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UNICELLULAR SHIFTS ON BANACH SPACES

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INTRODUCTION

Recall that a bounded operator on a Banach space is *unicellular* if its lattice of closed invariant subspaces is totally ordered. In [6, Theorem 2.10, p. 21] we showed that every separable Banach space has a unicellular unilateral weighted shift. In the present paper we show that every separable Banach space also has a unicellular backward shift (Theorem 1.1) and a unicellular bilateral shift (Theorem 2.3). We also prove an analogous result for diagonal operators (Theorem 3.3). We construct a unicellular backward shift by an extension of the method we used in [6, Theorem 4.1, p. 27] to construct unicellular forward shifts. Our construction of a unicellular bilateral shift is based partly on the existence of unicellular forward and backward shifts, and partly on a theorem of Domar about scalar sequences [2, Theorem 5]; Domar used his theorem to construct unicellular bilateral shifts on ℓ^p [2, Theorem 2].

Our shifts and diagonal operators will all be defined with respect to M-bases. Recall that an *M-basis* (or *Markushevich basis*) for a locally convex space X is a biorthogonal sequence $\{x_n, x_n^*\}_{n=0}^{\infty}$ for which the span of $\{x_n\}_0^{\infty}$ is dense in X, and the linear functionals $\{x_n^*\}_0^{\infty} \subseteq X$ are total over X. Since $\{x_n\}_0^{\infty}$ has dense span, the associated linear functionals are completely determined so that $\{x_n\}_0^{\infty}$ alone is sometimes called an M-basis. We can use any countably infinite set, not just the non-negative integers, as an index set for an M-basis. Notice that if $\{x_n, x_n^*\}$ is an M-basis and if $\{c_n\}$ is a sequence of non-zero scalars, then $\{c_n x_n, (x_n^*/c_n)\}$ is also an M-basis.

In the case that X is a Banach space, the M-basis $\{x_n, x_n^*\}$ always satisfies $\|x_n\|\,\|x_n^*\| \geqslant |x_n^*(x_n)| = 1$. The M-basis is said to be *bounded* if in addition there is an $M > 0$ with $\|x_n\|\,\|x_n^*\| \leqslant M$ for all n. It is easy to construct an M-basis in any separable Banach space [10, Proposition 1.f.3, p. 43], and one can in fact always construct a bounded M-basis [10, Theorem 1.f.4, p. 44], [11].

1. UNILATERAL SHIFTS

Suppose that $\{x_n, x_n^*\}_0^\infty$ is an M-basis for the locally convex space X and that $\{c_n\}_0^\infty$ is a sequence of non-zero scalars. A continuous linear operator T on X is a *forward shift* with *weights* $\{c_{n+1}/c_n\}_0^\infty$ (for the M-basis) if $Tx_n = (c_{n+1}/c_n)x_{n+1}$ for all n; and it is a *backward shift* with *weights* $\{c_{n+1}/c_n\}_0^\infty$ if $T(x_{n+1}) = (c_{n+1}/c_n)x_n$ for all $n \geqslant 0$ and $T(x_0) = 0$. Since the span of $\{x_n\}_0^\infty$ is dense, there can be no more than one forward or backward shift with a given sequence of weights; but no shift of either type with weights $\{c_{n+1}/c_n\}$ need exist.

We now construct unicellular backward shifts for an arbitrary M-basis.

THEOREM 1.1. *Suppose that $\{x_n, x_n^*\}_0^\infty$ is an M-basis for the Banach space X and that $\{c_n\}_0^\infty$ is a sequence of non-zero scalars. If $\{\|x_n^*\|\}_0^\infty$ is bounded below, and if*

(i) *there is a $k > 0$ for which $\{|c_{n+k}|/|c_n|\}_0^\infty$ is eventually non-increasing,*

(ii) $\sum_0^\infty (|c_{n+1}|/|c_n|)\|x_n\|\ \|x_{n+1}^*\| < \infty$,

(iii) *the sequence $\{(|c_{n+1}|/|c_n|)\|x_{n+1}^*\|\}_0^\infty$ is bounded,*

then the backward shift with weights $\{c_{n+1}/c_n\}_0^\infty$ is unicellular.

Before proving Theorem 1.1, let us observe that Condition (ii) is precisely the condition needed to guarantee that there is a backward shift T with weights $\{c_{n+1}/c_n\}$. The shift T is then compact, in fact nuclear, and therefore quasinilpotent. Also, notice that Condition (iii) follows from Condition (ii) if $\{\|x_n^*\|\}$ is bounded or if $\{\|x_n\|\}$ is bounded below. However, we will need Condition (iii) as stated above in our application to bilateral shifts in the next section.

Proof of Theorem 1.1. By multiplying the M-basis by a sequence of unimodular scalars if necessary, we can assume all c_n are positive.

In the dual space X^*, $\{x_n^*, x_n\}_0^\infty$ is an M-basis for the weak*-topology, and T^* is the forward shift with respect to $\{x_n^*, x_n\}$ with weights $\{c_{n+1}/c_n\}_0^\infty$. It will be convenient to let $z^n = c_n x_n^*$ and to let $\pi_n = x_n/c_n$ and to identify π_n with its image in X^{**}.

Then X^* can be considered as a space of formal power series by identifying f in X^* with the formal power series $\sum_0^\infty \pi_n(f)z^n$. With this identification, $T^*f = zf$ (for more information on this identification, see [5, p. 81] or [6, pp. 18--20]).

The map $E \to E^\perp$ sets up a one-one correspondence between the closed T-invariant subspaces of X and the weak*-closed T^*-invariant subspaces of X^*. So to show that T is unicellular, we must show that the only non-zero weak*-closed T^*-invariant subspaces of X^* are the spaces

$$(1.2)\qquad B_k = \{x_0, x_1, \ldots, x_{k-1}\}^\perp = \{f \in X^*: \pi_0(f) = \pi_1(f) \ldots \pi_{k-1}(f) = 0\},$$

where we interpret B_0 as X^*.

Suppose now that L is a non-zero weak*-closed T^*-invariant subspace of X^*. We break the proof that L is one of the spaces B_k into three steps.

STEP 1. Some z^n belongs to L.

STEP 2. Whenever z^n belongs to L, then $B_n \subseteq L$.

STEP 3. Some $B_k = L$.

STEP 1. Let $K^\#$ be the Banach space of all formal power series $f = \sum_0^\infty \lambda_n z^n$ for which the norm

$$\|f\|_K = |\lambda_0| + \sum_1^\infty |\lambda_n| c_{n-1}$$

is finite, and let K be the series in $K^\#$ with constant term equal to 0. Condition (i) implies that, under an equivalent norm, K and $K^\#$ are Banach algebras with K radical (see [3, Lemma 2.4, p. 643] and the proof and remark following [3, Theorem 2.10, pp. 645—646]), and it also implies that every non-zero ideal in K contains a power of z [3, pp. 644—645]. Since the polynomials are dense in K, every closed non-zero subspace of K which is invariant under multiplication by z is an ideal, and therefore contains a power of z.

Condition (iii) implies that K is continuously imbedded in B. Therefore $L \cap K$ is a closed ideal in K. Condition (ii) and the assumption that $\{\|x_n^*\|\}$ is bounded below together imply that $Bz^2 \subseteq K$. Therefore $L \cap K$ contains z^2L and is non-zero. Hence $L \supseteq L \cap K$ contains a power of z.

STEP 2. Now suppose that z^n belongs to L. We must show that if x in X is annihilated by all f in the weak*-closed subspace L of X^*, then x is annihilated by all f in B_n. Since L is T^*-invariant, x is annihilated by $z^n, z^{n+1}, z^{n+2} \dots$. But $\{z_n\}_0^\infty$ is total over x, so x must therefore equal $\sum_{k=1}^{n-1} z^k(x)\pi_k$, which is annihilated by all f in B_n.

STEP 3. There is a $k > 0$ for which $L \subseteq B_k$ and for which L contains an f with $\pi_k(f) \neq 0$. By Step 2, there is a g in L and a polynomial p with non-zero constant term for which $f = z^kp + g$. Then, in the notation of Step 1, z^kp belongs to the ideal $K^\# \cap L$ of $K^\#$. Since K is radical and p has non-zero constant term, p is invertible in $K^\#$, so that $z^k = z^kpp^{-1} \in K^\# \cap L \subseteq L$. By Step 2, we then have $L \supseteq B_k$, so $L = B_k$. This completes the proof of the theorem.

There are various ways of weakening the hypothesis of Theorem 1.1. Condition (i) is used only in the proof of Step 1 to guarantee that K is a radical algebra in which every closed ideal contains a power of z. Condition (i) could therefore be replaced by this assumption about K or by any of the many known conditions which imply that K is a radical algebra in which every closed ideal contains a power of z.

Other possible generalizations include changing $|c_{n+1}|$ and $\|x_{n+1}^*\|$ in (iii) to $|c_{n+j}|$ and $\|x_{n+j}^*\|$ for some fixed $j > 0$, and, in addition, when a backward shift with weights $\{c_{n+1}/c_n\}$ is known to exist, changing $|c_{n+1}|$ and $\|x_{n+1}^*\|$ in (ii) to $|c_{n+m}|$ and $\|x_{n+m}^*\|$ for some $m > 0$. Then in the proof of Theorem 1.1, K is defined using c_{n-j}, and then one shows $Bz^{m+J} \subseteq K$.

One can also modify Theorem 1.1, or its generalizations, to give a slightly more general construction of forward weighted shifts than is given in [6]. In the hypothesis of Theorem 1.1, interchange the roles of $\{x_n\}$ and $\{x_n^*\}$ (e.g. assume $\{\|x_n\|\}$ is bounded below). The proof for forward shifts does not use duality arguments; instead, one identifies X itself as a space of power series by letting $z^n = c_n x_n$. The proofs of Steps 1 and 3 are exactly the same as in the proof of Theorem 1.1; and Step 2 is now trivial.

2. BILATERAL SHIFTS

Suppose that X is a Banach space and that $\{c_n\}_{-\infty}^{\infty}$ is a sequence of non-zero scalars. The bounded operator T is a *bilateral shift* for the M-basis $\{x_n, x_n^*\}_{-\infty}^{\infty}$ with *weights* $\{c_{n+1}/c_n\}_{-\infty}^{\infty}$ provided $Tx_n = (c_{n+1}/c_n)x_{n+1}$ for all integers n. As with unilateral shifts, no bilateral shift with a given sequence of weights need exist; but if one does exist it is unique.

The bilateral shift T is unicellular if and only if its only proper non-zero closed invariant subspaces are the spaces

$$X_k = \mathrm{cl}\,[\mathrm{span}\{x_n\}_{n=k}^{\infty}] \tag{2.1}$$

defined for each integer k. Thus in order for $\{x_n, x_n^*\}_{-\infty}^{\infty}$ to have any unicellular shifts, it is necessary that

$$X_k = [\{x_n^*\}_{n=-\infty}^{k-1}]^{\perp} \tag{2.2}$$

for any, and hence all, integers k. We will call those M-bases for which formula (2.2) holds for all k, *splitting bases.*

It is clear that every Schauder basis is a splitting basis, but we will see in the next section that every separable Banach space has an M-basis $\{x_n, x_n^*\}_{-\infty}^{\infty}$ which is not a splitting basis. Formula (2.2) is equivalent to the assertion that the biorthogonal sequence induced by the M-basis $\{x_n, x_n^*\}_{-\infty}^{\infty}$ on the quotient space X/X_k is an M-basis for the quotient space. Hence the construction by Gurariĭ and Kadec in [7, Theorem 1, p. 966] provides a splitting basis for each separable Banach space.

We now show that every splitting basis has a unicellular shift, and hence that every separable Banach space has a unicellular bilateral shift for some M-basis.

THEOREM 2.3. *The* M-*basis* $\{x_n, x_n^*\}_{-\infty}^{\infty}$ *has a unicellular bilateral shift if and only if the* M-*basis is splitting.*

Proof. If the M-basis is not splitting then the spaces in formula (2.2) are distinct sequences of invariant subspaces for any bilateral shift, so no such shift can be unicellular. Assume that $\{x_n, x_n^*\}_{-\infty}^{\infty}$ is a splitting basis. Without loss of generality, we can normalize so that all $\|x_n\| = 1$.

Choose a sequence of positive numbers $\{w_n\}_{-\infty}^{\infty}$ satisfying:

(i) $\sum_{-\infty}^{\infty} (w_{n+1}/w_n)\|x_n^*\| < \infty$.

(ii) $\{w_{n+1}/w_n\}_0^{\infty}$ is non-increasing and $\{w_{n+1}/w_n\}_{-\infty}^{0}$ is non-decreasing.

(iii) $\liminf_{n\to\infty}(w_n)^{1/n^2} < 1/3$ and $\limsup_{n\to-\infty}(w_n)^{1/n^2} > 3$.

Condition (i) implies that there exists a bilateral shift T with weights $\{w_{n+1}/w_n\}_{-\infty}^{\infty}$. Suppose that L is a proper non-zero T-invariant subspace. We must show that L is one of the spaces X_k of formula (2.1).

We first assume that there is an integer p for which $L \cap X_p \neq \{0\}$. Then the restriction of T to X_p is a forward unilateral shift with non-zero invariant subspace $L \cap X_p$. It follows from [6, Theorem 2.10, p. 21], or from the remarks at the end of the previous section, that the restriction of T to X_p is unicellular. Hence there is a $j \geqslant n$ with $L \supseteq L \cap X_p = X_j$.

The assumption that $\{x_n, x_n^*\}_{-\infty}^{\infty}$ is a splitting basis means that the biorthogonal system $\{y_n, y_n^*\}_0^{\infty}$ that it induces on the quotient space X/X_j is an M-basis. If we let $c_n = 1/(w_{j-1-n})$, then T induces on X/X_j the backward shift for $\{y_n, y_n^*\}_0^{\infty}$ with weights $\{c_{n+1}/c_n\}_0^{\infty}$ and with proper invariant subspace L/X_j.

Let θ be the natural projection on X/X_j. Then, for all $n \geqslant 0$, we have

$$\|y_n\| = \|\theta x_{j-1-n}\| \leqslant \|x_{j-1-n}\| = 1$$

and

$$\|y_{n+1}^*\| = \|y_{n+1}^*\theta\| = \|y_{j-n}^*\| \geqslant 1.$$

Hence it follows from (i) and (ii) that the induced backward shift on X/X_j satisfies the hypothesis of Theorem 1.1 and is therefore unicellular. Thus the invariant subspace L/X_j equals X_k/X_j for some $k \leqslant j$, so that $L = X_k$. This finishes the proof in the case that some $L \cap X_p$ is non-zero.

To complete the proof, we show that if $x \neq 0$ belongs to $L \cap T(X)$ then x belongs to some X_p. It will be convenient to define $z_n = w_n x_n$ so that $Tz_n = z_{n+1}$. Since x belongs to the range of T, it follows from (i) that x is the sum of an absolutely converging series $\sum_{-\infty}^{\infty} a_n z_n$. The convergence of the series implies, in particular, that the sequence $\{a_n\|z_n\|\} = \{a_n w_n\}_{-\infty}^{\infty}$ is bounded. Since L is a proper closed subspace of X, there is a non-zero f in X^* which annihilates L. Let $b_n = f(z_n)$; then in particular $|b_n| \leqslant \|f\|\|z_n\|$. Hence the sequence $\{b_n/w_n\}_{-\infty}^{\infty}$ is bounded and not identically zero. Since L is T-invariant and contains x, it also contains $T^m x$ for each $m \geqslant 0$. Thus for each $m \geqslant 0$ we have

$$0 = fT^m x = fT^m\left(\sum_{-\infty}^{\infty} a_n z_n\right) = \sum_{-\infty}^{\infty} a_n f T^m(z_n) = \sum_{-\infty}^{\infty} a_n b_{n+m}.$$

Hence it follows from Domar's [2, Theorem 5] that there is an integer p for which $a_n = 0$ for $n < p$. Therefore $x = \sum_p^\infty a_n z_n = \sum_p^\infty a_n w_n x_n$ belongs to $L \cap X_p$. This completes the proof.

If the M-basis $\{x_n, x_n^*\}_{-\infty}^{\infty}$ is not a splitting basis, the above proof can be adapted to show that L is a proper non-zero invariant subspace for the bilateral shift if and only if there is an integer k for which $[\{x_n^*\}_{-\infty}^{k-1}]^\perp \supseteq L \supseteq X_{k+1}$ (cf. [6, Theorem 4.4, p. 29]).

When the sequence $\{\|x_n\| \, \|x_n^*\|\}_{-\infty}^{\infty}$ is bounded or does not become unbounded too rapidly, it is possible to write down simple explicit $\{w_n\}$ satisfying the conditions (i), (ii), and (iii) in the proof of Theorem 2.3. For instance, if $\limsup_{|n|\to\infty}(\|x_n\| \, \|x_n^*\|)^{1/|n|} < 4$ and if we normalize so that all $\|x_n\| = 1$, then we can define $w_n = (1/4)^{n^2}$ for $n \geqslant 0$ and $w_n = (4)^{n^2}$ for $n \leqslant 0$. I do not know if every separable Banach space has a bounded splitting basis. But if in the construction of the splitting basis in [7, Theorem 1, p. 966], the M-bases which start the construction are taken to be bounded, which can always be done [10, Theorem 1.f.4, p. 44], then for the resulting splitting bases the sequence $\{(\|x_n\| \, \|x_n^*\|)/|n|\}_{-\infty}^{\infty}$ is bounded.

The construction in [7, Theorem 1, p. 966] starts with an arbitrary subspace, which is X_1 in our notation, of infinite dimension and codimension. Thus any subspace of infinite dimension and co-dimension in X is an invariant subspace of some unicellular bilateral shift on X.

3. DIAGONAL OPERATORS

Suppose that $\{x_n, x_n^*\}_0^\infty$ is an M-basis for the Banach space X and that $\{\lambda_n\}_0^\infty$ is a sequence of scalars. The bounded operator T is a *diagonal operator* for the M-basis with *weights* $\{\lambda_n\}_0^\infty$ if $Tx_n = \lambda_n x_n$ for all n. As with shifts no such diagonal operator need exist, but if one does exist it is unique.

For each set A of non-negative integers, we let

$$X_A = \text{cl}\,[\text{span}\{x_n\}_{n\in A}]. \tag{3.1}$$

Each X_A is an invariant subspace of every diagonal operator T for the M-basis $\{x_n, x_n^*\}_0^\infty$. When there are no other invariant subspaces, the invariant subspace lattice of T is isomorphic to the lattice of subsets of a countably infinite set. This cannot happen for any diagonal operator unless

$$X_A = [\{x_n^*\}_{n\notin A}]^\perp \tag{3.2}$$

for all sets of non-negative integers A. We will call M-bases for which (3.2) holds for all A, *synthesis bases.* Using the terminology of the previous section, an M-basis is a synthesis basis if and only if it is a splitting basis whenever it is rearranged into a doubly infinite sequence.

It is clear that every Schauder basis is a synthesis basis. In fact if an M-basis has practically any summability property, then it is a synthesis basis [12, pp. 722 and 731]. However, every separable Banach space has an M-basis which is not a synthesis basis [2, p. 732] (and hence has an M-basis $\{x_n, x_n^*\}_{-\infty}^{\infty}$ which is not splitting). It is an open problem whether every separable Banach space has a synthesis basis [1, Problem 2, p. 189], but every separable Banach space whose dual has the approximation property is known to have a synthesis basis [8, Theorem 1, p. 481].

We now show how to construct diagonal operators whose only invariant subspaces are the spaces X_A of formula (3.1).

THEOREM 3.3. *Suppose that $\{x_n, x_n^*\}_0^\infty$ is a synthesis basis for the Banach space X and that $\{\lambda_n\}_0^\infty$ is a sequence of distinct non-zero scalars. If $\sum_0^\infty |\lambda_n| \, \|x_n\| \, \|x_n^*\| < \infty$, then the diagonal operator with weights $\{\lambda_n\}_0^\infty$ has its invariant subspace lattice isomorphic with the lattice of subsets of a countably infinite set.*

Proof. For each non-negative integer n, let E_n be the rank one projection $E_n(x) = x_n^*(x)x_n$. Then $\sum_0^\infty \lambda_n E_n$ converges absolutely in the uniform norm to the diagonal operator T with weights $\{\lambda_n\}_0^\infty$. T is therefore compact and has the points $\{\lambda_n\}$ as the non-zero points of its spectrum, with $\{E_n\}$ as the associated spectral projection. Since the spectrum of T does not disconnect the plane, each E_n is a limit of polynomials in T, and therefore any T-invariant subspace is invariant under all E_n.

Suppose that L is a closed T-invariant subspace, and let

$$A = \{n \geqslant 0 : x_n^*(x) \neq 0 \text{ for some } x \text{ in } L\}.$$

If $n \notin A$, then $x_n^*(L) = \{0\}$, so it follows from formula (3.2) that $L \subseteq X_A$. On the other hand if $n \in A$, $x_n \in E_n(L) \subseteq L$, so that $L \supseteq X_A$. Hence $L = X_A$, and the proof is complete.

If the M-basis $\{x_n, x_n^*\}_0^\infty$ is not a synthesis basis, then the above proof shows that the closed subspace L is T-invariant if and only if there is a set of non-negative integers A for which $X_A \subseteq L \subseteq [\{x_n^*\}_{n \notin A}]^\perp$.

We use the term synthesis basis because of analogies with spectral synthesis in Banach algebras [9, pp. 192—194]. Explicitly, if we let $\{d_n\}_0^\infty$ be a sequence of non-zero scalars satisfying $\sum d_n \|x_n^*\|^2 \|x_n\| < \infty$, then there is an equivalent norm on X under which X becomes a commutative semisimple Banach algebra when multiplication is defined by $yz = \sum_0^\infty x_n^*(y)x_n^*(z)x_n$ [4], [13, Theorem 8, p. 447]. The closed subspace L is an ideal of the algebra if and only if it is T-invariant. The maximal ideal space of the algebra is the non-negative integers, and the set A of non-negative integers satisfies (3.2) precisely when A is a set of spectral synthesis [9, Definition 8.3.1, p. 194].

When $\{x_n, x_n^*\}_0^\infty$ is an unconditional basis or a Schauder basis, we can relax the assumption that $\sum_0^\infty |\lambda_n| \, \|x_n\| \, \|x_n^*\| < \infty$. In the case of an unconditional basis, we need only assume that $\{\lambda_n\}_0^\infty$ is a bounded sequence of distinct scalars and that each λ_k is in the unbounded component of the complement of the set of limit points of $\{\lambda_n\}_0^\infty$. For Schauder bases we add the assumption that $\{\lambda_n\}_0^\infty$ is of bounded variation. In each case we still have that each λ_n is an isolated point in the spectrum of T with a deleted neighborhood in the unbounded component of the spectrum. Thus each E_n is still a limit of polynomials in T, and the proof of Theorem 3.3 goes through as before. With a somewhat more complicated proof, we can allow a finite number of $\{\lambda_n\}_0^\infty$ to be limit points of the sequence or to be in a bounded component or the complement of the set of limit points.

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Note added in proof. Two of the results used above to construct unicellular bilateral shifts have recently been sharpened [14], [15], allowing us to prove unicellularity under much weaker conditions on $\{w_n\}$ than we assumed in the proof of Theorem 2.3. Yngve Domar has sharpened his [2, Theorem 5] to the stronger result [14, Theorem 3], which allows us to replace (iii) by the weaker condition $nw_n^{1/n} \to 0$ as $|n| \to \infty$. Because of the monotonicity assumptions in (ii), this new condition is also equivalent to $nw_{n+1}/w_n \to 0$ as $|n| \to \infty$. Paolo Terenzi has shown that, in the terminology we used above, every separable Banach space has a bounded splitting basis [15, Theorem II]. For a bounded splitting basis we need assume only that $\{w_n\}$ satisfies conditions (i) and (ii) in the proof of Theorem 2.3 above. This is because, for a bounded M-basis, (i) says that $\sum_{-\infty}^{\infty} w_{n+1}/w_n < \infty$, which, when combined with the monotonicity assumptions in (ii), implies that $nw_{n+1}/w_n \to 0$ as $|n| \to \infty$.

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