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GAUGE INVARIANCE OF SCHRÖDINGER OPERATORS AND RELATED SPECTRAL PROPERTIES

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INTRODUCTION

In this paper we consider Schrödinger operators

$$\mathscr{H}(\vec{a}, V) = \left(\frac{1}{\mathrm{i}}\vec{\nabla} - \vec{a}\right)^2 + V =$$

$$= -\Delta + 2\,\mathrm{i}\vec{a}\cdot\vec{\nabla} + \mathrm{i}\vec{\nabla}\cdot\vec{a} + \vec{a}^2 + V,$$

where $(\vec{a}, V)\colon \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}$ is a measurable function, and look for a most general condition on $(\vec{a}, V)$ so that the essential spectrum of $\mathscr{H}(\vec{a}, V)$ coincides with the interval $[0, \infty)$. We set the spaces of potentials $\mathscr{K}_1, \mathscr{K}_2, \mathscr{L}_1, \mathscr{L}_2, \mathscr{M}_1$ and $\mathscr{M}_2$ as follows:

$$\mathscr{K}_1 = \{(\vec{a}, V) \mid \vec{a} \in L^2_{\mathrm{loc}},\ V \in L^1_{\mathrm{loc}}\},$$

$$\mathscr{K}_2 = \{(\vec{a}, V) \mid \vec{a} \in L^4_{\mathrm{loc}},\ \vec{\nabla}\cdot\vec{a} \in L^2_{\mathrm{loc}},\ V \in L^2_{\mathrm{loc}}\},$$

$$\mathscr{L}_1 = \{(\vec{a}, V) \mid (\vec{a}, V) \in \mathscr{K}_1, \text{the form } (\mathscr{H}(\vec{a}, V)\cdot, \cdot) \text{ is closable}^{*)} \text{ on } C_0^\infty\},$$

$$\mathscr{L}_2 = \{(\vec{a}, V) \mid (\vec{a}, V) \in \mathscr{K}_2, \mathscr{H}(\vec{a}, V) \text{ is essentially self-adjoint on } C_0^\infty\},$$

$$\mathscr{M}_1 = \{(\vec{a}, V) \mid (\vec{a}, V) \in \mathscr{K}_1, V_- \text{ is } \Delta\text{-formbounded with relative bound less than } 1\},$$

$$\mathscr{M}_2 = \{(\vec{a}, V) \mid (\vec{a}, V) \in \mathscr{K}_2, V_- \text{ is } \Delta\text{-bounded with relative bound less than } 1\}.$$

Here V_- denotes the nonnegative function max $(-V, 0)$. We note that $\mathscr{K}_1$ (respectively $\mathscr{K}_2$) is the largest possible set of potentials $(\vec{a}, V)$ such that the quadratic form $(\mathscr{H}(\vec{a}, V)\cdot, \cdot)$ (respectively the operator $\mathscr{H}(\vec{a}, V)$) can be defined on C_0^∞; that

*) In this note a closable form is assumed to be bounded from below.

$\mathscr{M}_i \subset \mathscr{L}_i \subset \mathscr{K}_i$ (see [2; 22; 14]) and if $(\vec{a}, V) \in \mathscr{L}_i$ there corresponds a unique self-adjoint operator $H(\vec{a}, V)$ which we abbreviate simply by $H(\vec{a})$ if $V = 0$.

The purpose of this note is two-fold. On one hand we prove that any self--adjoint Schrödinger operator $H(\vec{a}, V)$ is invariant under gauge transformations $\vec{a} \to \vec{a} + \vec{\nabla}\lambda$. On the other hand we apply the freedom of gauge transformations to the investigation of the essential spectrum of $H(\vec{a}, V)$ and show that $\sigma_{ess}(H(\vec{a}, V)) = [0, \infty)$ under very mild conditions on the potentials $(\vec{a}, V)$.

We show about the gauge transformation that:

(1) $\mathscr{L}_i$ is closed with respect to gauge transformations among $\mathscr{K}_i$, i.e. if $(\vec{a}, V) \in \mathscr{L}_i, (\vec{b}, V) \in \mathscr{K}_i$ and $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ with some $\lambda \in \mathscr{D}'$, then $(\vec{b}, V) \in \mathscr{L}_i$ and the Schrödinger operators $H(\vec{a}, V)$, $H(\vec{b}, V)$ are unitarily equivalent,

$$e^{i\lambda} H(\vec{a}, V)\, e^{-i\lambda} = H(\vec{b}, V).$$

(2) On each $(\vec{a}, V) \in \mathscr{L}_i$ we may impose the so called "Coulomb gauge condition $\vec{\nabla} \cdot \vec{a} = 0$", i.e. for any $(\vec{a}, V) \in \mathscr{L}_i$ we can find a gauge transformation $\lambda \in \mathscr{D}'$ such that $(\vec{b}, V) \in \mathscr{L}_i$ and $\vec{\nabla} \cdot \vec{b} = 0$, where $\vec{b} = \vec{a} + \vec{\nabla}\lambda$.

Using these results we show that the spectrum of $H(\vec{a}, V)$ is distributed in a specific way in $\mathbf{R}^+$ provided $(\vec{a}, V) \in \mathscr{L}_1$ and $Q(H(\vec{a}, V)) \subset Q(H(\vec{a}))$ (see Theorem 2.1 for a precise statement). As a simple consequence of this we obtain:

(1) Either $\sigma_{ess}(H(\vec{a}, V)) = \emptyset$ or $\sigma_{ess}(H(\vec{a}, V))$ is unbounded.

(2) If $H(\vec{a}) \leqslant \gamma H(\vec{a}, V)$ (some $\gamma > 0$) and $\inf \sigma(H(\vec{a}, V)) = 0$, then $\sigma(H(\vec{a}, V)) = [0, \infty)$.

Our main result states that $\sigma_{ess}(H(\vec{a}, V)) = [0, \infty)$ if $(\vec{a}, V) \in \mathscr{M}_1$ and if for some $\vec{b} \in L^2_{loc}$ with $\operatorname{curl} \vec{a} = \operatorname{curl} \vec{b}$ in Ω

$$\lim_{|x| \to \infty} \left[\int_{S_x} V_- + \int_{\Omega \cap S_x} (\vec{b}^2 + V_+) \right] = 0. \tag{0.1}$$

Here S_x is the sphere $\{y \mid |x - y| < 1\}$ and Ω is any open subset of $\mathbf{R}^m$ containing spheres of any radius. In $\mathbf{R}^3$ (or $\mathbf{R}^2$) condition (0.1) is satisfied if e.g.

$$\lim_{\substack{|x| \to \infty \\ x \in \Omega}} \operatorname{curl} \vec{a}(x) = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \int_{S_x} (V_- + X_\Omega V_+) = 0$$

and we recover a result in [15; 16; 17]. Finally we remark that most results mentioned above can be extended to admit highly singular potentials (see Concluding Remarks).

It seems to us that the usefulness of gauge transformations though well known to physicists, has not been extensively used in many mathematical treatments of Schrödinger operators and we know only [2; 13; 19; 24] where rigorous mathematical statements concerning gauge transformations are made.

While there exists an enormous literature about the investigation of the essential spectrum of Schrödinger operators without magnetic fields (see [19; vol. IV, pp. 342—343]) the number of results including magnetic fields is very limited. Without giving a complete list we mention those in [3; 4; 11; 12; 13; 15; 16; 17; 21; 25].

After having checked the literature we believe that the results presented in this note have not yet been proved elsewhere. Certainly, Theorem 2.5 is known in principle [11; 13; 21; 25] but only under considerably more restrictive assumptions.

We list here the notation and the conventions used in this paper. For $1 \leqslant p < \infty$ and $\Omega \subset \mathbf{R}^m$ open, $L^p(\Omega)$ is the space of (equivalence classes of) complex-valued functions $u: \Omega \to \mathbf{C}$ which are measurable and satisfy $\int |u|^p < \infty$. In case $p = 2$, $L^2(\Omega)$ is a complex Hilbert space with inner product (u, v) and norm $\|u\| = (u, u)^{1/2}$. $(L^2(\Omega))^m$ is the m-fold Cartesian product of $L^2(\Omega)$ and is equipped with the scalar product $(\vec{u}, \vec{v}) = \sum_{j=1}^{m} (u_j, v_j)$ and norm $\|\vec{u}\| = (\vec{u}, \vec{u})^{1/2}$. The space of infinitely differentiable functions with compact support in Ω will be denoted by $C_0^\infty(\Omega)$ or $\mathscr{D}(\Omega)$. $\mathscr{D}'(\Omega)$ is the space of distributions on Ω. For $1 \leqslant j \leqslant m$ let $\partial_j = \partial/\partial x_j$ be the j-th partial derivative, $\vec{\nabla} = (\partial_1, \ldots, \partial_m)$, each acting on $\mathscr{D}'(\Omega)$. For $n \in \mathbf{N}$, $1 \leqslant p < \infty$ the Sobolev space $W^{n,p}(\Omega)$ is defined as the space of those $u \in L^p(\Omega)$, for which all partial derivatives up to order n are in $L^p(\Omega)$. $W^{n,p}(\Omega)$ is a Banach space with norm $\|u\|_{n,p} = \left(\sum_{|\alpha| \leqslant n} \int_\Omega |\vec{\nabla}_u^\alpha|^p \right)^{1/p}$. For a linear subspace $F \subset \mathscr{D}'(\Omega)$ we define F_{loc} as the space of those $u \in \mathscr{D}'(\Omega)$ such that $\varphi u \in F$ whenever $\varphi \in \mathscr{D}(\Omega)$. If in addition F is normed, then $u_n \to u$ in F_{loc} means $\varphi u_n \to \varphi u$ in F for all $\varphi \in \mathscr{D}(\Omega)$. Finally $\Delta = \sum_{j=1}^{m} \partial_j^2$ is the Laplacian and $\vec{P}(\vec{a}) = \frac{1}{\mathrm{i}} \vec{\nabla} - \vec{a}$, $\vec{\nabla} \cdot \vec{a} = \sum_{j=1}^{m} \partial_j a_j$ with $\mathrm{i} = \sqrt{-1}$ and $\vec{a} \in L^2_{\mathrm{loc}}(\Omega)$. If $p = 2$ or $\Omega = \mathbf{R}^m$ we usually omit the symbols "p" respectively "Ω". For any real-valued function we set $V_\pm = \max(\pm V, 0)$.

Concerning notations and results in the theory of linear operators we refer to [19].

1. GAUGE INVARIANCE OF SCHRÖDINGER OPERATORS

Since we are dealing with Schrödinger operators with singular vector potentials, we first make precise what we mean by a gauge transformation.

Let Ω be any open subset of $\mathbf{R}^m$ and $\vec{a}, \vec{b} \in L^1_{\text{loc}}(\Omega)$. We say that $\vec{a}$ and $\vec{b}$ are related by a gauge transformation, $\vec{a} \underset{\Omega}{\sim} \vec{b}$, if there is a $\lambda \in \mathscr{D}'(\Omega)$ satisfying

$$\vec{b} = \vec{a} + \vec{\nabla}\lambda. \tag{1.1}$$

By $\operatorname{curl} \vec{a}$ we denote the skew-symmetric, matrix-valued distribution having $\partial_k a_l - \partial_l a_k \in \mathscr{D}'(\Omega)$ as matrix elements.

Lemma 1.1. *Let Ω be any open subset of $\mathbf{R}^m$, $1 \leq p < \infty$ and $\vec{a}, \vec{b} \in L^p_{\text{loc}}(\Omega)$ Then the following assertions hold:*

(i) *Every $\lambda \in \mathscr{D}'(\Omega)$ satisfying $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ belongs to $W^{1,p}_{\text{loc}}(\Omega)$. If Ω is simply-connected, then:*

$$\vec{a} \underset{\Omega}{\sim} \vec{b} \Leftrightarrow \operatorname{curl} \vec{a} = \operatorname{curl} \vec{b}.$$

(ii) *For any $\vec{a} \in L^p_{\text{loc}}(\Omega)$ $(2 \leq p < \infty)$ there exist a $\vec{c} \in L^p_{\text{loc}}(\Omega)$ such that*

$$\vec{a} \underset{\Omega}{\sim} \vec{c} \quad \textit{and} \quad \vec{\nabla} \cdot \vec{c} = 0 \quad (\textit{Coulomb gauge condition}). \tag{1.2}$$

Remarks. (1) The statements of Lemma 1.1 (i) are well known facts of vector analysis if $\vec{a}$ and $\vec{b}$ are smooth (see [5]). In case of nonsmooth $\vec{a}, \vec{b}$ we refer the reader to [20, Theorem 14].

(2) Suppose $\vec{a}(x) = a(|x|) \dfrac{x}{|x|}$, $a \in L^1_{\text{loc}}(\mathbf{R}^+)$ and $a(r) = 0$ in a neighborhood of 0. Then $\operatorname{curl} \vec{a} = \vec{0}$ and thus $\vec{a} \sim \vec{0}$ by Lemma 1.1. This example shows that there exist $\vec{a} \in L^1_{\text{loc}}(\mathbf{R}^m)$ satisfying $\vec{a} \sim \vec{0}$ where $|\vec{a}|$ may behave "very badly".

Proof of Lemma 1.1. (i). Clearly $\vec{a} \sim \vec{b}$ implies $\operatorname{curl} \vec{a} = \operatorname{curl} \vec{b}$. Now suppose $\operatorname{curl} \vec{a} = \operatorname{curl} \vec{b}$. We first show that for any point $x \in \Omega$ there exist an open sphere $\mathbf{S}_x \subset \overline{\mathbf{S}}_x \subset \Omega$ and a $\lambda_x \in W^{1,p}(\mathbf{S}_x)$ such that

$$\vec{b} = \vec{a} + \vec{\nabla}\lambda_x \quad \text{in } \mathbf{S}_x. \tag{1.3}$$

To see (1.3), choose vector fields $\vec{a}_n$ and $\vec{b}_n$ in $C^\infty(\overline{\mathbf{S}}_x)$ such that

$$\vec{a}_n \to \vec{a}, \vec{b}_n \to \vec{b} \text{ in } L^p_{\text{loc}}(\mathbf{S}_x) \quad \text{and} \quad \operatorname{curl} \vec{a}_n = \operatorname{curl} \vec{b}_n. \tag{1.4}$$

By a classical result in vector analysis (see Remark 1) there exist $\lambda_n \in C^\infty(\overline{\mathbf{S}}_x)$ such that

$$\vec{b}_n = \vec{a}_n + \vec{\nabla}\lambda_n \quad \text{in } \mathbf{S}_x. \tag{1.5}$$

Without loss of generality we may assume in addition

$$\int_{S_x} \lambda_n = 0. \tag{1.6}$$

Applying Poincaré's inequality (see [9; (7.45)]) to $\lambda_n - \lambda_l$ and using (1.5) we get

$$\begin{aligned} \|\lambda_n - \lambda_l\|_{W^{1,p}(S_x)} &\leqslant c \, \|\vec{\nabla}\lambda_n - \vec{\nabla}\lambda_l\|_{L^p(S_x)} \leqslant \\ &\leqslant c[\|\vec{b}_n - \vec{b}_l\|_{L^p(S_x)} + \|\vec{a}_n - \vec{a}_l\|_{L^p(S_x)}] \to 0 \quad (n, l \to \infty) \end{aligned} \tag{1.7}$$

where c is a constant which depends only on m, p and the radius of S_x. From (1.4), (1.7) we conclude that there is a $\lambda_x \in W^{1,p}(S_x)$ such that $\lambda_n \to \lambda_x$ in $W^{1,p}(S_x)$. Taking $n \to \infty$ in (1.5) we get (1.3).

To obtain the global result choose some fixed sphere $S \subset \Omega$ with center x_0, any $x \in \Omega$ and connect x and x_0 by some continuous curve $\gamma_x \subset \Omega$. Using the preceding step and a finite covering argument it is easy to find a connected neighborhood Ω_x of γ_x containing S and a $\gamma_x \in W^{1,p}(\Omega_x)$ satisfying

$$\vec{b} = \vec{a} + \vec{\nabla}\lambda_x \quad \text{in } \Omega_x; \quad \int_S \lambda_x = 0. \tag{1.8}$$

Since $\vec{\nabla}\lambda_x = \vec{\nabla}\lambda_{x'}$ in $\Omega_x \cap \Omega_{x'}$ and since Ω is simply connected it follows $\lambda_x = \lambda_x'$ in $\Omega_x \cap \Omega_{x'}$. Thus $\lambda = \bigcup_{x \in \Omega} \lambda_x$ is a function which belongs to $W^{1,p}_{\text{loc}}(\Omega)$ and satisfies $\vec{a} \underset{\Omega}{\sim} \vec{b}$.

Finally, if $\mu \in \mathscr{D}'(\Omega)$ satisfies $\vec{b} = \vec{a} + \vec{\nabla}\mu$, then $\mu - \lambda$ is locally constant and thus $\mu \in W^{1,p}_{\text{loc}}(\Omega)$.

(ii) Let λ be any distributional solution of

$$-\Delta\lambda = \vec{\nabla}\cdot\vec{a}. \tag{1.9}$$

By a result of [10; Corollaries 3.5.3, 3.7.1] solutions of (1.9) really do exist. Since $\vec{a} \in L^p_{\text{loc}}(\Omega)$ and $2 \leqslant p < \infty$ we have $\vec{\nabla}\cdot\vec{a} \in W^1(\Omega)'_{\text{loc}}$ and thus every solution of (1.9) belongs to $W^1_{\text{loc}}(\Omega)$ (see [10, Theorem 4.1.5]), in particular $\lambda \in L^2_{\text{loc}}(\Omega)$. Let p' be the exponent conjugate to p, $1 = p^{-1} + p'^{-1}$, and let Ω' be any sphere strictly contained in Ω. For all $\varphi \in \mathscr{D}'(\Omega)$ we have

$$|(\lambda, -\Delta\varphi)| = |(\vec{a}, \vec{\nabla}\varphi)| \leqslant \|\vec{a}\|_{L^p(\Omega')}\|\varphi\|_{W^{1,p'}(\Omega')}.$$

By elliptic regularity [1; Theorem 6.1] we conclude $\lambda \in W^{1,p}_{loc}(\Omega)$. Define $\vec{c} = \vec{a} + \vec{\nabla}\lambda$, then $\vec{c} \in L^p_{loc}(\Omega)$ and $\vec{\nabla} \cdot \vec{c} = \vec{\nabla} \cdot \vec{a} + \Delta\lambda = 0$. ▨

We now concentrate on the gauge invariance of Schrödinger operators. Let us recall the fact that if $(\vec{a}, V) \in \mathscr{K}_1$, the symmetric quadratic form $(\mathscr{H}(\vec{a}, V)\cdot,\cdot)$ on C_0^∞ is well defined by

$$(1.10) \qquad (\mathscr{H}(\vec{a}, V)\varphi, \varphi) = (\vec{P}(\vec{a})\varphi, \vec{P}(\vec{a})\varphi) + (|V|^{1/2}\operatorname{sgn} V\varphi, |V|^{1/2}\varphi).$$

If $(\vec{a}, V) \in \mathscr{L}_1$, then by the representation theorem for forms a unique self-adjoint operator $H(\vec{a}, V)$ corresponds to $(\mathscr{H}(\vec{a}, V)\cdot,\cdot)$. On the other hand if $(\vec{a}, V) \in \mathscr{K}_2$

$$(1.11) \qquad \mathscr{H}(\vec{a}, V) := -\Delta + 2i\vec{a}\cdot\vec{\nabla} + i\vec{\nabla}\cdot\vec{a} + \vec{a}^2 + V, \quad \mathscr{D}(\mathscr{H}(\vec{a}, V)) = C_0^\infty$$

is a symmetric (hence closable) operator. Let us denote by $H_0(\vec{a}, V)$ the closure of $\mathscr{H}(\vec{a}, V)$. Notice that we have $H_0(\vec{a}, V) \subset H(\vec{a}, V)$ if $(\vec{a}, V) \in \mathscr{L}_1 \cap \mathscr{K}_2$ and $H_0(\vec{a}, V) = H(\vec{a}, V)$ in case $(\vec{a}, V) \in \mathscr{L}_1 \cap \mathscr{L}_2$.

Our results on the gauge invariance of Schrödinger operators can now be stated as follows.

THEOREM 1.2. *Let $(\vec{a}, V)$, $(\vec{b}, V) \in \mathscr{K}_1$ and assume* $\operatorname{curl}\vec{a} = \operatorname{curl}\vec{b}$. *Then $(\mathscr{H}(\vec{a}, V)\cdot,\cdot)$ is closable if and only if $(\mathscr{H}(b, V)\cdot,\cdot)$ is closable. Moreover*

$$(1.12) \qquad e^{i\lambda}H(\vec{a}, V)e^{-i\lambda} = H(\vec{b}, V)$$

whenever $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ in $\mathscr{D}'$.

THEOREM 1.3. *Let $(\vec{a}, V)$, $(\vec{b}, V) \in \mathscr{K}_2$ and assume* $\operatorname{curl}\vec{a} = \operatorname{curl}\vec{b}$. *Then*

$$(1.13) \qquad e^{i\lambda}H_0(\vec{a}, V)e^{-i\lambda} = H_0(\vec{b}, V)$$

whenever $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ in $\mathscr{D}'$. In particular $\mathscr{H}(\vec{a}, V)$ is essentially self-adjoint if and only if $\mathscr{H}(\vec{b}, V)$ is essentially self-adjoint.

REMARKS. (1). In view of Lemma 1.1 any gauge transformation λ appearing in Theorem 1.2 respectively 1.3 belongs to $W^{1,2}_{loc}$ respectively $W^{1,4}_{loc}$. Thus λ is always a measurable function and $e^{\pm i\lambda}$ are unitary operators.

(2). Theorems 1.2 and 1.3 tell us that the uniqueness of Schrödinger operators depends only on the "magnetic field" $\operatorname{curl}\vec{a}$. Moreover, any spectral property of a Schrödinger operator $H(\vec{a}, V)$ may be proved by choosing any suitable gauge transformation.

(3). One can impose on Schrödinger operators $H(\vec{a}, V)$ whose potentials $(\vec{a}, V)$ belong to $\mathscr{L}_t$ the so called "Coulomb gauge condition $\vec{\nabla}\cdot\vec{a} = 0$" without loosing generality.

Before proving Theorems 1.2 and 1.3 we give an application.

COROLLARY 1.4. *Let $\vec{a} \in L^4_{\text{loc}}$, $V \in L^2_{\text{loc}}$ and suppose V_- is Δ-bounded with relative bound less than 1. Let $\lambda \in \mathscr{D}'$ be any solution of $-\Delta\lambda = \vec{\nabla}\cdot\vec{a}$. Then $\mathscr{H}(\vec{a}, V)$ is essentially self-adjoint on $\mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D}$.*

REMARKS. (1) Since $\mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D} \subset W^{1,4}$ and $-\Delta\lambda = \vec{\nabla}\cdot\vec{a}$,

$$\mathscr{H}(\vec{a}, V)\varphi = -\Delta\varphi + 2\mathrm{i}\,\vec{a}\cdot\vec{\nabla}\varphi + \mathrm{i}(\vec{\nabla}\cdot\vec{a})\varphi + \vec{a}^2\varphi + V\varphi$$

belong to L^2 for all $\varphi \in \mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D}$.

(2) Corollary 1.4 has been shown by Simon [24, Theorem 3] under the assumptions $\vec{a} \in L^q$, $q > m$, $q \geqslant 4$ and $V_- = 0$. But Simon remarks that it may be possible to extend his result to allow $\vec{a} \in L^q_{\text{loc}}$ and $V_- \neq 0$.

(3) Notice that on one hand $\mathscr{H}(\vec{a}, V)$ is essentially self-adjoint even on C_0^∞ if $\vec{\nabla}\cdot\vec{a} \in L^2_{\text{loc}}$ [14, Theorem 3] and that on the other hand $\mathscr{H}(\vec{a}, V)$ can not be defined on C_0^∞ if $\vec{\nabla}\cdot\vec{a} \notin L^2_{\text{loc}}$. Hence, in case $\vec{\nabla}\cdot\vec{a} \notin L^2_{\text{loc}}$, $\mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D}$ seems to be a natural domain of essential self-adjointness for $\mathscr{H}(\vec{a}, V)$.

Proof of Corollary 1.4. (using Theorem 1.2). Let $\lambda \in \mathscr{D}'$ be any solution of $-\Delta\lambda = \vec{\nabla}\cdot\vec{a}$ (such solutions really exist) and define $\vec{b} = \vec{a} + \vec{\nabla}\lambda$. Then by Lemma 1.1 (ii) (see proof) we have $\vec{b} \in L^4_{\text{loc}}$ and $\vec{\nabla}\cdot\vec{b} = 0$, hence $(\vec{a}, V), (\vec{b}, V) \in \mathscr{L}_1$. Moreover $(\vec{b}, V) \in \mathscr{L}_2$ by a result in [14; Theorem 3]. Applying Theorem 1.2 we obtain

$$\mathrm{e}^{\mathrm{i}\lambda}H(\vec{a}, V)\,\mathrm{e}^{-\mathrm{i}\lambda} = H(\vec{b}, V) = H_0(\vec{b}, V).$$

Thus $\mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D}$ is an operator core of $H(\vec{a}, V)$. Since $\mathrm{e}^{-\mathrm{i}\lambda}\mathscr{D} \subset W^{1,4}$, it is not difficult to see that for all $\varphi, \psi \in \mathscr{D}$

$$(H(\vec{a}, V)\,\mathrm{e}^{-\mathrm{i}\lambda}\varphi, \psi) = (\vec{P}(\vec{a})\mathrm{e}^{-\mathrm{i}\lambda}\varphi, \vec{P}(\vec{a})\psi) + (|V|^{1/2}\mathrm{sgn}V\mathrm{e}^{-\mathrm{i}\lambda}\varphi, |V|^{1/2}\psi) =$$

$$= (\mathscr{H}(\vec{a}, V)\mathrm{e}^{-\mathrm{i}\lambda}\varphi, \psi^*)_{\text{dist.}}.$$

Thus $\mathscr{H}(\vec{a}, V)\mathrm{e}^{-\mathrm{i}\lambda}\varphi = H(\vec{a}, V)\,\mathrm{e}^{-\mathrm{i}\lambda}\varphi$ for all $\varphi \in \mathscr{D}$ and the proof is complete. ▨

Proof of Theorem 1.2. Let $(\vec{a}, V), (\vec{b}, V) \in \mathscr{K}_1$ and assume $\mathrm{curl}\,\vec{a} = \mathrm{curl}\,\vec{b}$. By Lemma 1.1 there exists $\lambda \in W^1_{\text{loc}}$ such that $\vec{b} = \vec{a} + \vec{\nabla}\lambda$. Take a sequence (λ_n)

of C^∞-functions such that $\lambda_n \to \lambda$ in W^1_{loc} and define $\vec{b}_n = \vec{a} + \vec{\nabla}\lambda_n$. Let $\varphi \in C_0^\infty$. Then

$$(\mathscr{H}(\vec{a}, V)\mathrm{e}^{-i\lambda_n}\varphi, \mathrm{e}^{-i\lambda_n}\varphi) = \|\vec{P}(\vec{a})\mathrm{e}^{-i\lambda_n}\varphi\|^2 + (|V|^{1/2}\operatorname{sgn} V\mathrm{e}^{-i\lambda_n}\varphi, |V|^{1/2}\mathrm{e}^{-i\lambda_n}\varphi).$$

But $\vec{P}(\vec{a})\,\mathrm{e}^{-i\lambda_n}\varphi = \mathrm{e}^{-i\lambda_n}\vec{P}(\vec{b}_n)\,\varphi$ and $\mathrm{e}^{-i\lambda_n}$ is a unitary operator, thus

$$(\mathscr{H}(\vec{a}, V)\mathrm{e}^{-i\lambda_n}\varphi, \mathrm{e}^{-i\lambda_n}\varphi) = (\mathscr{H}(\vec{b}_n, V)\varphi, \varphi). \tag{1.14}$$

Since $\vec{b}_n \to \vec{b}$ in L^2_{loc}, we conclude $\vec{P}(\vec{b}_n)\varphi \to \vec{P}(\vec{b})\,\varphi$ in L^2 and then

$$(\mathscr{H}(\vec{b}, V)\,\varphi, \varphi(= \lim_{n\to\infty} (\mathscr{H}(\vec{b}_n, V)\,\varphi, \varphi). \tag{1.15}$$

Hence, if $(\mathscr{H}(\vec{a}, V)\cdot,\cdot) \geq \gamma$, then by (1.14), (1.15)

$$(\mathscr{H}(\vec{b}, V)\varphi, \varphi) = \lim_{n\to\infty} (\mathscr{H}(\vec{a}, V)\mathrm{e}^{-i\lambda_n}\varphi, \mathrm{e}^{-i\lambda_n}\varphi) \geq \gamma(\varphi, \varphi)$$

for all $\varphi \in C_0^\infty$. Suppose now $(\mathscr{H}(\vec{a}, V)\cdot,\cdot)$ is closable. Since $\mathrm{e}^{-i\lambda_n}\varphi \to \mathrm{e}^{-i\lambda}\varphi$ and

$$(\mathscr{H}(\vec{a}, V)(\mathrm{e}^{-i\lambda_n}\varphi - \mathrm{e}^{-i\lambda_k}\varphi), \mathrm{e}^{-i\lambda_n}\varphi - \mathrm{e}^{-i\lambda_k}\varphi) =$$

$$= \|\mathrm{e}^{-i\lambda_n}\vec{P}(\vec{b}_n)\varphi - \mathrm{e}^{-i\lambda_k}\vec{P}(\vec{b}_k)\varphi\|^2 + \int V|\mathrm{e}^{-i\lambda_n} - \mathrm{e}^{-i\lambda_k}|^2|\varphi|^2 \to 0 \quad (n, k \to \infty)$$

we see that $\mathrm{e}^{-i\lambda}\varphi \in Q(H(\vec{a}, V))$ and (using 1.14, 1.15)

$$(H(\vec{a}, V)\mathrm{e}^{-i\lambda}\varphi, \mathrm{e}^{-i\lambda}\varphi) = (\mathscr{H}(\vec{b}, V)\varphi, \varphi) \tag{1.16}$$

for all $\varphi \in C_0^\infty$.

Let $\varphi_n \in C_0^\infty$, $\varphi_n \to u \in L^2$ and $(\mathscr{H}(\vec{b}, V)(\varphi_n - \varphi_k), \varphi_n - \varphi_k) \to 0$ as $n, k \to \infty$. We have $\mathrm{e}^{-i\lambda}\varphi_n \in Q(H(\vec{a}, V))$, $\mathrm{e}^{-i\lambda}\varphi_n \to \mathrm{e}^{-i\lambda}u$ and

$$(H(\vec{a}, V)(\mathrm{e}^{-i\lambda}\varphi_n - \mathrm{e}^{-i\lambda}\varphi_k), \mathrm{e}^{-i\lambda}\varphi_n - \mathrm{e}^{-i\lambda}\varphi_k) \to 0 \text{ as } n, k \to \infty$$

due to (1.16). Since $(\mathscr{H}(\vec{a}, V)\cdot,\cdot)$ is closable we conclude $\mathrm{e}^{-i\lambda}u \in Q(H(\vec{a}, V))$ and

$$\lim_{n\to\infty} (\mathscr{H}(\vec{b}, V)\,\varphi_n, \varphi_n) = \lim_{n\to\infty} (H(\vec{a}, V)\mathrm{e}^{-i\lambda}\varphi_n, \mathrm{e}^{-i\lambda}\varphi_n) = 0,$$

last equation being valid if $u = 0$. Hence $(\mathscr{H}(\vec{b}, V)\cdot,\cdot)$ is closable, $\mathrm{e}^{-i\lambda}Q(H(\vec{b}, V)) \subset \subset Q(H(\vec{a}, V))$ and taking the limit $n \to \infty$ in (1.16) (with φ replaced by φ_n) one gets

$$(H(\vec{a}, V)\mathrm{e}^{-i\lambda}u, \mathrm{e}^{-i\lambda}u) = (H(\vec{b}, V)\,u, u)$$

for all $u \in Q(H(\vec{b}, V))$. Finally, interchanging $\vec{a}$ and $\vec{b}$ we obtain $e^{i\lambda}(H(\vec{a}, V)e^{-i\lambda} = H(\vec{b}, V)$ as quadratic forms and thus as an operator equality, too. ▨

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2. Let $(\vec{a}, V)$, $(\vec{b}, V) \in \mathscr{K}_2$ and $\operatorname{curl}\vec{a} = \operatorname{curl}\vec{b}$. By Lemma 1.1 there exists an $\lambda \in W^{1,4}_{loc}$ satisfying $\vec{b} = \vec{a} + \vec{\nabla}\lambda$. Choose $\lambda_n \in C^\infty$ such that $\lambda_n \to \lambda$ in $W^{1,4}_{loc}$ and $\Delta\lambda_n \to \Delta\lambda$ in L^2_{loc} (recall $\Delta\lambda = \vec{\nabla}\cdot\vec{b} - \vec{\nabla}\cdot\vec{a} \in L^2_{loc}$). Let $\varphi \in C_0^\infty$, then $e^{-i\lambda_n}\varphi \in C_0^\infty$ and

$$\mathscr{H}(\vec{a}, V)e^{-i\lambda_n}\varphi = e^{-i\lambda_n}\mathscr{H}(\vec{b}_n, V)\varphi \tag{1.17}$$

where $\vec{b}_n = \vec{a} + \vec{\nabla}\lambda_n$ and $\mathscr{H}(\vec{b}_n, V) = -\Delta + 2i\vec{b}_n\cdot\vec{\nabla} + i\vec{\nabla}\cdot\vec{b}_n + \vec{b}_n^2 + V$. From $\vec{b}_n \to \vec{b}$ in L^4_{loc} and $\vec{\nabla}\cdot\vec{b}_n = \vec{\nabla}\cdot\vec{a} + \nabla\lambda_n \to \vec{\nabla}\cdot\vec{a} + \Delta\lambda = \vec{\nabla}\cdot\vec{b}$ in L^2_{loc} we conclude $\mathscr{H}(\vec{b}_n, V)\varphi \to \mathscr{H}(\vec{b}, V)\varphi$. Moreover, $e^{-i\lambda_n}\varphi \to e^{-i\lambda}\varphi$ and $e^{-i\lambda_n}\mathscr{H}(\vec{b}_n, V)\varphi \to e^{-i\lambda}\mathscr{H}(\vec{b}, V)\varphi$.

Thus $e^{-i\lambda}\varphi \in \mathscr{D}(H_0(\vec{a}, V))$ and

$$H_0(\vec{a}, V)e^{-i\lambda}\varphi = e^{-i\lambda}H_0(\vec{b}, V)\varphi. \tag{1.18}$$

But (1.18) means $H_0(\vec{b}, V) \subset e^{i\lambda}H_0(\vec{a}, V)e^{-i\lambda}$. Interchanging $\vec{a}$ and $\vec{b}$ we get $e^{i\lambda}H_0(\vec{a}, V)e^{-i\lambda} = H_0(b, V)$. ▨

2. GAUGE TRANSFORMATIONS AND SOME SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS

We begin this section by showing first a result about the distribution of the spectral measure of Schrödinger operators $H(\vec{a}, V)$.

THEOREM 2.1. *Let $(\vec{a}, V) \in \mathscr{L}_1$ and assume*

$$H(\vec{a}) \leqslant \gamma^2[H(\vec{a}, V) + c] \tag{2.1}$$

with some positive constants c, γ. Set $d(\lambda, \mu) = \mu + 2\lambda\gamma(\mu + c)^{1/2}$ and let E be the spectral measure of $H(\vec{a}, V)$. Then for any $\lambda, \mu \geqslant 0$

$$\dim R(E[\lambda^2 - d(\lambda, \mu), \lambda^2 + d(\lambda, \mu)]) \geqslant \dim R(E[-\mu, \mu]). \tag{2.2}$$

Proof. Let $\vec{b} := \vec{a} + \nabla v$ with $v(x) = \lambda x_1$. By Theorem 1.2 we have $(\vec{b}, v) \in \mathscr{L}_1$ and $e^{iv} H(\vec{a}, V) e^{-iv} = H(\vec{b}, V)$. In particular

$$\sigma_{ess}(H(\vec{a}, V)) = \sigma_{ess}(H(\vec{b}, V)). \tag{2.3}$$

We show first $\mathscr{D}(H(\vec{a}, V)) = \mathscr{D}(H(\vec{b}, V))$ and

$$H(\vec{b}, V) - \lambda^2 = H(\vec{a}, V) - 2\lambda P_1(\vec{a}). \tag{2.4}$$

For $\varphi \in C_0^\infty$, a simple calculation shows

$$((H(\vec{b}, V) - \lambda^2)\varphi, \varphi) = ((H(\vec{a}, V) - 2\lambda P_1(\vec{a}))\varphi, \varphi). \tag{2.5}$$

Since, by virtue of the assumption (2.1) both sides of (2.5) are closed forms with common core C_0^∞, we have (2.4). Let $\varphi \in R(E[-\mu, \mu])$. By (2.4)

$$\begin{aligned} \|H(\vec{b}, V)\varphi - \lambda^2\varphi\| &\leqslant \|H(\vec{a}, V)\varphi\| + 2\lambda\|P_1(\vec{a})\varphi\| \leqslant \\ &\leqslant \|H(\vec{a}, V)\varphi\| + 2\lambda(H(\vec{a})\varphi, \varphi)^{1/2} \leqslant \\ &\leqslant \mu\|\varphi\| + 2\lambda\gamma[(H(\vec{a}, V)\varphi, \varphi) + c(\varphi,\varphi)]^{1/2} \leqslant \\ &\leqslant (\mu + 2\lambda\gamma(\mu + c)^{1/2})\|\varphi\| =: d(\lambda, \mu)\|\varphi\|. \end{aligned} \tag{2.6}$$

Let F be the spectral measure associated with $H(\vec{b}, V)$ and let $I := [\lambda^2 - d(\lambda, \mu), \lambda^2 + d(\lambda, \mu)]$. The spectral theorem and estimate (2.6) show that, if $F(I)\varphi = 0$, we have $\varphi = 0$. Thus $F(I)$ is injective on $R(E[-\mu, \mu])$ which gives

$$\dim R(F(I)) \geqslant \dim R(E[-\mu, \mu]).$$

Since $E(I) = e^{-iv} F(I) e^{iv}$, the result follows. ▨

COROLLARY 2.2. *Let $(\vec{a}, V) \in \mathscr{L}_1$ and assume $Q(H(\vec{a}, V)) \subset Q(H(\vec{a}))$. Then either $\sigma_{ess}(H(\vec{a}, V)) = \emptyset$ or $\sigma_{ess}(H(\vec{a}, V))$ is unbounded.*

Proof. By the closed graph theorem, (2.1) holds with some $\gamma, c > 0$. Suppose $\sigma_{ess}(H(\vec{a}, V)) \neq \emptyset$. Take $\mu > 0$ large enough so that $\dim R(E[-\mu, \mu]) = \infty$. Then by (2.2) we have

$$\dim R(E[\lambda^2 - d(\lambda, \mu), \lambda^2 + d(\lambda, \mu)]) = \infty \tag{2.7}$$

for all $\lambda \geqslant 0$. Since $d(\lambda, \mu)$ is linear in λ infinitely many nonoverlapping intervals $I_n = [\lambda_n^2 - d(\lambda_n, \mu), \lambda_n^2 + d(\lambda_n, \mu)]$ exist if the λ_n are chosen suitably. Hence

$$\sigma_{\text{ess}}(H(\vec{a}, V)) \supset \bigcup_{n=1}^{\infty} (\sigma_{\text{ess}}(H(\vec{a}, V)) \cap I_n)$$

is an unbounded set. ▨

COROLLARY 2.3. *Let* $(\vec{a}, V) \in \mathscr{L}_1$ *and suppose* (2.1) *holds with* $c = 0$. *Then either* $\inf \sigma(H(\vec{a}, V)) > 0$ *or* $\sigma(H(\vec{a}, V)) = [0, \infty)$.

Proof. By hypothesis $H(\vec{a}, V) \geqslant 0$. Suppose $\inf \sigma(H(\vec{a}, V) = 0$, then (use 2.2)

$$\dim R(E[\lambda^2 - d(\lambda, \mu), \lambda^2 + d(\lambda, \mu)]) \geqslant \dim R(E[-\mu, \mu]) > 0$$

for all $\lambda \geqslant 0$, $\mu > 0$. Since $d(\lambda, \mu) \to 0$ as $\mu \to 0$ we conclude $\lambda^2 \in \sigma(H(\vec{a}, V))$. Hence $\sigma(H(\vec{a}, V)) = [0, \infty)$. ▨

REMARKS. (1) Notice that the operator $H(\vec{a}, V)$ of Corollary 2.3 is always positive and thus $\inf \sigma(H(\vec{a}, V) \geqslant 0$. Moreover we mention that 0 is never an eingenvalue of $H(\vec{a}, V)$, because $H(\vec{a}, V)\varphi = 0$ would imply $\vec{P}(\vec{a})\varphi = 0$ which is in fact impossible since each $P_k(\vec{a})$ is unitarily equivalent to ∂_k (see [22]).

(2) The assumptions of Corollary 2.3 are satisfied if $\vec{a} \in L^2_{\text{loc}}$ and $0 \leqslant V \in L^1_{\text{loc}}$, but slightly negative potentials are allowed, too. For instance, if $m = 3$, $\vec{a} \in L^2_{\text{loc}}$, $V \in L^1_{\text{loc}}$ and if $V_- \leqslant \varepsilon r^{-2}$ for some $0 < \varepsilon < 1/4$, then $(\vec{a}, V) \in \mathscr{M}_1 \subset \mathscr{L}_1$ and $H(\vec{a}) \leqslant \gamma H(\vec{a}, V)$ with some $\gamma > 0$. This can be seen easily by using $(4r^2)^{-1} \leqslant H(\vec{a})$ (see [23], [19; vol 2, p. 169]).

To establish our main result we need Weyl's theorem for Schrödinger operators $H(\vec{a}, V)$ under very mild conditions on the electric potential V.

THEOREM 2.4. *Let* $(\vec{a}, V) \in \mathscr{M}_1$. *Then*

(i) $$\lim_{|x| \to \infty} \int_{S_x} V_- = 0 \text{ implies } \sigma_{\text{ess}}(H(\vec{a}, V)) = \sigma_{\text{ess}}(H(\vec{a}, V_+))$$

and

(ii) $$\lim_{|x| \to \infty} \int_{S_x} |V| = 0 \text{ implies } \sigma_{\text{ess}}(H(\vec{a}, V)) = \sigma_{\text{ess}}(H(\vec{a})),$$

where S_x *denotes the open sphere of center* x *and radius* 1.

Proof. (i) Set $A = H(\vec{a}, V_+)$, $B = H(\vec{a}, V)$ and $C = V_- \geqslant 0$. By Weyl's essential spectrum theorem [19; Theorem XIII. 14 and Corollary 4] it is sufficient to show that C is A-formbounded with relative bound less than 1 and that $C^{1/2}(A+1)^{-n/2}$ is compact for some integer n. By hypothesis, if $\lambda > 0$ is sufficiently large, $C^{1/2}(-\Delta + \lambda)^{-1/2}$ is a bounded operator with norm less than 1 and $C^{1/2}\colon W^{n,2} \to L^2$ is compact whenever $n > m/2$ [25; Theorem 10.20]. On the other hand the operator $(-\Delta + \lambda)^{-n/2}\colon L^2 \to W^{n,2}$ is bounded. Hence $C^{1/2}(-\Delta + \lambda)^{-n/2}$ is compact. As a consequence of Kato's inequality

$$|(A + \lambda)^{-1}\varphi| \leqslant (-\Delta + \lambda)^{-1}|\varphi| \tag{2.8}$$

holds pointwise a.e. for all $\varphi \in L^2$ (see [22], [14, Lemma 6]). Thus, iterating (2.8) and multiplying by $C^{1/2}$ we obtain

$$|C^{1/2}(A + \lambda)^{-k}\varphi| \leqslant C^{1/2}(-\Delta + \lambda)^{-k}|\varphi| \tag{2.9}$$

for all integers k and all $\varphi \in L^2$. As a consequence of (2.9) we see that C is A-formbounded with relative bound less than 1 and that $C^{1/2}(A + \lambda)^{-n/2}$ is compact [8; 18], hence (i) is proved.

(ii) Set $A = H(\vec{a})$ and $B = H(\vec{a}, V)$. By a result in [7; Theorem 13] it suffices to show that A and B are mutually subordinated and that $E_B(I)(B - A)E_A(I)$ is a compact operator for all bounded intervals. Here $E_A(I)$ (respectively $E_B(I)$) denotes the spectral projection of A (respectively B) for the interval I. As in the proof of case (i) we conclude that $|V|^{1/2}(A + \lambda)^{-n/2}$ is compact if $n > m/2$. Thus, $|V|^{1/2}E_A(I) = |V|^{1/2}(A + \lambda)^{-n/2}(A + \lambda)^{n/2}E_A(I)$ is a compact operator, since $(A + \lambda)^{n/2}E_A(I)$ is bounded. Moreover, we have $R((A + \lambda)^{-n/2}) \subset Q(A) \cap Q(V)$ and using (2.8) it is not hard to see that $Q(B) = Q(A) \cap Q(V)$. Thus, by the closed graph theorem

$$\|(B + \lambda)^{1/2}(A + \lambda)^{-n/2}\| + \|(A + \lambda)^{1/2}(B + \lambda)^{-1/2}\| < \infty$$

if λ is chosen sufficiently large. This shows that A and B are mutually subordinated. Since $R(E_A(I)) \cup R(E_B(I)) \subset Q(A) \cap Q(B)$ we have the equation

$$E_B(I)(B - A)E_A(I) = [|V|^{1/2}E_B(I)]^* \operatorname{sgn} V[|V|^{1/2}E_A(I)].$$

Hence $E_B(I)(B - A)E_A(I)$ is compact which completes the proof. ▨

An open set $\Omega \subset \mathbf{R}^m$ is called quasi-conical if it contains spheres of any radius. Clearly, every conical region is quasi-conical.

THEOREM 2.5. *Let* $(\vec{a}, V) \in \mathcal{M}_1$. *Assume there exist a quasi-conical region* Ω *and a magnetic vector potential* $\vec{b} \in L^2_{\mathrm{loc}}$ *such that* $\operatorname{curl}\vec{a} = \operatorname{curl}\vec{b}$ *in* Ω *and*

$$\lim_{|x|\to\infty} \left[\int_{S_x} V_- + \int_{\Omega \cap S_x} (\vec{b}^2 + V_+) \right] = 0. \tag{2.10}$$

Then $\sigma_{\mathrm{ess}}(H(\vec{a}, V)) = [0, \infty)$.

Proof. First we show that without loss of generality we may assume that $\vec{a}$ itself satisfies (2.10). To see this it suffices to notice that there exist a quasi-conical region $\Omega' \subset \Omega$ and a gauge transformation $\lambda \in W'_{\mathrm{loc}}(\mathbf{R}^m)$ such that $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ in Ω' (use Lemma 1.1 and a suitable cut-off function!). Next we remark that in view of Theorem 2.4 we may assume $V_- = 0$. Finally, applying Corollary 2.3, it is sufficient to show $\inf\sigma(H(\vec{a}, V)) = 0$.

Define $W = (\vec{a}^2 + V)\chi_\Omega$ and remember $\vec{a} = \vec{b}$ in Ω as well as $V \geqslant 0$ due to our reductions made above. Then

$$(H(\vec{a}, V)\varphi, \varphi) = ((-\Delta + W)\varphi, \varphi) = \|\vec{\nabla}\varphi\|^2 + \|W^{1/2}\varphi\|^2$$

holds for all real-valued $\varphi \in C_0^\infty(\Omega)$. Choose pairwise disjoint spheres S_{x_n} with center x_n and radius n, all contained in Ω, and define $\varphi_n = n^{-m/2}\varphi\left(\frac{\cdot - x_n}{n}\right)$, where φ is any real-valued C^∞-function with support in $\{x \mid |x| < 1\}$ and with norm 1. Then $\varphi_n \in C_0^\infty(\Omega)$, $\|\varphi_n\| = 1$, $\varphi_n \xrightarrow{w} 0$ and $\vec{\nabla}^\alpha\varphi_n \xrightarrow{s} 0$ for every multiindex $\alpha \neq 0$. In particular, $\varphi_n \xrightarrow{w} 0$ in each Sobolev space W^k. Since $\lim_{|x|\to\infty} \int_{S_x} W = 0$, we have $W^{1/2}\varphi \xrightarrow{s} 0$ [25; Theorem 10.20] and thus $(H(\vec{a}, V)\varphi_n, \varphi_n) \to 0$ as $n \to \infty$. It follows $\inf\sigma(H(\vec{a}, V)) = 0$ which proves the theorem. ▨

COROLLARY 2.6. *Let* $m = 3$, $(\vec{a}, V) \in \mathcal{M}_1$ *and assume* $\vec{a} \in C^1$. *Suppose there exists a quasi-conical region* Ω *such that*

$$\lim_{\substack{|x|\to\infty \\ x\in\Omega}} \operatorname{curl}\vec{a}(x) = 0 \quad \textit{and} \quad \lim_{|x|\to\infty} \int_{S_x} (V_- + X_\Omega V_+) = 0.$$

Then $\sigma_{\mathrm{ess}} H((\vec{a}, V)) = [0, \infty)$.

REMARK. In case $V = 0$ this result has been proved in [15; 16; 17].

Proof. By hypothesis there exist pairwise disjoint spheres $S_n \subset \Omega$ with center c_n and radius n such that $|\operatorname{curl}\vec{a}| \leqslant n^{-2}$ on each S_n. It is easy to find a vector potential $\vec{b} \in L^2_{\text{loc}}$ which satisfies $\text{curl}\,\vec{a} = \text{curl}\,\vec{b}$ and $|\vec{b}| \leqslant \text{const}\, n^{-1}$ on each S_n. For instance, with the notations $\vec{B} = (B_1, B_2, B_3) = \text{curl}\,\vec{a}$ and $c_n = (x_n, y_n, z_n)$, the vector potential

$$\vec{b}(x, y, z) := \left(\int_{z_n}^{z} B_2(x, y, t)\,dt + \int_{y_n}^{y} B_3(x, t, 0)\,dt,\ -\int_{z_n}^{z} B_1(x, y, t)\,dt,\ 0 \right)$$

on S_n and $\vec{b}(x, y, z) = 0$ would do it.

Apply Theorem 2.5 $\left(\text{with } \Omega = \bigcup_{n=1}^{\infty} S_n\right)$ and Corollary 2.6 follows. ▨

COROLLARY 2.7. *Let $(\vec{a}, V) \in \mathscr{M}_1$. Suppose there is a conical region Ω and a vector potential $\vec{b} \in L^2_{\text{loc}}$ such that $\text{curl}\,\vec{a} = \text{curl}\,\vec{b}$ in Ω. Assume further that with some $0 < \varepsilon < 2$ the following relations hold*

$$\lim_{|x| \to \infty} \int_{S_x} V_- = 0, \quad \inf_{x \in \Omega} |x|^\varepsilon V_-(x) > 0$$

$$\lim_{\substack{|x| \to \infty \\ x \in \Omega}} |x|^\varepsilon (\vec{b}^2(x) + V_+(x)) = 0.$$

Then $\sigma_{\text{ess}}(H(\vec{a}, V)) = [0, \infty)$ and $\sigma_{\text{disc}}(H(\vec{a}, V))$ is infinite.

Proof. Without loss of generality we may assume that Ω is an open cone with center 0 and that $\vec{a} = \vec{b}$ in Ω (see the proof of Theorem 2.5). As a consequence of Theorem 2.5 we have $\sigma_{\text{ess}}(H(\vec{a}, V)) = [0, \infty)$. Therefore infinitely many eigenvalues exist below 0, if we can show that $(H(\vec{a}, V)\,\varphi_n, \varphi_n) < 0$ for a sequence of real-valued functions in $C_0^\infty(\Omega)$ having pairwise disjoint support. Define W by $W = \vec{b}^2 + V$, then (recall $\vec{a} = \vec{b}$ in Ω)

$$(H(\vec{a}, V)\,\varphi, \varphi) = ((-\Delta + W)\,\varphi, \varphi)$$

for all real-valued $\varphi \in C_0^\infty(\Omega)$. Choose a real-valued function $\varphi \in C_0^\infty(\Omega)$ with norm 1 and put $\varphi_n = n^{-m/2}\varphi(\cdot/n)$. Then $\varphi_n \in C_0^\infty(\Omega)$ and a short calculation shows

$$\lim_{n \to \infty} n^\varepsilon(-\Delta\varphi_n, \varphi_n) = 0$$

$$\limsup_{n \to \infty} n^\varepsilon(W\varphi_n, \varphi_n) \leqslant -\text{const.} \inf_{x \in \Omega} |x|^\varepsilon V_-(x).$$

Hence $\limsup_{n \to \infty} n^\varepsilon(H(\vec{a}, V)\,\varphi_n, \varphi_n) < 0$, which proves our theorem. ▨

3. CONCLUDING REMARKS

We like to mention that most results of this note can be extended to Schrödinger operators including highly singular electric potentials:

Define the class

$\mathcal{N} := \{(\vec{a}, V) \mid \vec{a} \in L^2_{\mathrm{loc}},\ 0 < V$ measurable, $Q(H(\vec{a})) \cap Q(V)$ dense in $L^2\}$ and let $H(\vec{a}, V)$ be the self-adjoint Schrödinger operator associated with the quadratic form

$$Q(h) = Q(H(\vec{a})) \cap Q(V)$$

$$h(\varphi, \psi) = (\vec{P}(\vec{a})\varphi, \vec{P}(\vec{a})\psi) + (V^{1/2}\varphi, V^{1/2}\psi).$$

Using the results of this note the following theorems can be shown.

THEOREM 3.1. *Let* $(\vec{a}, V) \in \mathcal{N}$, $\vec{b} \in L^2_{\mathrm{loc}}$ *and* $\vec{b} = \vec{a} + \vec{\nabla}\lambda$ *in* $\mathcal{D}'$. *Then* $(b, V) \in \mathcal{N}$ *and* $e^{i\lambda}H(\vec{a}, V)e^{-i\lambda} = H(\vec{b}, V)$.

THEOREM 3.2. *Let* $(\vec{a}, V) \in \mathcal{N}$ *and let* E *be the spectral measure of* $H(\vec{a}, V)$ *Then the following assertions hold:*

(i) $\dim R(E[(\lambda - \mu)^2 - 2\mu^2, (\lambda + \mu)^2]) \geqslant \dim R(E[-\mu^2, \mu^2])$ $(\lambda, \mu \geqslant 0)$;

(ii) *either* $\sigma_{\mathrm{ess}}(H(\vec{a}, V)) = \emptyset$ *or* $\sigma_{\mathrm{ess}}(H(\vec{a}, V))$ *is unbounded;*

(iii) *either* $\inf\sigma(H(\vec{a}, V)) > 0$ *or* $\sigma(H(\vec{a}, V)) = [0, \infty)$.

THEOREM 3.3. *Let* $(\vec{a}, V) \in \mathcal{N}$. *Assume there is a quasi-conical region* Ω *and a vector potential* $\vec{b} \in L^2_{\mathrm{loc}}$ *such that* $\operatorname{curl}\vec{a} = \operatorname{curl}\vec{b}$ *in* Ω *and*

$$\lim_{|x|\to\infty} \int_{\Omega \cap S_x} (\vec{b}^2 + V) = 0.$$

Then $\sigma_{\mathrm{ess}}(H(\vec{a}, V)) = [0, \infty)$.

Moreover, Corollary 2.6 remains valid if $\mathcal{M}_1$ is replaced by $\mathcal{N}$. We mention that the potential V in Theorem 3.3 may have severe singularities on any closed subset $\Gamma \subset \mathbf{R}^m \setminus \Omega$ of measure 0. Thus Theorem 3.3 generalizes a result of Colgen [6] and shows in particular that no perturbation theory is needed to prove it.

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