D.

3.3. COROLLARY. *There is no non-trivial projection in* $C_r^*(F_n)$.

Proof. There is a faithful trace-state on $C_r^*(F_n)$. The range of the homomorphism $K_0(C_r^*(F_n)) \to \mathbf{R}$ induced by this trace-state is $\mathbf{Z}$ because of Corollary 3.2. Hence the trace of a projection in $C_r^*(F_n)$ can be only 0 or 1 and hence the only projections in $C_r^*(F_n)$ are 0 and 1. Q.E.D.

There is also another six terms exact sequence for reduced crossed products by F_n, which can be also viewed as a generalization of the exact sequence for crossed products by $\mathbf{Z}$. To establish this second exact sequence we need some preparations.

The Toeplitz algebra for $A \times_{\alpha r} F_n$ arising from Γ_k will be denoted by $\mathscr{T}_{n,k}$ and the corresponding connecting homomorphism $K_j(A \times_{\alpha r} F_n) \to K_{j+1}(A \otimes \mathscr{K})$ will be denoted by $\partial_{n,k}$. By B_n we shall denote the fibered product of the algebras $\mathscr{T}_{n,k}$ over $A \times_{\alpha r} F_n$, i.e. we define:

$$B_n = \{x_1 \oplus \dots \oplus x_n \in \bigoplus_{k=1}^{n} \mathscr{T}_{n,k} \mid p_{n,1}(x_1) = \dots = p_{n,n}(x_n)\}$$

where $p_{n,k} : \mathscr{T}_{n,k} \to A \times_{\alpha r} F_n$ are the corresponding surjections ($k = 1, \dots, n$). Thus, there is an exact sequence

$$0 \to (A \times \mathscr{K})^n \to B_n \to A \times_{\alpha r} F_n \to 0.$$

Consider $\rho_{n,k} : A \to \mathscr{T}_{n,k}$ the homomorphisms corresponding to ρ in case $k = n$ and consider

$$\rho_n : A \to B_n$$

defined by

$$\rho_n(a) = \rho_{n,1}(a) \oplus \dots \oplus \rho_{n,n}(a).$$

Thus $\mathscr{T}_{n,k}$ will be generated by

$$\rho_{n,k}(A) \cup \{S_{n,k}\} \cup \{U_{n,k,j} \mid 1 \leqslant j \leqslant n,\ j \neq k\}.$$

It is then easy to see that B_n is generated by the isometries

$$\Sigma_{n,j} = U_{n,1,j} \oplus \dots \oplus U_{n,j-1,j} \oplus S_{n,j} \oplus U_{n,j+1,j} \oplus \dots \oplus U_{n,n,j}$$

$(1 \leqslant j \leqslant n)$ and $\rho_n(A)$. It is also easily seen, using Lemma 1.2. that there are unital *-homomorphisms $d_n : B_{n-1} \to B_n$ such that

$$d_n(\rho_{n-1}(a)) = \rho_n(a)$$

$$d_n(\Sigma_{n-1,j}) = \Sigma_{n,j} \quad (1 \leqslant j \leqslant n-1).$$

3.4. LEMMA. *The homomorphisms*

$$\rho_{n*} : \mathrm{K}_j(A) \to \mathrm{K}_j(B_n)$$

are isomorphisms.

Proof. We have $\rho_n = d_n \circ \ldots \circ d_1$, so that it will be sufficient to prove that d_{n*} is an isomorophism for all n.

The closed bilateral ideal in B_n generated by $\{1 - \Sigma_{n,1}\Sigma_{n,1}^*, \ldots, 1 - \Sigma_{n,n-1}\Sigma_{n,n-1}^*\}$ is isomorphic to $(A \otimes \mathscr{K})^{n-1}$ and we have an exact sequence

$$0 \to (A \otimes \mathscr{K})^{n-1} \to B_n \to \mathscr{T}_{n,n} \to 0.$$

This exact sequence is part of a commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & (A \otimes \mathscr{K})^{n-1} & \to & B_n & \longrightarrow & \mathscr{T}_{n,n} & \longrightarrow 0 \\ & \uparrow j & & \uparrow d_n & & \uparrow d & \\ 0 \to & (A \otimes \mathscr{K})^{n-1} & \to & B_{n-1} & \to & A \times_{\alpha' r} F_{n-1} & \to 0 \end{array}$$

where the map $j : (A \otimes \mathscr{K})^{n-1} \to (A \otimes \mathscr{K})^{n-1}$ is an injection which induces the obvious isomorphism for the K-groups.

Thus we have a commutative diagram

$$\begin{array}{ccccccccc} \mathrm{K}_{j+1}(\mathscr{T}_{n,n}) & \to & \mathrm{K}_j((A \otimes \mathscr{K})^{n-1}) & \to & \mathrm{K}_j(B_n) & \to & \mathrm{K}_j(\mathscr{T}_{n,n}) & \longrightarrow & \mathrm{K}_{j-1}((A \otimes K)^{n-1}) \\ \uparrow d_* & & \uparrow j_* & & \uparrow d_{n*} & & \uparrow d_* & & \uparrow j_* \\ \mathrm{K}_{j+1}(A \times_{\alpha' r} F_{n-1}) & \to & \mathrm{K}_j((A \otimes \mathscr{K})^{n-1}) & \to & \mathrm{K}_j(B_{n-1}) & \to & \mathrm{K}_j(A \times_{\alpha' r} F_{n-1}) & \to & \mathrm{K}_{j-1}((A \otimes \mathscr{K})^{n-1}). \end{array}$$

Since j_* are isomorphisms and d_* is an isomorphism by § 2, we infer that d_{n*} is also an isomorphism. Q.E.D.

3.5. THEOREM. *The diagram*

$$\begin{array}{ccccc} (\mathrm{K}_0(A))^n & \xrightarrow{\beta} & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(A \times_{\alpha r} F_n) \\ \uparrow \delta & & & & \downarrow \delta \\ \mathrm{K}_1(A \times_{\alpha r} F_n) & \xleftarrow{\pi_*} & \mathrm{K}_1(A) & \xleftarrow{\beta} & (\mathrm{K}_1(A))^n \end{array}$$

where

$$\beta(\gamma_1 \oplus \ldots \oplus \gamma_n) = \sum_{j=1}^{n} (\gamma_j - (\alpha(g_j^{-1}))_* \gamma_j)$$

and $\delta\chi = \bigoplus_{j=1}^{n} \partial_{n,j}\chi$, *is an exact sequence.*

Proof. The extension

$$0 \to (A \otimes \mathcal{K})^n \to B_n \to A \times_{\alpha r} F_n \to 0$$

gives rise to an exact sequence

$$\begin{array}{ccccc} (\mathrm{K}_0(A))^n & \longrightarrow & \mathrm{K}_0(B_n) & \longrightarrow & \mathrm{K}_0(A \times_{\alpha r} F_n) \\ \uparrow & & & & \uparrow \\ \mathrm{K}_1(A \times_{\alpha r} F_n) & \longleftarrow & \mathrm{K}_1(B_n) & \longleftarrow & (\mathrm{K}_1(A))^n \end{array}$$

where the vertical arrows are easily seen to coincide with δ. Using Lemma 3.4 we have isomorphisms $\rho_{n*} : \mathrm{K}_j(A) \to \mathrm{K}_j(B_n)$ so that all we have to prove is the commutativity of

$$\begin{array}{ccc} \mathrm{K}_j((A \otimes \mathcal{K})^n) & \longrightarrow & \mathrm{K}_j(B_n) \\ \wr\wr & & \uparrow \rho_{n*} \\ (\mathrm{K}_j(A))^n & \xrightarrow{\beta} & \mathrm{K}_j(A) \end{array}$$

since it is obvious that the composition of homomorphisms $A \xrightarrow{\rho_n} B_n \to A \times_{\alpha r} F_n$ is just π. To prove the commutativity of the above diagram one proceeds as in the proof of Lemma 1.3, considering the C^*-algebra generated by $\rho_n(A) \cup \Sigma_{n,j}$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(A, \alpha_j, \mathbf{Z})$ for the action $\alpha_j : \mathbf{Z} \to \mathrm{Aut}(A)$ defined by $\alpha_j(m) = \alpha(g_j^m)$. Q.E.D.

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