J. OPERATOR THEORY
8(1982), 319–326

ABSOLUTE VALUES OF COMPLETELY HYPONORMAL OPERATORS

C. R. PUTNAM

1. INTRODUCTION

Only bounded operators on a separable Hilbert space H will be considered. If A is selfadjoint with the spectral resolution $A = \int t \mathrm{d}E_t$ then A is said to be absolutely continuous if $\|E_t x\|^2$ is an absolutely continuous function of t for all x in H. A similar notion holds for unitary operators. An operator $T = A + \mathrm{i}B$ (A, B selfadjoint) is said to be hyponormal if

$$(1.1)\qquad T^*T - TT^* = D \geqslant 0, \text{ equivalently, } AB - BA = -\mathrm{i}C \ \left(C = \frac{1}{2} D \geqslant 0\right),$$

and completely hyponormal if, in addition, T has no normal part. (See [1] for a recent survey of hyponormal operators.) If T is completely hyponormal then $A = \mathrm{Re}(T)$ (as well as $B = \mathrm{Im}(T)$) is absolutely continuous; see [2], p. 42. Further, in case T is completely hyponormal and has a polar factorization

$$(1.2)\qquad T = UP, \ U \text{ unitary}, \ P = |T| = (T^*T)^{\frac{1}{2}},$$

then U is also absolutely continuous; see [2], p. 21 and Lemma 4 of [4]. In addition, every absolutely continuous selfadjoint operator is the real part of some completely hyponormal operator and every absolutely continuous unitary operator is the unitary factor in the factorization (1.2) of some completely hyponormal operator; see [5], p. 323.

Spectral properties of the absolute value $|T| = (T^*T)^{\frac{1}{2}}$ of a completely hyponormal operator T are much less clear-cut. In the consideration of $|T|$ below it will always be supposed that T has a factorization (1.2). (It turns out that a completely hyponormal T has such a (unique) factorization if and only if $0 \notin \sigma_{\mathrm{p}}(T^*)$; see the discussion in [3].) In case $\sigma(U)$ is a proper subset of the unit circle, $|T|$ is surely absolutely continuous ([2], p. 21), but simple examples show that the latter assertion is not true in general. Thus, for example, if $\sigma(U)$ is the entire unit circle, then $\sigma(|T|)$ may consist of two points, necessarily (see below) of infinite multiplicity in $\sigma_p(|T|)$. Even if $\sigma(U)$ is the entire circle, nevertheless, under certain circumstances involving the spectrum of T, $|T|$ is absolutely continuous, or at least has an absolutely continuous part. Some of these questions have been studied in [6] and [7].

In this paper, the following necessary conditions (N) and sufficient conditions (S) in order that a selfadjoint A be the absolute value of a completely hyponormal T with a factorization (1.2) will be established.

(N) *If A is the absolute value $|T|$ of a completely hyponormal T with the factorization* (1.2) *then*

(1) $A \geq 0$ *and* $\sigma(A)$ *contains at least two points;*

(2) $0 \notin \sigma_p(A)$*;*

(3) *neither* $\max \sigma(A)$ *nor* $\min \sigma(A)$ *is in* $\sigma_p(A)$ *with finite multiplicity.*

(S) *If A satisfies* (1) *and* (2) *above and if also* (*whenever* $\sigma_p(A)$ *is nonempty*),

(3′) *neither* $\inf \sigma_p(A)$ *nor* $\sup \sigma_p(A)$ *is in* $\sigma_p(A)$ *with finite multiplicity,*

then $A = |T|$ *for some completely hyponormal T with a factorization* (1.2).

REMARKS. Just how much the sufficiency conditions (S) can be weakened, that is, how much (3′) can be moderated in the direction of (3), will remain undecided. Conceivably, the conditions (1), (2) and (3) of (N) are sufficient as well as necessary.

2. PROOF OF (N)

The first part of (1) is obvious while the second part of (1) and also (2) follow readily from the complete hyponormality of T.

In order to prove (3) note that by (1.1) and (1.2),

$$P^2 - UP^2U^* = D \geq 0. \tag{2.1}$$

Suppose that $a = \min \sigma(P)$ is an eigenvalue of P of finite multiplicity, hence a^2 is an eigenvalue of P^2 of finite multiplicity. Let

$$\Gamma = \{x : Px = ax\} = \{x : P^2x = a^2x\}.$$

If $x \in \Gamma$, then, by (2.1),

$$a^2\|x\|^2 - (UP^2U^*x, x) = (Dx, x) = \|D^{\frac{1}{2}}x\|^2$$

and, since $\min \sigma(P^2) = \min \sigma(UP^2U^*)$, it follows that $Dx = 0$. Thus, by (2.1) and $P^2x = a^2x$, $UP^2U^*x = a^2x$, that is, $P^2U^*x = a^2U^*x$, so that Γ is invariant under U^*. Since Γ is finite-dimensional then U^* has an eigenvalue, in contradiction with the absolute continuity of U (and of U^*). Similarly, by interchanging the roles of P^2 and UP^2U^* in (2.1), it is easy to see that $\max \sigma(P)$ cannot be an eigenvalue of P of finite multiplicity. This proves (N).

3. PROOF OF (S)

Note that the assertions (1), (2), (3) and (3′) of (N) and (S) for A are equivalent to the corresponding assertions for A^2. Further, if $A_1 \geqslant 0$, then $A_1 = UAU^*$, where U is unitary, holds if and only if $A_1^2 = UA^2U^*$ (for the same U). These and similar observations will often be used in the sequel.

It is convenient to collect certain facts of spectral multiplicity theory for general selfadjoint operators. For a concise summary see, e.g., [8], especially pp. 230–234. Let A be an arbitrary selfadjoint operator on H. Then there exists a subset α of $\{1,2, \ldots, \aleph_0\}$ with the property that $A = \sum_{n\in\alpha} \oplus A_n$ on $H = \sum_{n\in\alpha} \oplus H_n$, where $A_n = A \mid H_n$ is (unitarily equivalent to) mutiplication by t on $H_n = L^2(\mathrm{d}\mu_n) \oplus \oplus \ldots \oplus L^2(\mathrm{d}\mu_n)$ (n times). The positive Borel measures $\{\mu_n\}$ are pairwise mutually singular (that is, have disjoint supports) on $(-\infty, \infty)$, and each μ_n is uniquely determined modulo its equivalence class.

(I) CONTINUOUS SPECTRUM. First, consider the case where $B = A^2$ is the selfadjoint operator on $L^2(0,1)$ of multiplication by $m(t)$, where $m(t)$ is strictly increasing on $[0,1]$ and $0 \leqslant a = m(0) < b = m(1)$. In particular, $m(t)$ has no intervals of constancy and hence B has no point spectrum, although, of course, $m(t)$ need not be continuous. If $\mu(x)$ denotes the inverse of $m(t)$, so that

$$\mu(x) = \operatorname{meas}\{t : 0 \leqslant t \leqslant 1,\ m(t) < x,\ a \leqslant x \leqslant b\}, \tag{3.1}$$

then it is seen that $\mu(x)$ is continuous and that

$$(Bf, g) = \int_0^1 m(t)f(t)\overline{g}(t)\mathrm{d}t = \int_a^b xf\overline{g}\mathrm{d}\mu$$

for $f, g \in L^2(0,1)$. Thus, B is unitarily equivalent to multiplication by x on $L^2(\mathrm{d}\mu)$.

Next, define B_1 to be the selfadjoint operator on $L^2(0, 1)$ of multiplication by $m_1(t) = m(t^2)$. If μ_1 denotes the corresponding inverse function then B_1 is seen to be unitarily equivalent to multiplication by x on $L^2(\mathrm{d}\mu_1)$. It is clear that if $D = B - B_1$ then $D \geqslant 0$ and $0 \notin \sigma_p(D)$. In addition, B_1 is unitarily equivalent to B. To see this, it is enough to show that μ and μ_1 are equivalent measures. But this is clear from the fact that for a Borel set E,

$$\mu(E) = \operatorname{meas}\{t : m(t) \in E\} = \operatorname{meas}\{t^2 : m(t^2) \in E\} =$$

$$= \operatorname{meas}\{t^2 : m_1(t) \in E\}.$$

Consequently, if $\mu(E) = 0$ then

$$\mu_1(E) = \operatorname{meas}\{t : m_1(t) \in E\} = 0,$$

and consversely. Thus, there exists a unitary U for which $B_1 = UBU^*$. If $T = UP$, where $P = B^{\frac{1}{2}} = A$, then clearly T is completely hyponormal.

Conversely, if one starts with any finite, positive, continuous Borel measure on $(-\infty, \infty)$ with support contained in $[a, b]$ (and, as can be supposed, satisfying $\mu([a, b]) = 1$) then its "inverse" is of the form $m(t)$ considered above. (Since μ is continuous, $m(t)$ has no intervals of constancy.) It follows that if $B = A^2$ is the operator of multiplication by $m(t)$ on $L^2(0, 1)$ and if B_1 is defined as before then $T = UP$, where $P = B^{\frac{1}{2}} = A$, is completely hyponormal and $B = P^2$ is unitarily equivalent to multiplication by x on $L^2(d\mu)$.

In view of the structure (to within unitary equivalence) outlined at the beginning of Section 3, it is clear that by taking direct sums one can define a completely hyponormal $T = UA$ where now A is an arbitrary selfadjoint operator which satisfies (1) and (2) of (N) and which has no point spectrum.

As above, and correspondingly in the sequel, unitary equivalence will often be replaced by equality.

(II) Pure Point Spectrum. First, consider the case where A (hence $B = A^2$) of (S) has a spectrum consisting of two points. Then $\sigma(B) = \sigma_p(B) = \{a, b\}$, where $0 < a = \inf \sigma_p(B) < \sup \sigma_p(B)$ and (cf. (3')) a, b each is of infinite multiplicity. Let H denote the bilateral sequence ℓ^2 space $\{\ldots, x_{-2}, x_{-1}, (x_0), x_1, x_2, \ldots\}$. Then B can be taken to be

$$B = \mathrm{diag}(\ldots, b, b, (b), a, a, \ldots). \tag{3.2}$$

If U is the unitary bilateral shift then

$$C = UBU^* = \mathrm{diag}(\ldots, b, b, (a), a, a, \ldots)$$

and

$$B - C = D = \mathrm{diag}(\ldots, 0, (b-a), 0, \ldots) \geq 0.$$

Obviously, $T = UB^{\frac{1}{2}} = UA$ is completely hyponormal. (Note that since $a > 0$ then $0 \notin \sigma_p(A)$ and so the least space reducing T and containing the range of D is the same as the least space invariant under A, U, U^* and containing the range of D.)

Similarly, if A (hence $B = A^2$) has a pure point spectrum consisting of three points $\{a, c, b\}$ where $0 < a < c < b$ and a, b each has infinite multiplicity while c has multiplicity 1, then B can be taken to be

$$B = \mathrm{diag}(\ldots, b, b, (c), a, a, \ldots). \tag{3.3}$$

If U is the bilateral shift then

$$C = UBU^* = \mathrm{diag}(\ldots, b, b, c, (a), a, \ldots)$$

and, as before, $T = UA$ is completely hyponormal.

It is now easy to treat the case where A (hence $B = A^2$) has a pure point spectrum and where $a = \inf\sigma_p(B)$ (>0) and $b = \sup\sigma_p(B) > a$ each is an eigenvalue of infinite multiplicity. For it is clear that B is (unitarily equivalent to) either a single operator as in (3.2), thus $T = UB^{\frac{1}{2}}$, or a direct sum of operators $B_n = \operatorname{diag}(\ldots, b, b, (c_n), a, a, \ldots)$ of the form (3.3), in which case T is the direct sum of the corresponding $T_n = UB_n^{\frac{1}{2}}$.

Next, suppose that neither $a = \inf\sigma_p(B)$ nor $b = \sup\sigma_p(B)$ belongs to $\sigma_p(B)$, where $B = A^2$. Then the eigenvalues of B (including the multiplicities) can be arranged as a countable set

$$E = \{a_1, a_2, \ldots\}, \tag{3.4}$$

where $a < a_n < b$ and $\liminf_{n\to\infty} a_n = a$ and $\limsup_{n\to\infty} a_n = b$. Then consider $\{b_1, b_2, \ldots\}$, where $b_k = a_{n(k)} < a_k$ and $\{n(1), n(2), \ldots\}$ is a permutation of $\{1,2, \ldots\}$. Clearly, such a permutation exists. For instance, let $n(1)$ denote the least positive integer j for which $a_j < a_1$, $n(2)$ the least $j \neq n(1)$ for which $a_j < a_2$, $n(3)$ the least $j \neq n(1), n(2)$ for which $a_j < a_3$, and so on. Let H be the unilateral sequence ℓ^2 space and put $B = \operatorname{diag}(a_1, a_2, \ldots)$ and $C = \operatorname{diag}(b_1, b_2, \ldots)$. Clearly, C is unitarily equivalent to B, thus $C = UBU^*$, and $B - C = D \geqslant 0$ with $0 \notin \sigma_p(D)$. Thus $T = UB^{\frac{1}{2}} = UA$ is completely hyponormal.

The last possibility in case A (hence $B = A^2$) has a pure point spectrum occurs when either

$$\begin{aligned} &a = \inf\sigma_p(B) \notin \sigma_p(B) \quad \text{and} \\ &b(> a) = \sup\sigma_p(B) \in \sigma_p(B) \text{ with infinite multiplicity,} \end{aligned} \tag{3.5}$$

or vice versa. It is sufficient to treat the first case only, that is, (3.5), since the second (is similar. Let H be the bilateral sequence ℓ^2 space. Next, arrange the eigenvalues of B tincluding multiplicities) which are less than b as a matrix (a_{ij}) in such a way that he elements of the i-th row satisfy $a_{i1} > a_{i2} > \ldots \to a$, for each $i = 1, 2, \ldots$. That this can be done (in many ways) is clear. Then put $B_i = \operatorname{diag}(\ldots, b, b, (b), a_{i1}, a_{i2}, \ldots)$ and $C_i = \operatorname{diag}(\ldots, b, b, (a_{i1}), a_{i2}, a_{i3}, \ldots)$. Obviously, each $C_i = UB_iU^*$, where U is a unitary bilateral shift and $B_i - C_i = D_i \geqslant 0$. Clearly, $T_i = UB_i^{\frac{1}{2}}$ is completely hyponormal on H and so $T = \sum_i \oplus T_i$ is completely hyponormal on $\sum_i \oplus H_i$ ($H_i = H$) where $\sum_i \oplus B_i^{\frac{1}{2}}$ is unitarily equivalent to the given A.

Part (S) of the theorem has now been proved in case $\sigma_p(A)$ is either empty (case (I) above) or contains at least two points (case (II), or, via direct sums, a combination of cases (I) and (II)). There remains then the case where $\sigma_p(A)$ consists of a single point, say c, of infinite multiplicity and, in view of (1), A also contains some continuous spectrum.

Clearly, A (hence $B = A^2$) is (unitarily equivalent to) the countable direct sum of at least one operator of the type occurring in (I) (that is, multiplication by x on $L^2(d\mu)$, where μ is continuous and has its support in $[a, b]$) on H and multiplication by c on H. In each case, H is an infinite-dimensional Hilbert space. By isolating just one of the operators of the type in (I) (to form B_2 below), it is seen that one can suppose that

$$B = B_1 \oplus B_2 \oplus B_3 \quad \text{on } H \oplus H \oplus H, \tag{3.6}$$

where H is infinite-dimensional, B_1 (if, of course, it is present) is a direct sum of operators of the type in (I),

$$B_2 = \text{multiplication by } m(t) \text{ on } L^2(0, 1), \tag{3.7}$$

where $m(t)$ is defined by (3.1), and

$$B_3 = \text{multiplication by } c \text{ on } H. \tag{3.8}$$

First, suppose that

$$\min \sigma(B_2) < c < \max \sigma(B_2). \tag{3.9}$$

Then the operator $B_2 \oplus B_3$ is (unitarily equivalent to)

$$E \equiv B_2 \oplus B_3 = \text{multiplication by } m^*(t) \text{ on } L^2(0, 1), \tag{3.10}$$

where $m^*(t)$ is nondecreasing on $[0,1]$, $m^*(0) = a\ (\geqslant 0)$, $m^*(1) = b > a$, $m^*(t) = c$ on $[\alpha, \beta]$, where $0 < \alpha < \beta < 1$, and $m^*(t)$ is strictly increasing on $(0, \alpha) \cup (\beta, 1)$. (Essentially, this amounts to replacing B_2 of (3.7) by multiplication by $m^*(t)$ on $L^2((0, \alpha) \cup (\beta, 1))$ and B_3 of (3.8) by multiplication by c on $L^2(\alpha, \beta)$ in such a way that $\mu(x)$ of (3.1) and $\mu^*(x) = \text{meas}\{t : 0 \leqslant t < \alpha, \beta < t \leqslant 1, m^*(t) < x, a \leqslant x \leqslant b\}$ are equivalent measures.) Next, choose γ so that $\gamma > \beta$ and $\alpha = \gamma^n$ for some constant $n > 1$ and then define $m_1(t) = m^*(t^n)$. If E_1 is multiplication by $m_1(t)$ on $L^2(0, 1)$ then $E - E_1$ is multiplication by $m^*(t) - m_1(t)$ on $L^2(0, 1)$ and, in particular, $E - E_1 = D \geqslant 0$ and $0 \notin \sigma_p(D)$. Also, $E_1 = UEU^*$ for some unitary U. In fact, if (c) denotes multiplication by c on an infinite-dimensional Hilbert space then E is unitarily equivalent to multiplication by x on $L^2(d\mu) \oplus (c)$ and E_1 is unitarily equivalent to multiplication by x on $L^2(d\mu_1) \oplus (c)$, where μ and μ_1 are equivalent continuous measures. (Cf. the discussion of case (I) above.) Thus, $S = UE^{1/2}$ is completely hyponormal on some infinite-dimensional Hilbert space H. In view of (3.6),

$$B = B_1 \oplus E \tag{3.11}$$

and $T = T_1 \oplus S$ where both T_1 (which corresponds to the T at the end of section (I) above) and S are completely hyponormal.

Finally if, instead of (3.9), c is an endpoint of $\sigma(B_2)$, there is no loss of generality in supposing that

$$c = \max \sigma(B_2). \tag{3.12}$$

(Note that if $c = \min \sigma(B_2)$, then, of course, $\min \sigma(B_2) > 0$.) One may suppose that E of (3.10) is multiplication by $m^*(t)$ on $L^2(0, 1)$ where for some α, $0 < \alpha < 1$, $m^*(t)$ is strictly increasing on $(0, \alpha)$, and $m^*(t) = c$ on $(\alpha, 1)$. Let E_1 be multiplication by $m_1(t) = m^*(t^2)$ on $L^2(0, 1)$. Then, as before, $E_1 = UEU^*$ for some unitary U and $E - E_1 = D \geqslant 0$, so that $S = UE^{1/2}$ is hyponormal. However, $0 \in \sigma_p(D)$ and S may not now be completely hyponormal.

Suppose then that S is not completely hyponormal and let Ω be the largest subspace reducing S, that is, reducing both E and U (note that $0 \notin \sigma_p(E)$), on which S is normal. It is clear that $\Omega \subset L^2(\alpha^{1/2}, 1)$, where $L^2(\alpha^{1/2}, 1)$ is regarded in the usual way as a subspace of $L^2(0, 1)$. It will be shown that

$$L^2(\alpha^{1/2}, 1) \ominus \Omega \text{ is infinite-dimensional.} \tag{3.13}$$

For

$$L^2(0, 1) \ominus \Omega = L^2(0, \alpha^{1/2}) \oplus (L^2(\alpha^{1/2}, 1) \ominus \Omega)$$

and, since $L^2(0, \alpha^{1/2})$ is invariant under E and E_1, it is clear that $1 \in \sigma_p(E|L^2(0, \alpha^{1/2}))$ with infinite multiplicity and that $1 \notin \sigma_p(E_1|L^2(0, \alpha^{1/2}))$. Since $E_1 = UEU^*$ on $L^2(0, 1)$ and since $L^2(0, 1) \ominus \Omega$ reduces S, relation (3.13) follows.

Thus, one need only consider $R = S|\Omega^\perp$, where $\Omega^\perp = L^2(0, 1) \ominus \Omega$. Clearly, R is completely hyponormal and hence (cf. the sentence containing formula line (3.11)), the existence of a B, when (3.12) is assumed, and hence an A ($B = A^2$) which satisfies (S), is proved. This completes the proof of (S).

This work was supported by a National Science Foundation research grant.

REFERENCES

1. Clancey, K., *Seminormal operators*, Lecture Notes in Mathematics, **742**, Springer-Verlag, 1979.
2. Putnam, C. R., *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Math., **36**, Springer-Verlag, 1967.
3. Putnam, C. R., Spectra of polar factors of hyponormal operators, *Trans. Amer. Math. Soc.*, **188**(1974), 419–428.
4. Putnam, C. R., A polar area inequality for hyponormal operators, *J. Operator Theory*, **4**(1980), 191–200.
5. Putnam, C. R., Absolute continuity and hyponormal operators, *Internat. Jour. Math. & Math. Sci.*, **4**(1981), 321–335.

6. PUTNAM, C. R., Absolute continuity of polar factors of hyponormal operators, *Amer. J. Math.*, Suppl. 1981, 277–283.
7. PUTNAM, C. R., *The spectrum of the absolute value of a hyponormal operator*, Operator Theory: Advances and Applications, to appear.
8. REED, M.; SIMON. B., *Methods of modern mathematical physics.* I: *Functional analysis*, Academic Press, 1972.

C. R. PUTNAM
Department of Mathematics,
Purdue University,
West Lafayette, IN 47907,
U.S.A.

Received September 23, 1981.