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W^*-DYNAMICAL SYSTEMS AND REFLEXIVE OPERATOR ALGEBRAS

JON KRAUS

Let α be a σ-weakly continuous action of a locally compact abelian group G on a von Neumann algebra $\mathscr{R}$ by $*$-automorphisms. If α is inner, and Σ is a positive semigroup in the dual group of G, then to each inner unitary implementation of α, we can associate (via Stone's Theorem) a commutative subspace lattice (CSL) $\mathscr{L} \subset \mathscr{R}$ such that $(\mathrm{alg}\mathscr{L}) \cap \mathscr{R}$ coincides with the algebra of Σ-analytic operators of the action. Such a lattice $\mathscr{L}$ is said to be (G, Σ)-analytic. Loebl and Muhly have shown that every totally ordered CSL is $(\mathbf{R}, [0, \infty))$-analytic. We extend their result, and show that any width n CSL is $(\mathbf{R}^n, [0, \infty)^n)$-analytic. We also show that if $\mathscr{L}$ is any CSL whose core $\mathscr{L}''$ is totally atomic, then there is a second countable compact abelian group G, and a positive semigroup Σ in the dual group of G, such that $\mathscr{L}$ is (G, Σ)-analytic. These results allow us to use spectral subspace techniques to study certain reflexive operator algebras. As an application, we prove that if $\mathscr{L}_i \subset \mathscr{R}_i$, $i = 1, 2$ are commuting CSL's with totally atomic cores, then

$$(1) \qquad (\mathrm{alg}\mathscr{L}) \cap \mathscr{R} = [(\mathrm{alg}\mathscr{L}_1) \cap \mathscr{R}_1] \mathbin{\overline{\otimes}} [(\mathrm{alg}\mathscr{L}_2) \cap \mathscr{R}_2],$$

where $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2$, $\mathscr{R} = \mathscr{R}_1 \overline{\otimes} \mathscr{R}_2$, and the tensor product on the right hand side of (1) is the σ-weak closure (in $\mathscr{R}$) of the algebraic tensor product. This result is related to Tomita's commutation theorem for tensor products of von Neumann algebras, and to a result of Gilfeather, Hopenwasser and Larson (who show, by other methods, that (1) holds when $\mathscr{R}_1$ and $\mathscr{R}_2$ are type I factors, and $\mathscr{L}_1$ and $\mathscr{L}_2$ are totally ordered).

1. PRELIMINARIES AND NOTATION

In this paper, all Hilbert spaces will be assumed separable. Let H be a Hilbert space. We write $B(H)$ for the algebra of all bounded operators on H. "Projection" will always mean self-adjoint projection, and we identify projections with their ranges. The projections in $B(H)$ form a lattice under the operations $\wedge$ (intersection) and $\vee$ (closed linear span). A *subspace lattice* $\mathscr{L}$ is a lattice of projections in $B(H)$ that contains 0 and the identity operator I, and is closed in the strong operator topology. If the elements of $\mathscr{L}$ pairwise commute, $\mathscr{L}$ is called a *commutative subspace lattice* (CSL).

If $\mathscr{L}$ is a subspace lattice, alg$\mathscr{L}$ denotes the algebra of operators in $B(H)$ leaving the elements of $\mathscr{L}$ invariant. If $\mathscr{A}$ is a subset of $B(H)$, lat$\mathscr{A}$ denotes the subspace lattice of projections in $B(H)$ left invariant by all of the elements of $\mathscr{A}$. If $\mathscr{A}$ and $\mathscr{L}$ are contained in a von Neumann algebra $\mathscr{R}$, we write $\mathscr{R}$-alg$\mathscr{L}$ for (alg$\mathscr{L}$) $\cap \mathscr{R}$ and $\mathscr{R}$-lat$\mathscr{A}$ for (lat$\mathscr{A}$) $\cap \mathscr{R}$. We say $\mathscr{L}$ (resp. $\mathscr{A}$) is *$\mathscr{R}$-reflexive* if $\mathscr{L} = \mathscr{R}$-lat$\mathscr{R}$-alg$\mathscr{L}$ (resp. $\mathscr{A} = \mathscr{R}$-alg$\mathscr{R}$-lat$\mathscr{A}$), and that $\mathscr{L}$ or $\mathscr{A}$ is *reflexive* if it is $B(H)$-reflexive. Note that if $\mathscr{L} \subset \mathscr{R}$ is a subspace lattice, the associated algebra $\mathscr{R}$-alg$\mathscr{L}$ is automatically $\mathscr{R}$-reflexive.

A *chain* is a totally ordered subspace lattice. (Note that in Ringrose's terminology [17], the chains are precisely the complete nests containing 0 and I.) A subspace lattice $\mathscr{L}$ is *finite width* if there is a finite set of chains $\mathscr{L}_1, \ldots, \mathscr{L}_n$ such that $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$, where $\mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$ denotes the subspace lattice generated by $\mathscr{L}_1, \ldots, \mathscr{L}_n$. The width of $\mathscr{L}$ is the smallest such integer n. The *core* of a subspace lattice $\mathscr{L}$ is the von Neumann algebra $\mathscr{L}''$ generated by $\mathscr{L}$, and $\mathscr{L}$ is *multiplicity free* if its core is a maximal abelian von Neumann algebra.

Commutative subspace lattices were first studied systematically by Arveson [2], who developed powerful techniques for analyzing these lattices and their associated reflexive algebras. One of the main results in [2] is that all CSL's are reflexive. Ringrose had previously shown in [17] that chains are reflexive. He called the associated algebras nest algebras, and introduced them as a generalization of the maximal hyperintransitive triangular algebras of Kadison and Singer [11]. An excellent survey and bibliography of nest algebras and more general CSL algebras can be found in [6].

If $\mathscr{L} \subset \mathscr{R}$ is a chain, $\mathscr{R}$-alg$\mathscr{L}$ is called a *nest-subalgebra* of $\mathscr{R}$. Nest-subalgebras of von Neumann algebras have been extensively studied by Gilfeather and Larson [8, 9]. Their results show that these algebras form a very interesting class of reflexive operator algebras. In [14, § 4.2], Loebl and Muhly established a connection between nest-subalgebras of von Neumann algebras and W^*-dynamical systems. A *W^*-dynamical system* $(\mathscr{R}, G, \alpha)$ consists of a von Neumann algebra $\mathscr{R}$ (on H), a locally compact abelian group G, and a homomorphism α of G into the group of $*$-automorphisms of $\mathscr{R}$, such that all of the maps $g \to \rho(\alpha_g(A))$ are continuous ($A \in \mathscr{R}$, $\rho \in \mathscr{R}_*$). We say α is *inner* if there is a $U \in \mathrm{Rep}(G, H)$ (the set of strongly continuous unitary representations of G on H) such that $\alpha = \mathrm{ad}\,U$ (i.e. $\alpha_g(A) = U_g A U_g^*$, $A \in \mathscr{R}$, $g \in G$), and $U_g \in \mathscr{R}$ for all g in G. The dual group of G is denoted by Γ.

A powerful tool in studying dynamical systems is the theory of spectral subspaces, developed in its present form by Arveson [1]. For the convenience of the reader we include some of the basic definitions and facts about spectral subspaces below. Detailed expositions can be found in [1, 3, 4, 15, 16].

If $(\mathscr{R}, G, \alpha)$ is a dynamical system, then for each $f \in L^1(G)$ and $A \in \mathscr{R}$, the $\mathscr{R}$-valued integral

$$\alpha(f)(A) = \int_G f(g)\alpha_g(A)\mathrm{d}g$$

(where $\mathrm{d}g$ is Haar measure) is well defined, $\alpha(f)$ is a σ-weakly continuous bounded linear operator on $\mathscr{R}$, and the action $(f, A) \to \alpha(f)A$ makes $\mathscr{R}$ into an $L^1(G)$-module, with $\|\alpha(f)(A)\| \leqslant \|f\| \, \|A\|$. For $A \in \mathscr{R}$, the *spectrum* of A is defined by

$$\mathrm{sp}_\alpha(A) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \text{ for all } f \in L^1(G) \text{ with } \alpha(f)(A) = 0\},$$

where

$$\hat{f}(\gamma) = \int_G f(g)\,(g, \gamma)\mathrm{d}g,$$

and $g, \gamma \to (g, \gamma)$ is the dual pairing of G and Γ. The spectrum of A can also be characterized as the support of the map $\hat{f} \to f(\alpha)(A)$, i.e., the complement of the largest open set $\Omega \subset \Gamma$ such that supp $\hat{f} \subset \Omega$ implies $\alpha(f)(A) = 0$. If $E \subset \Gamma$, the *spectral subspace* $\mathscr{R}^\alpha(E)$ is the σ-weakly closed linear span of $\{A \in \mathscr{R} : \mathrm{sp}_\alpha(A) \subset E\}$. If E is closed, $\{A \in \mathscr{R} : \mathrm{sp}_\alpha(A) \subset E\}$ is a σ-weakly closed linear subspace of $\mathscr{R}$, and so equals $\mathscr{R}^\alpha(E)$. If $E, F \subset \Gamma$ then $\mathscr{R}^\alpha(E)\mathscr{R}^\alpha(F) \subseteq \mathscr{R}^\alpha(E + F)$.

A subset $\Sigma \subset \Gamma$ is a *positive semigroup* if (i) $\Sigma + \Sigma \subset \Sigma$, (ii) $\Sigma \cap (-\Sigma) = \{0\}$, and (iii) $\Sigma = \overline{\mathrm{int}\Sigma}$. Note that if we define a binary relation $\geqslant$ on Γ by $\gamma \geqslant \lambda$ iff $\gamma - \lambda \in \Sigma$, then $\geqslant$ is a partial order, and $\Sigma = \{\gamma \in \Gamma : \gamma \geqslant 0\}$. If $\Sigma \subset \Gamma$ is a positive semigroup, an operator $A \in \mathscr{R}$ is *analytic* (or *Σ-analytic*) if $\mathrm{sp}_\alpha(A) \subset \Sigma$. The set of all analytic operators, $\mathscr{R}^\alpha(\Sigma)$, is a σ-weakly closed subalgebra of $\mathscr{R}$ containing the identity operator I. Algebras of analytic operators were first studied by Loebl and Muhly in [14]. In the special case when $G = \mathbf{R}$ (so $\Gamma = \mathbf{R}$), and $\Sigma = [0, \infty)$, A is analytic iff all of the maps $t \to \rho(\alpha_t(A))$, $\rho \in \mathscr{R}_*$, are in $H^\infty(\mathbf{R})$.

One of the results of [14] is that the nest-subalgebras of von Neumann algebras are precisely the algebras of analytic operators of inner one-parameter W^*-dynamical systems. In the next section, we prove an analogous result for "finite width subalgebras" of von Neumann algebras (Theorem 2.6).

2. ANALYTIC CSL'S

Let G be a locally compact abelian group. If $U \in \mathrm{Rep}(G, H)$, then by Stone's Theorem, there is a unique projection-valued measure $P(\cdot)$ on the Borel sets of Γ such that

$$U_g = \int_\Gamma (g, \gamma)\,\mathrm{d}P(\gamma) \quad (g \in G). \tag{2}$$

We write $U \sim P$ if (2) holds. If Σ is a positive semigroup in Γ, and $\mathscr{L} \subset \mathscr{R}$ is a CSL, we say $\mathscr{L}$ is *(G, Σ)-analytic* if there is a $U \in \operatorname{Rep}(G, H)$ such that $U_g \in \mathscr{R}$ for all $g \in G$, and $\{P(\gamma + \Sigma) : \gamma \in \Gamma\}$ generates $\mathscr{L}$ as a subspace lattice (where $P \sim U$). We write $\mathscr{L} \sim U$ or $\mathscr{L} \sim P$ in this case.

PROPOSITION 2.1. *Suppose $\mathscr{L} \subset \mathscr{R}$ is (G, Σ)-analytic, $U \sim \mathscr{L}$ and $\alpha = \operatorname{ad} U$. Then*

$$\mathscr{R}\text{-alg}\mathscr{L} = \mathscr{R}^{\alpha}(\Sigma). \tag{3}$$

Proof. Let $P \sim U$. By [1, Corollary 2, p. 231], $A \in \mathscr{R}$ is in $\mathscr{R}^{\alpha}(\Sigma)$ iff $A[P(\gamma + \Sigma)H] \subset [P(\gamma + \Sigma)H]$ for all $\gamma \in \Gamma$, i.e. iff $\{P(\gamma + \Sigma) : \gamma \in \Gamma\} \subseteq \operatorname{lat}\{A\}$. But $\operatorname{lat}\{A\}$ is a subspace lattice, so $A \in \mathscr{R}^{\alpha}(\Sigma)$ iff $\mathscr{L} \subset \operatorname{lat}\{A\}$, and (3) follows.

REMARK 2.2. The converse of Proposition 2.1 is *not* true. For example, if $\mathscr{R}$ is a nontrivial abelian von Neumann algebra, and $(\mathscr{R}, \mathbf{R}, \alpha)$ is the trivial dynamical system (i.e. each α_t is the identity automorphism) then every CSL $\mathscr{L} \subset \mathscr{R}$ satisfies (3), since both sides equal $\mathscr{R}$. On the other hand, the only inner one-parameter dynamical system is the trivial one ($\mathscr{R}$ is abelian), and so the only $(\mathbf{R}, [0, \infty))$-analytic CSL is the trivial lattice $\{0, I\}$. However, it follows easily from [1, Corollary 2] and the fact that all CSL's are reflexive that the converse to Proposition 2.1 does hold in the special case $\mathscr{R} = B(H)$.

A positive semigroup $\Sigma \subset \Gamma$ is said to be *generating* if the σ-algebra generated by $\{\gamma + \Sigma : \gamma \in \Gamma\}$ is the σ-algebra of Borel sets of Γ.

EXAMPLE 2.3. (1) Suppose Γ is a countable discrete group (this is true iff G is compact and second countable [18, 2.2.6]). Let $\mathscr{S}$ denote the σ-algebra generated by $\{\gamma + \Sigma : \gamma \in \Gamma\}$. Then for each $\gamma_0 \in \Gamma$,

$$\{\gamma_0\} = (\gamma_0 + \Sigma) - \Big(\bigcup_{\gamma > \gamma_0} (\gamma + \Sigma)\Big) \in \mathscr{S}.$$

Since Γ is countable, $\mathscr{S}$ contains all subsets of Γ, and so Σ is generating.

EXAMPLE 2.3. (2) Let $\Gamma = \mathbf{R}^n$, $\Sigma = [0, \infty)^n$. Then if $(a_1, \ldots, a_n) = a \in \mathbf{R}^n$, and $\delta > 0$,

$$\{(x_1, \ldots, x_n) : a_i \leqslant x_i < a_i + \delta,\ 1 \leqslant i \leqslant n\} =$$

$$= a + \Sigma - \left(\bigcup_{i=1}^{n} (a_1, \ldots, a_i + \delta, \ldots, a_n) + \Sigma\right)$$

is in the σ-algebra $\mathscr{S}$ generated by $\{\gamma + \Sigma : \gamma \in \Gamma\}$, and so $\mathscr{S}$ contains all open sets in $\mathbf{R}^n$, and hence Σ is generating.

PROPOSITION 2.4. *Suppose $\Sigma \subset \Gamma$ is generating, $U \in \operatorname{Rep}(G, H)$ and $U \sim P$. Then $A \in \{U_g : g \in G\}'$ iff $AP(\gamma + \Sigma) = P(\gamma + \Sigma)A$ for all $\gamma \in \Gamma$.*

Proof. Since $P(\cdot)$ is a projection-valued measure on the Borel sets of Γ, $\{E : P(E)A = AP(E)\}$ is a σ-algebra. But $A \in \{U_g\}'$ iff $P(E)A = AP(E)$ for all Borel sets E, and Σ generates, so the result follows.

THEOREM 2.5. *Let $\mathscr{L}_1, \ldots, \mathscr{L}_n$ be mutually commuting* CSL's *in a von Neumann algebra $\mathscr{R}$, and suppose $\mathscr{L}_i$ is (G_i, Σ_i)-analytic, $i = 1, 2, \ldots, n$, where each Σ_i is generating. Then $\mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$ is $\left(\prod_{i=1}^{n} G_i, \prod_{i=1}^{n} \Sigma_i\right)$-analytic.*

Proof. We will prove the theorem for $n = 2$. An easy induction argument then gives the general case. So let $\mathscr{L}_1 \sim U^{(i)} \sim P_i$, $i = 1, 2$, and let $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$, $\Sigma = \Sigma_1 \times \Sigma_2$, $\mathscr{L} = \mathscr{L}_1 \vee \mathscr{L}_2$. If $\gamma_i \in \Gamma_i$, then $P_i(\gamma_i + \Sigma_i) \in \mathscr{L}_i$, $i = 1, 2$. Since $\mathscr{L}_1$ and $\mathscr{L}_2$ commute, it follows from Proposition 2.4 that

$$U^{(1)}_{g_1} U^{(2)}_{g_2} = U^{(2)}_{g_2} U^{(1)}_{g_1}, \quad g_i \in G_i,\ i = 1, 2.$$

Let

$$W_{(g_1, g_2)} = U^{(1)}_{g_1} U^{(2)}_{g_2}.$$

Then $W \in \mathrm{Rep}(G, H)$. Moreover, if $W \sim P$, then combining [15, 2.5.2] and [16, 8.1.5] we have: if $\gamma \in \Gamma_1$, $\lambda \in \Gamma_2$, then

$$P((\gamma, \lambda) + \Sigma) = P((\gamma + \Sigma_1) \times (\lambda + \Sigma_2)) = P_1(\gamma + \Sigma_1) P_2(\lambda + \Sigma_2).$$

Hence if $\mathscr{L}_0$ is the CSL generated by $\{P(\gamma + \Sigma) : \gamma \in \Gamma\}$, then $\mathscr{L}_0 \subset \mathscr{L}$. On the other hand, for each fixed $\gamma \in \Gamma_1$, $P_1(\gamma + \Sigma_1) P_2(\lambda + \Sigma_2) \in \mathscr{L}_0$ for *all* $\lambda \in \Gamma_2$, and hence $P_1(\gamma + \Sigma_1) Q \in \mathscr{L}_0$ for all $Q \in \mathscr{L}_2$. In particular, $P_1(\gamma + \Sigma_1) \in \mathscr{L}_0$ for all $\gamma \in \Gamma_1$, so $\mathscr{L}_1 \subset \mathscr{L}_0$. Similarly, $\mathscr{L}_2 \subset \mathscr{L}_0$, and so $\mathscr{L} = \mathscr{L}_0$. Hence $W \sim \mathscr{L}$, and $\mathscr{L}$ is (G, Σ)-analytic.

THEOREM 2.6. *Let $\mathscr{L}_1, \ldots, \mathscr{L}_n$ be mutually commuting chains in a von Neumann algebra $\mathscr{R}$, and let $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$. Then there is an inner n-parameter W^*-dynamical system $(\mathscr{R}, \mathbf{R}^n, \alpha)$ such that*

$$\mathscr{R}\text{-alg}\mathscr{L} = \mathscr{R}^{\alpha}([0, \infty)^n). \tag{4}$$

Conversely, if $(\mathscr{R}, \mathbf{R}^n, \alpha)$ is an inner n-parameter dynamical system, then there are n commuting chains $\mathscr{L}_1, \ldots, \mathscr{L}_n$ in $\mathscr{R}$ such that (4) *holds for $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$.*

Proof. We will first prove the converse. So assume $(\mathscr{R}, \mathbf{R}^n, \alpha)$ is an n-parameter inner dynamical system, and suppose $\alpha = \mathrm{ad}U$, where $U_g \in \mathscr{R}$, $g \in \mathbf{R}^n$. Let $P \sim U$, and let $\Sigma = [0, \infty)^n$. For each $t \in \mathbf{R}$, and each i, $1 \leqslant i \leqslant n$, let $E_{t,i} = \{(t_1, \ldots, t_n) \in \mathbf{R}^n : t_i \geqslant t\}$. Then $(t_1, \ldots, t_n) + \Sigma = \bigcap_{i=1}^{n} E_{t_i, i}$, and so

$$P((t_1, \ldots, t_n) + \Sigma) = \prod_{i=1}^{n} P(E_{t_i, i}). \tag{5}$$

Let $\mathscr{L}_i$ be the subspace lattice generated by $\{P(E_{t,i}) : t \in \mathbf{R}\}$, $1 \leqslant i \leqslant n$. Then the $\mathscr{L}_i$ are commuting chains in $\mathscr{R}$, and if we let $\mathscr{L}$ be the CSL generated by $\{P(\gamma + \Sigma) : \gamma \in \mathbf{R}^n\}$, then $\mathscr{L} \subset \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$ by (5). On the other hand, the identity operator I is the strong limit of $P(E_{m,i})$ as $m \to -\infty$, $1 \leqslant i \leqslant n$, so another application of (5) shows that $\mathscr{L}_i \subset \mathscr{L}$ for all i. Hence $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$, and (4) follows from Proposition 2.1.

Now suppose $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$, where the $\mathscr{L}_i$ are commuting chains. To prove (4), it suffices to show that $\mathscr{L}$ is $(\mathbf{R}^n, \Sigma)$-analytic. It is well known (see, e.g. [11] or [10, Theorem 1.1, p. 214]) that to each $\mathscr{L}_i$ there is associated a projection-valued measure $P_i(\cdot)$ on $\mathbf{R}$, with support in $[0, 1]$, such that $\mathscr{L}_i$ coincides with $\{P_i([\lambda, \infty)) : \lambda \in \mathbf{R}\}$. Hence each $\mathscr{L}_i$ is $(\mathbf{R}, [0, \infty))$-analytic, and so $\mathscr{L}$ is $(\mathbf{R}^n, \Sigma)$-analytic as required.

REMARK 2.7. If $\mathscr{L}_1, \ldots, \mathscr{L}_n$ are commuting chains in $\mathscr{R}$, then we can find $U^{(i)} \in \mathrm{Rep}(\mathbf{R}, H)$, $1 \leqslant i \leqslant n$, such that $U^{(i)} \sim \mathscr{L}_i$ and $\mathrm{sp}(U^{(i)}) \subset [0, 1]$ (where $\mathrm{sp}(U^{(i)})$ is the spectrum of $U^{(i)}$, i.e. the support of the projection-valued measure associated with $U^{(i)}$). Then with $U_{(t_1, \ldots, t_n)} = U^{(1)}_{t_1} U^{(2)}_{t_2} \ldots U^{(n)}_{t_n}$, $U \sim \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$ by the proof of Theorem 2.5, and $\mathrm{sp}(U) \subset [0, 1]^n$. Hence, if $\alpha = \mathrm{ad}U$, then α is uniformily continuous, and so we can find an *uniformily* continuous dynamical system such that (4) holds.

3. A REPRESENTATION THEOREM FOR ANALYTIC CSL'S

If $(X, \leqslant, m)$ is a standard, partially ordered measure space [2], let $L(X, \leqslant)$ denote the lattice of increasing Borel sets of $(X, \leqslant)$, and let $\mathscr{L}(X, \leqslant, m) = \{P_E : E \in L(X, \leqslant)\}$, where P_E is the operator on $L^2(X, m)$ which multiplies functions in $L^2(X, m)$ by the characteristic function of E. Arveson showed in [2] that if $\mathscr{L}$ is a CSL in $B(H)$, then there is a partially ordered measure space $(X, \leqslant, m)$ such that $\mathscr{L}$ is unitarily equivalent to $\mathscr{L}(X, \leqslant, m)$. In this section we show that if $\mathscr{L}$ is (G, Σ)-analytic (with Σ generating and Γ second countable) and multiplicity free, and if $U \sim \mathscr{L}$, then we can take X to be $\mathrm{sp}(U)$, $\leqslant$ the partial order induced on X by Σ, and m to be a finite positive Borel measure on X. This generalizes a result of Laurie for finite width multiplicity free CSL's.

LEMMA 3.1. *Let $\Sigma \subset \Gamma$ be a positive semigroup, with Γ second countable. Let $\gamma_1, \gamma_2, \ldots$ be a dense subset of Γ, and $E_i = \gamma_i + \Sigma$, $i = 1, 2, \ldots$. Then $\gamma \leqslant \lambda$ iff $\chi_{E_i}(\gamma) \leqslant \chi_{E_i}(\lambda)$, $i = 1, 2, \ldots$.*

Proof. Suppose $\gamma, \lambda \in \Gamma$ and $\gamma \nleqslant \lambda$, i.e. $\lambda \notin \gamma + \Sigma$. Since $\gamma + \Sigma$ is closed, we can find a neighborhood W of 0 in Γ such that $\overline{\lambda + W - W} \cap \gamma + \Sigma = \emptyset$. Then $\lambda + W \cap \gamma + \Sigma + W = \emptyset$. Since $0 \in \overline{\mathrm{int}\Sigma}$, $(-W) \cap \mathrm{int}\Sigma \neq \emptyset$, so $\gamma + W \cap \gamma - \mathrm{int}\Sigma$ is a nonempty open set, and thus $\gamma_i \in \gamma + W \cap \gamma - \mathrm{int}\Sigma$ for some i.

Hence $\gamma \in \gamma_i + \text{int}\Sigma \subset \gamma_i + \Sigma$, and $\chi_{E_i}(\gamma) = 1$. Moreover, $\gamma_i \in \gamma + W \subset \gamma + \Sigma + W$, so $\lambda \notin \gamma_i + \Sigma$. Thus $0 = \chi_{E_i}(\lambda) < \chi_{E_i}(\gamma)$. Therefore, if $\chi_{E_i}(\gamma) \leqslant \chi_{E_i}(\lambda)$ for all i, then $\gamma \leqslant \lambda$. Conversely, if $\gamma \leqslant \lambda$, and $\chi_{E_i}(\gamma) = 1$, then $\gamma_i \leqslant \gamma \leqslant \lambda$, so $\chi_{E_i}(\lambda) = 1$. Hence $\chi_{E_i}(\gamma) \leqslant \chi_{E_i}(\lambda)$ for all i.

REMARK 3.2. Let m be Haar measure on Γ, and $\leqslant$ denote the ordering induced by Σ. Then it follows from Lemma 3.1 and [2, 1.2.2] that $\mathscr{L}(\Gamma, \leqslant, m)$ is generated as a subspace lattice by $\{P_{E_i} : i = 1, 2, \ldots\}$.

THEOREM 3.3. *Let* $\mathscr{L} \subset B(H)$ *be a* (G, Σ)*-analytic* CSL, *with* Σ *generating and* Γ *second countable. Suppose* $U \sim \mathscr{L}$, $X = \text{sp}(U)$, $\mathscr{M} = \mathscr{L}''$, v *is a unit separating vector for* $\mathscr{M}$, *and* Q *is the projection onto* $K = [\mathscr{M}v]$. *Then there is a finite, positive, regular Borel measure* m *on* X, *and a unitary operator* $V : L^2(X, m) \to K$ *such that*

$$V\mathscr{L}(X, \leqslant, m)V^{-1} = \mathscr{L}Q,$$

where $x \leqslant y$ *iff* $y - x \in \Sigma$ $(x, y \in X)$.

Proof. Let $P \sim U$, and for $u, w \in H$, define $m_{u,w}$ on X by $m_{u,w}(E) = (P(E)u, w)$, E Borel. Let $m = m_{v,v}$. Then m is a positive, regular Borel measure, and $m(X) = \|v\|^2 = 1$. Moreover, if E is Borel, $m(E) = 0$ iff $P(E) = 0$, since v is separating for $\mathscr{M}$. (Note that by Proposition 2.4, $\mathscr{M} = \{U_g : g \in G\}''$.) Hence the formula

$$(T(f)\,u, w) = \int_X f \,\mathrm{d}m_{u,w} \qquad (u \in H,\ w \in H)$$

defines an isometric $*$-isomorphism T of the Banach algebra $L^\infty(X, m)$ onto a closed normal subalgebra $\mathscr{A}$ of $B(H)$ [19, 12.21]. Define V on $L^\infty(X, m) \subseteq L^2(X, m)$ by $Vf = T(f)v$. Then $\|T(f)v\|^2 = \int_X |f|^2\,\mathrm{d}m$, so V extends to an isometry of $L^2(X, m)$ into H. Moreover, $A \in \mathscr{A}'$ iff $A \in \{P(E) : E \text{ is Borel}\}'$ iff $A \in \mathscr{L}'$, so $\mathscr{A}'' = \mathscr{L}'' = \mathscr{M}$, and hence the range of V is $K = [\mathscr{M}v]$. Routine calculations show that $T(\chi_E) = P(E)$, E Borel, and $VL_fV^{-1} = T(f)Q$, $f \in L^\infty(X, m)$, where L_f denotes the operator "multiplication by f", and $T(f)Q$ is looked at as an operator on K. Hence $VP_EV^{-1} = P(E)Q$, E Borel, and so

$$\mathscr{L}_0 \equiv V\mathscr{L}(X, \leqslant, m)V^{-1} = \{P(E)Q : E \in L(X, \leqslant)\}.$$

Since $\mathscr{L}_0$ is a CSL, $\{P(\gamma + \Sigma) : \gamma \in \Gamma\}$ generates $\mathscr{L}$, and $P(\gamma + \Sigma) = P((\gamma + \Sigma) \cap X)$, $\gamma \in \Gamma$, we have $\mathscr{L}Q \subset \mathscr{L}_0$. Let $\gamma_1, \gamma_2, \ldots$ be a dense subset of Γ, and let $F_i = (\gamma_i + \Sigma) \cap X$. Then for $x, y \in X$, $x \leqslant y$ iff $\chi_{F_i}(x) \leqslant \chi_{F_i}(y)$ (Lemma 3.1), and so $\{P_{F_i} : i = 1, 2, \ldots\}$ generates $\mathscr{L}(X, \leqslant, m)$ as a subspace lattice [2, 1.2.2].

Hence $\{VP_{F_i}V^{-1} : i = 1, 2, \ldots\}$ generates $\mathscr{L}_0$. But $VP_{F_i}V^{-1} = P(F_i)Q = P(\gamma_i \times \Sigma)Q$, so $\mathscr{L}_0 \subset \mathscr{L}Q$. Hence $\mathscr{L}_0 = \mathscr{L}Q$.

Note that if $\mathscr{L}$ is multiplicity free (so $\mathscr{M}$ is a m.a.s.a.), every separating vector for $\mathscr{M}$ is cyclic, and so $\mathscr{L}$ is unitarily equivalent to $\mathscr{L}(X, \leqslant, m)$.

COROLLARY 3.4. ([13, 3.3(2)]) *Let $\mathscr{L} = \mathscr{L}_1 \vee \ldots \vee \mathscr{L}_n$ be a finite width multiplicity free* CSL. *Then there exists a finite Borel measure m on $[0, 1]^n$ and a unitary operator $V : L^2([0, 1]^n, m) \to H$ such that $V\mathscr{L}([0, 1]^n, \leqslant, m)V^{-1} = \mathscr{L}$, where $\leqslant$ is the product partial ordering on $[0, 1]^n$.*

Proof. By Remark 2.7, $\mathscr{L} \sim U$, with $\mathrm{sp}(U) \subset [0, 1]^n$. Apply Theorem 3.3, and set $m = 0$ on the complement of $\mathrm{sp}(U)$.

4. TENSOR PRODUCTS

In this section we will study tensor products of CSL's and reflexive algebras using the results of Section 2. If $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras on Hilbert spaces H_1 and H_2, let $H = H_1 \otimes H_2$ be the Hilbert space tensor product of H_1 and H_2, and $\mathscr{R} = \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ be the von Neumann algebra tensor product of $\mathscr{M}$ and $\mathscr{N}$. If $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ are subspace lattices, let $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2$ be the subspace lattice generated by the elementary tensors $\{P_1 \otimes P_2 : P_i \in \mathscr{L}_i\}$. If $\mathscr{A} \subset \mathscr{M}$ and $\mathscr{B} \subset \mathscr{N}$ are linear subspaces of $\mathscr{M}$ and $\mathscr{N}$, let $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$ denote the σ-weak closure (in $\mathscr{R}$) of the algebraic tensor product $\mathscr{A} \otimes \mathscr{B}$.

QUESTION 1. For which subspace lattices $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ do we have

$$\mathscr{R}\text{-alg}\mathscr{L} = \mathscr{M}\text{-alg}\mathscr{L}_1 \mathbin{\overline{\otimes}} \mathscr{N}\text{-alg}\mathscr{L}_2? \tag{6}$$

It is shown in [7] that (6) holds if $\mathscr{M} = B(H_1)$, $\mathscr{N} = B(H_2)$, and $\mathscr{L}_1$ and $\mathscr{L}_2$ are chains. As noted in [7], if $\mathscr{M} = B(H_1)$, $\mathscr{N} = B(H_2)$, and $\mathscr{L}_1$ and $\mathscr{L}_2$ are the projection lattices of von Neumann algebras $\mathscr{M}_1$ and $\mathscr{N}_1$, (6) becomes

$$(\mathscr{M}_1 \mathbin{\overline{\otimes}} \mathscr{N}_1)' = \mathscr{M}_1' \mathbin{\overline{\otimes}} \mathscr{N}_1',$$

which is just the statement of Tomita's commutation theorem for tensor products of von Neumann algebras. More generally, if we assume $\mathscr{L}_1$ and $\mathscr{L}_2$ are the projection lattices of von Neumann subalgebras $\mathscr{M}_1 \subset \mathscr{M}$ and $\mathscr{N}_1 \subset \mathscr{N}$, (6) becomes

$$(\mathscr{M}_1 \mathbin{\overline{\otimes}} \mathscr{N}_1)' \cap (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}) = (\mathscr{M}_1' \cap \mathscr{M}) \mathbin{\overline{\otimes}} (\mathscr{N}_1' \cap \mathscr{N}).$$

This relative commutant version of Tomita's theorem is also valid ([21, p. 167]). However, all of the proofs of Tomita's theorem use self-adjointness in an essential way, and so shed no light on the general validity of (6). A complete answer to

Question 1 is apparently quite difficult. The main result of this section is that (6) holds for a special class of subspace lattices: the CSL's with totally atomic cores. Note that any chain generated by an increasing sequence of projections has totally atomic core (although the converse is not true), and so, in particular, (6) is valid for any finite width CSL generated by chains of this form. Also note that if $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ are chains with totally atomic cores, then it follows from (6) that the tensor product of the associated nest-subalgebras is $\mathscr{R}$-reflexive.

Now suppose $\mathscr{L}_1 \subset \mathscr{M}$, $\mathscr{L}_2 \subset \mathscr{N}$ are CSL's, and $\mathscr{L}_i$ is (G_i, Σ_i)-analytic, $i = 1, 2$. If $U \sim \mathscr{L}_1$ and $V \sim \mathscr{L}_2$, define $W \in \mathrm{Rep}(G, H)$ by

$$W_{(g_1, g_2)} = U_{g_1} \otimes V_{g_2} \quad ((g_1, g_2) \in G_1 \times G_2 = G).$$

Let $\alpha = \mathrm{ad}U$, $\beta = \mathrm{ad}V$, and $\alpha \otimes \beta = \mathrm{ad}W$. Then $(\mathscr{R}, G, \alpha \otimes \beta)$ is an inner W^*-dynamical system. Note that

$$(\alpha \otimes \beta)_{(g_1, g_2)}(A \otimes B) = \alpha_{g_1}(A) \otimes \beta_{g_2}(B) \quad (g_i \in G_i\,,\ A \in \mathscr{M},\ B \in \mathscr{N}).$$

THEOREM 4.1. *Let $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ be CSL's, and suppose $\mathscr{L}_i$ is (G_i, Σ_i)-analytic, $i = 1, 2$. Let $G = G_1 \times G_2$, $\Sigma = \Sigma_1 \times \Sigma_2$, $U \sim \mathscr{L}_1$, $V \sim \mathscr{L}_2$, $\alpha \otimes \beta = \mathrm{ad}(U \otimes V)$. Then $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2$ is (G, Σ)-analytic, $\mathscr{L} \sim U \otimes V$, and*

$$\mathscr{R}^{\alpha\otimes\beta}(\Sigma) = \mathscr{R}\text{-alg}\mathscr{L}. \tag{7}$$

Proof. By Proposition 2.1, it suffices to show that $\mathscr{L} \sim U \otimes V$. Let $P_1 \sim U$, $P_2 \sim V$. It is not hard to check: if $P \sim U \otimes V$, then $P(E_1 \times E_2) = P_1(E_1) \otimes P_2(E_2)$, $E_i \subset G_i$ Borel. An argument similar to that in the proof of Theorem 2.5 shows that $\mathscr{L}$ is the subspace lattice generated by

$$\{P((\lambda, \gamma) + \Sigma) : (\lambda, \gamma) \in \Gamma_1 \times \Gamma_2\} =$$

$$= \{P_1(\lambda + \Sigma_1) \otimes P_2(\gamma + \Sigma_2) : (\lambda, \gamma) \in \Gamma_1 \times \Gamma_2\},$$

i.e. that $\mathscr{L} \sim U \otimes V$.

Thus in the case of (G, Σ)-analytic CSL's, Question 1 is a special case of:

QUESTION 2. Let $(\mathscr{M}, G_1, \alpha), (\mathscr{N}, G_2, \beta)$ be W^*-dynamical systems, Σ_i a positive semigroup in Γ_i, $i = 1, 2$, and let $\mathscr{R} = \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$, $\Sigma = \Sigma_1 \times \Sigma_2$. When do we have

$$\mathscr{R}^{\alpha\otimes\beta}(\Sigma) = \mathscr{M}^{\alpha}(\Sigma_1) \mathbin{\overline{\otimes}} \mathscr{N}^{\beta}(\Sigma_2)? \tag{8}$$

(Note that we can define $\alpha \otimes \beta$ whether or not α and β are inner. For if $g_1 \in G_1$ and $g_2 \in G_2$, there is a unique $*$-automorphism $\alpha_{g_1} \otimes \beta_{g_2}$ of $\mathscr{R}$ such that

$$(\alpha_{g_1} \otimes \beta_{g_2})(A \otimes B) = \alpha_{g_1}(A) \otimes \beta_{g_2}(B), \quad A \in \mathscr{M},\ B \in \mathscr{N}$$

[5, p. 56] and it can be verified that $(\mathscr{R}, G, \alpha \otimes \beta)$ is a W^*-dynamical system.)

If follows from [7, Theorem 2.6] and [14, Corollary 4.2.4] that (8) is valid when $\mathcal{M} = B(H_1)$, $\mathcal{N} = B(H_2)$, $G_1 = G_2 = \mathbf{R}$ and $\Sigma_1 = \Sigma_2 = [0, \infty)$. We will show that if G_1 and G_2 are compact, then (8) always holds. We would first like to note that the right-hand side of (8) is always contained in the left-hand side. This can be seen as follows: If G_1 and G_2 are locally compact abelian groups, and if $f_i \in L^1(G_i)$, $i = 1, 2$, we can define $f_1 \otimes f_2 \in L^1(G_1 \times G_2)$ by

$$(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1)f_2(g_2).$$

If $(\mathcal{M}, G_1, \alpha)$ and $(\mathcal{N}, G_2, \beta)$ are W^*-dynamical systems, a computation shows that

$$(\varphi \otimes \psi)([(\alpha \otimes \beta)(f_1 \otimes f_2)](A \otimes B)) =$$

$$= (\varphi \otimes \psi)(\alpha(f_1)(A) \otimes \beta(f_2)(B))$$

for all $\varphi \in \mathcal{M}_*$, $\psi \in \mathcal{N}_*$, $A \in \mathcal{M}$, $\beta \in \mathcal{N}$. Since the algebraic tensor product $\mathcal{M}_* \otimes \mathcal{N}_*$ is norm dense in $(\mathcal{M} \bar{\otimes} \mathcal{N})_*$, and $\mathcal{M} \otimes \mathcal{N}$ is σ-weakly dense in $\mathcal{M} \bar{\otimes} \mathcal{N}$, we conclude:

(9) $$(\alpha \otimes \beta)(f_1 \otimes f_2) = \alpha(f_1) \otimes \beta(f_2) \quad (f_i \in L^1(G_i)).$$

Hence if $A \in \mathcal{M}$ and $B \in \mathcal{N}$,

$$\mathrm{sp}_{\alpha\otimes\beta}(A \otimes B) \subseteq \mathrm{sp}_\alpha(A) \times \mathrm{sp}_\beta(B),$$

from which it follows that

(10) $$\mathcal{M}^\alpha(E) \bar{\otimes} \mathcal{N}^\beta(F) \subset \mathcal{R}^{\alpha\otimes\beta}(E \times F), \quad (E \subset \Gamma_1, F \subset \Gamma_2).$$

Let $(\mathcal{R}, G, \alpha)$ be a W^*-dynamical system, with G a compact abelian group. Let $e_\gamma(g) = \overline{(g, \gamma)}$, $g \in G$, $\gamma \in \Gamma$, and set $p_\gamma = \alpha(e_\gamma)$. It is easily verified that

$$p_\gamma \mathcal{R} = \mathcal{R}^\alpha(\gamma) = \{A \in \mathcal{R} : \alpha_g(A) = (g, \gamma)A, \ g \in G\},$$

that $p_\gamma^2 = p_\gamma$ and $p_\gamma p_\lambda = 0$, $\gamma \neq \lambda$. Moreover if E is any subset of Γ, then

(11) $$\mathcal{R}^\alpha(E) = \bigvee_{\gamma\in E} \mathcal{R}^\alpha(\gamma)$$

where $\bigvee$ denotes the σ-weakly closed linear span [15, 2.3.4 (ii)].

THEOREM 4.3. *Let $(\mathcal{M}, G_1, \alpha)$ and $(\mathcal{N}, G_2, \beta)$ be W^*-dynamical systems where G_1 and G_2 are compact abelian groups, and let $\mathcal{R} = \mathcal{M} \otimes \mathcal{N}$. If $E \subset \Gamma_1$ and $F \subset \Gamma_2$, then*

(12) $$\mathcal{R}^{\alpha\otimes\beta}(E \times F) = \mathcal{M}^\alpha \bar{\otimes}(E) \ \mathcal{N}^\beta(F).$$

Proof. Let $\gamma \in \Gamma_1$, $\lambda \in \Gamma_2$. Then $e_{(\gamma,\lambda)} = e_\gamma \otimes e_\lambda$, so by (9), $p_{(\gamma,\lambda)} = p_\gamma \otimes p_\lambda$. Hence

$$\mathcal{R}^{\alpha\otimes\beta}((\gamma, \lambda)) = p_{(\gamma,\lambda)}\mathcal{R} = \mathcal{M}^\alpha(\gamma) \otimes \mathcal{N}^\beta(\lambda),$$

so $\mathscr{R}^{\alpha\otimes\beta}((\gamma, \lambda)) \subset \mathscr{M}^{\alpha}(E) \overline{\otimes} \mathscr{N}^{\beta}(F)$ for all $\gamma \in E$, $\lambda \in F$, and thus, by (11), $\mathscr{R}^{\alpha\otimes\beta}(E \times F) \subset \subset \mathscr{M}^{\alpha}(E) \overline{\otimes} \mathscr{N}^{\beta}(F)$. Combining this with (10), we get (12).

In order to apply Theorem 4.3 to CSL's, we need to know which CSL's are (G, Σ)-analytic for some compact abelian group G. First note that if $U \in \mathrm{Rep}(G, H)$, and $U \sim P$, then, since Γ is discrete,

$$U_g = \sum_{\gamma \in \Gamma} (g, \gamma) P_\gamma \tag{13}$$

where $P_\gamma = P(\{\gamma\})$, $\gamma \in \Gamma$, and the sum in (13) converges in the strong topology Hence the abelian von Neumann algebra $\{U_g : g \in G\}''$ is totally atomic, with atoms $\{P_\gamma : P_\gamma \neq 0\}$. If $U \sim \mathscr{L}$ and Σ is generating, then $\mathscr{L}'' = \{U_g\}''$, so a necessary condition for $\mathscr{L}$ to be (G, Σ)-analytic, G compact abelian, and Σ generating, is that $\mathscr{L}''$ be totally atomic. This necessary condition is also sufficient.

THEOREM 4.4. *Let $\mathscr{L} \subset \mathscr{R}$ be a* CSL *with totally atomic core. Then there is a second countable compact abelian group G, and a positive semigroup Σ in the dual group of G, such that $\mathscr{L}$ is (G, Σ)-analytic.*

Proof. Let $\{E_j\}_{j \in J}$ be the atoms of $\mathscr{L}''$. We may assume $J = \{1, 2, \ldots\}$ (possibly finite), since $\mathscr{R}$ is separably acting. For each $j \in J$, let $P_j = \bigwedge \{P \in \mathscr{L} : PE_j = E_j\}$. Since $\mathscr{L}$ is complete, each P_j is in $\mathscr{L}$, and if $P \in \mathscr{L}$, either $PE_j = 0$ or $P \geqslant P_j$. Moreover, if $j, k \in J$, $j \neq k$, then $P_j \neq P_k$. (For suppose $P_j = P_k$, $j \neq k$. Then for any $P \in \mathscr{L}$, either $E_j \vee E_k \leqslant P$ or $E_j \vee E_k \leqslant I - P$, so any nonzero partial isometry in $B(H)$ with initial space in E_j and final space in E_k commutes with $\mathscr{L}$ but does not commute with E_j, contradicting $E_j \in \mathscr{L}''$.) If $P \in \mathscr{L}$, then $P = \sum_{j \in J} PE_j$, and so $P = \bigvee \{P_j : PE_j \neq 0\}$. Hence $\mathscr{L}$ is generated as a subspace lattice by $\{P_j : j \in J\}$.

Let $G = \prod_{j \in J} T_j$, where each T_j is a copy of the circle group $T = \{z \in \mathbf{C} : |z| = 1\}$. Then the dual Γ_j of T_j is $\mathbf{Z}$, and the dual Γ of G is $\prod_{j \in J}{}^* \Gamma_j$, i.e., the set of all $(n_j) \in \prod_{j \in J} \Gamma_j$ such that $n_j = 0$ for all but a finite number of indices ([18, 2.2.3]). For $j \in J$, let $\gamma_j = (0, \ldots, 0, 1, 0, \ldots) \in \Gamma$ (where the 1 is in the j^{th} place). Let $Q_{\gamma_j} = E_j$, $j \in J$, and $Q_\gamma = 0$, $\gamma \notin \{\gamma_j : j \in J\}$. Then $\sum_{\gamma \in \Gamma} Q_\gamma = I$, and so we can define a projection-valued measure on Γ (with values in $\mathscr{L}''$) by setting $Q(E) = \sum_{\gamma \in E} Q_\gamma$, $(E \subset \Gamma)$. Let Σ denote the set of all finite sums of the form $\sum_{i=1}^{n} \gamma_{\varphi(i)} - \gamma_{\psi(i)}$, where $\varphi(i), \psi(i) \in J$ and $P_{\varphi(i)} \leqslant P_{\psi(i)}$, $1 \leqslant i \leqslant n$. We will show that Σ is a positive semigroup, and $\{Q(\gamma + \Sigma) : \gamma \in \Gamma\}$ generates $\mathscr{L}$.

Claim. $\gamma_{j_1} - \gamma_{j_2} \in \Sigma$ if and only if $P_{j_1} \leqslant P_{j_2}$ $(j_1, j_2 \in J)$.

The "if" direction of the claim is immediate. So suppose $\gamma_{j_1} - \gamma_{j_2} \in \Sigma$, $j_1 \neq j_2$. Then we can find $\varphi(i)$ and $\psi(i)$ in J $(1 \leqslant i \leqslant n)$ such that $P_{\varphi(i)} \leqslant P_{\psi(i)}$, and $\gamma_{j_1} - \gamma_{j_2} = \sum_{i=1}^{n} \gamma_{\varphi(i)} - \gamma_{\psi(i)}$. Note that if $\psi(i) = \varphi(k)$ for some i and k, then

$$\gamma_{\varphi(i)} - \gamma_{\psi(i)} + \gamma_{\varphi(k)} - \gamma_{\psi(k)} = \gamma_{\varphi(i)} - \gamma_{\psi(k)},$$

and

$$P_{\varphi(i)} \leqslant P_{\psi(i)} = P_{\varphi(k)} \leqslant P_{\psi(k)},$$

so we may assume that $J_1 \cap J_2 = \emptyset$, where J_1 is the range of φ, and J_2 is the range of ψ. Then $\sum_{i=1}^{n} \gamma_{\varphi(i)} - \gamma_{\psi(i)} = (m_j)$, where $m_j = \operatorname{card}(\varphi^{-1}(j))$ if $j \in J_1$, $m_j = -\operatorname{card}(\psi^{-1}(j))$ if $j \in J_2$, and $m_j = 0$ otherwise. But $\gamma_{j_1} - \gamma_{j_2} = (n_j)$, where $n_{j_1} = 1$, $n_{j_2} = -1$ and $n_j = 0$ otherwise, and so we must have $n = 1$, $\varphi(1) = j_1$ and $\psi(1) = j_2$. Hence $P_{j_1} = P_{\varphi(1)} \leqslant P_{\psi(1)} = P_{j_2}$, as claimed.

If follows easily from the claim that $\Sigma \cap (-\Sigma) = \{0\}$. But Σ is a semigroup by definition, and $\Sigma = \overline{\operatorname{int} \Sigma}$ (Γ is discrete), so Σ is a positive semigroup. Let $\mathscr{L}_0$ be the subspace lattice generated by $\{Q(\gamma + \Sigma) : \gamma \in \Gamma\}$. If $\gamma \in \Gamma$, then

$$Q(\gamma + \Sigma) = \bigvee \{E_j : \gamma_j \in \gamma + \Sigma\} \leqslant \bigvee \{P_j : \gamma_j \in \gamma + \Sigma\}.$$

Moreover, if $\gamma_k \in \gamma + \Sigma$, and $E_j \leqslant P_k$, then $P_j \leqslant P_k$, and so $\gamma_j \in \gamma_k + \Sigma \subset \gamma + \Sigma$. Hence $P_k = \bigvee \{E_j : E_j \leqslant P_k\} \leqslant Q(\gamma + \Sigma)$, and so $Q(\gamma + \Sigma) = \bigvee \{P_j : \gamma_j \in \gamma + \Sigma\} \in \mathscr{L}$. On the other hand, $\gamma_j \in \gamma_k + \Sigma$ if and only if $P_j \leqslant P_k$, and so $Q(\gamma_k + \Sigma) = P_k$. Since $\mathscr{L}$ is generated by $\{P_j : j \in J\}$, we conclude that $\mathscr{L}_0 = \mathscr{L}$, and thus $\mathscr{L}$ is (G, Σ)-analytic.

REMARK 4.5. In particular, if $\mathscr{L}$ is a chain generated by a doubly-indexed sequence of projections $\{P_i\}_{i=-\infty}^{\infty}$, with $P_i \leqslant P_{i+1}$, $i \in \mathbf{Z}$, $\bigwedge_i P_i = 0$, and $\bigvee_i P_i = I$, then $\mathscr{L}$ is (G, Σ)-analytic for some compact group G. In this case though we can say more: $\mathscr{L}$ is (T, N)-analytic, where T is the circle group, and $N = \{0, 1, 2, \ldots\}$. For if we let $Q_{-n} = P_n - P_{n-1}$, $n \in \mathbf{Z}$, and let $U_z = \sum_{n=-\infty}^{\infty} z^n Q_n$, $z \in T$, then $P_{-n} = \sum_{m=n}^{\infty} Q_m$, and so $\mathscr{L} \sim U$.

THEOREM 4.6. *Let $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ be* CSL*'s with totally atomic cores, and let $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2$, $\mathscr{R} = \mathscr{M} \overline{\otimes} \mathscr{N}$. Then*

$$\mathscr{R}\text{-alg}\mathscr{L} = \mathscr{M}\text{-alg}\mathscr{L}_1 \overline{\otimes} \mathscr{N}\text{-alg}\mathscr{L}_2. \tag{14}$$

Proof. Apply Theorems 4.4, 4.1, and 4.3 and Proposition 2.1.

COROLLARY 4.7. *Suppose* $\mathscr{L}_1 \subset \mathscr{M}$ *and* $\mathscr{L}_2 \subset \mathscr{N}$ *are finite* CSL's. *Then* (14) *holds.*

5. FURTHER QUESTIONS

Question 2 is a special case of:

QUESTION 3. Let $(\mathscr{M}, G_1, \alpha)$ and $(\mathscr{N}, G_2, \beta)$ be W^*-dynamical systems, and let $\mathscr{R} = \mathscr{M} \overline{\otimes} \mathscr{N}$. For $E \subset \Gamma_1$, and $F \subset \Gamma_2$, when do we have

$$\mathscr{R}^{\alpha\otimes\beta}(E \times F) = \mathscr{M}^\alpha(E) \overline{\otimes} \mathscr{N}^\beta(F)? \tag{15}$$

By (10), the right-hand side of (15) is always contained in the left-hand side, and Theorem 4.3 states that we have equality when G_1 and G_2 are compact. The author can also show that (15) is valid whenever E and F are both open sets. In [20], Tomiyama proved that (15) holds if $E = \{0\}$ and $F = \{0\}$, i.e., the fixed-point algebra of $\alpha \otimes \beta$ is the tensor product of the fixed-point algebras of α and β. His proof uses slice map techniques. If $\varphi \in B(H_1)_*$, the right slice mapping induced by φ is the σ-weakly continuous linear map from $B(H_1) \overline{\otimes} B(H_2)$ to $B(H_2)$ defined on elementary tensors by $R_\varphi(\sum A_i \otimes B_i) = \sum \varphi(A_i)B_i$. The left slice mappings L_ψ ($\psi \in B(H_2)_*$) are similarly defined [20]. If $\mathscr{S} \subset B(H_1)$ and $\mathscr{T} \subset B(H_2)$ are σ-weakly closed subspaces, let

$$F(\mathscr{S}, \mathscr{T}) = \{A \in B(H_1)\overline{\otimes}B(H_2) : R_\varphi(A) \in \mathscr{T}, L_\psi(A) \in \mathscr{S}, \text{for all } \varphi \in B(H_1)_*, \psi \in B(H_2)_*\}.$$

If $\mathscr{S}$ and $\mathscr{T}$ are von Neumann algebras, then $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \overline{\otimes} \mathscr{T}$ [20, Theorem 2.1]. (This result is equivalent to Tomita's Theorem.) We always have $\mathscr{S} \overline{\otimes} \mathscr{T} \subset F(\mathscr{S}, \mathscr{T})$, and we can ask

QUESTION 4. Let $\mathscr{S} \subset B(H_1)$ and $\mathscr{T} \subset B(H_2)$ be σ-weakly closed subspaces. When do we have

$$\mathscr{S} \overline{\otimes} \mathscr{T} = F(\mathscr{S}, \mathscr{T})? \tag{16}$$

If $(\mathscr{M}, G_1, \alpha)$ and $(\mathscr{N}, G_2, \beta)$ are W^*-dynamical systems, one can show

$$\mathscr{R}^{\alpha\otimes\beta}(E \times F) \subset F(\mathscr{M}^\alpha(E), \mathscr{N}^\beta(F)) \quad (E \subset \Gamma_1, F \subset \Gamma_2). \tag{17}$$

One can also show: If $\mathscr{L}_1 \subset \mathscr{M}$ and $\mathscr{L}_2 \subset \mathscr{N}$ are subspace lattices, then

$$\mathscr{R}\text{-alg}\mathscr{L} = F(\mathscr{M}\text{-alg}\mathscr{L}_1,\ \mathscr{N}\text{-alg}\mathscr{L}_2). \tag{18}$$

Hence the universal validity of (16) would imply the universal validity of (6) and (15). The author can show that if $\mathscr{S}$ is any spectral subspace of a dynamical system $(B(H_1), G, \alpha)$, where α is inner, and G is compact, then (16) is valid for any $\mathscr{T}$. It then follows from (17) and (10) that (15) is valid whenever $\mathscr{M} = B(H_1)$, G_1 is compact and α is inner. It also follows from (18) and Theorem 4.4 that (6) is valid whenever $\mathscr{M} = B(H_1)$, $\mathscr{L}_1$ is a CSL with totally atomic core, and $\mathscr{L}_2$ is an *arbitrary* subspace lattice. Proofs of these results will appear in [12].

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JON KRAUS
Department of Mathematics,
State University of New York at Buffalo,
Buffalo, New York 14214,
U.S.A.

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