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PRIME ACTIONS OF COMPACT ABELIAN GROUPS ON THE HYPERFINITE TYPE II_1 FACTOR

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§ 1.

This note is intended as an appendix to Ocneanu's paper on amenable group actions. It seems likely that, using Ocneanu's results one may obtain a satisfactory classification of all actions of a separable compact abelian group on the hyperfinite type II_1 factor R. (Finite group actions are classified in [6]). Here we restrict ourselves to prime actions (i.e. ones whose fixed point algebra is a factor) because the classification is easy to understand, and, using Ocneanu's work and Takesaki duality, easy to prove. An understanding of abelian actions will certainly be important for the classification of classical group actions as the maximal torus is a compact abelian group. In this paper "action" will mean faithful action.

An ergodic action is certainly prime and we shall show that our classification in this case is the same as that of [8].

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§ 2.

The characteristic invariant was introduced in [5] and [6], but for an abelian group acting on a factor it takes a particularly simple form which we shall now describe. Let M be a factor, A a discrete abelian group and $\alpha: A \to \operatorname{Aut} M$ an action. Let $N < A$ be the subgroup of A acting by inner automorphisms and choose for each $h \in N$ a unitary v_h in M with $\alpha_h = \operatorname{Ad} v_h$. Then since M is a factor there are scalars $\lambda(g, h) \in \mathbf{T}$ $(=\{z \in \mathbf{C}: |z| = 1\})$ with $\alpha_g(v_h) = \lambda(g, h)v_h$. The function $\lambda: A \times N \to \mathbf{T}$ so defined is independent of the choice of the v_h's and is a homomorphism in both variables (i.e. is bilinear for the **Z**-module structure). The restriction of λ to $N \times N$ is antisymmetric, i.e. $\lambda(h, h) = 1$ for $h \in N$.

With this in mind, let $\Lambda(A, N)$ be the group of all functions from $A \times N$ to $\mathbf{T}$ which are bilinear and antisymmetric when restricted to $N \times N$. The characteristic invariant $\Lambda(\alpha)$ of the action α is the element λ of $\Lambda(A, N)$ defined above. (Strictly speaking N is also a part of the characteristic invariant.) λ is clearly invariant to outer conjugacy.

Now let us return to compact abelian groups. Let G be compact abelian and let M be a von Neumann algebra acting on the separable Hilbert space $\mathscr{H}$; let $\alpha: G \to \operatorname{Aut} M$ be an action continuous in the sense that, for each $x \in M$, the map $g \mapsto \alpha_g(x)$ is strongly continuous. The crossed product $M \times_\alpha G$ of M by G is the von Neumann algebra on $L^2(G, H)$ generated by the operators $\pi(x)$ for $x \in M$ and u_g for $g \in G$ where $(\pi(x)f)((g) = \alpha_{g^{-1}}(x)f(g)$ and $u_g f(h) = f(gh)$ for $f \in L^2(G, H)$. The dual action $\hat{\alpha}$ of $\hat{G}$ on $M \times_\alpha G$ is the unique action defined by

$$\hat{\alpha}_\gamma(\pi(x)) = \pi(x) \quad \text{and} \quad \hat{\alpha}_\gamma(u_g) = \gamma(g)u_g.$$

The dual action may be spatially implemented, which shows its existence. It is well known (see [9]) that the fixed point algebra M^α $(=\{x \in M : \alpha_g(x)=x \quad \forall g \in G\})$ is a factor iff $M \times_\alpha G$ is a factor.

We will call an action α *prime* if either of these conditions holds. In [1], prime actions were called "minimal". We shall prove the following theorem.

THEOREM. *Two prime actions α and β of the compact abelian group G on the hyperfinite type* II_1 *factor R are conjugate iff*

(a) $$R^\alpha \cong R^\beta$$

and

(b) $$\Lambda(\hat{\alpha}) = \Lambda(\hat{\beta}).$$

The proof of the theorem is based on the following two results:

TAKESAKI DUALITY. (see [4] or [10]). *The second crossed product $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is ismorphic to $M \overline{\otimes} \mathscr{L}(L^2(G))$ under an isomorphism sending the second dual action $\hat{\hat{\alpha}}$ to the action $\alpha \otimes \rho$, $\rho: G \to \operatorname{Aut}(\mathscr{L}(L^2(G)))$ being conjugation by the unitaries of the regular representation.*

OCNEANU'S THEOREM. ([7]). *Let M be a factor with separable predual and such that $M \cong M \overline{\otimes} R$. Let $\overline{\operatorname{Int}} M$ and $\operatorname{Ct} M$ be as in* [2], *and let G be a countable discrete amenable group. Let α and β be actions of G on M with $\alpha(G)$ and $\beta(G) \subset \overline{\operatorname{Int}} M$ and*

$$\alpha^{-1}(\operatorname{Ct} M) = \alpha^{-1}(\operatorname{Int} M) = \beta^{-1}(\operatorname{Ct} M) = \beta^{-1}(\operatorname{Int} M) = N \lhd G.$$

Then there is a $\theta \in \overline{\mathrm{Int}}\, M$ *and an* α*-cocycle* $g \mapsto v_g$ *with*

$$\theta\beta_g\theta^{-1} = \mathrm{Ad}\, v_g\alpha_g$$

iff α *and* β *have the same characteristic invariant.*

In particular abelian groups are amenable so we may apply Ocneanu's theorem to the dual $\hat{G}$ of the compact (separable) abelian group G.

Proof of the theorem. If $\alpha \otimes \rho$ denotes the stabilized action mentioned in Takesaki duality above, then we know (e.g. [4], p. 22) that $R\times_\alpha G=[R\overline{\otimes}\mathscr{L}(L^2(G))]^{\alpha\otimes\rho}$. If $p = \int_G u_g \mathrm{d}g$ (where dg is normalised Haar measure on G) then it is an easy calculation to show that $p \in R\times_\alpha G$ and $p(R\times_\alpha G)p = R^\alpha p$. Moreover p is a minimal projection in $\mathscr{L}(L^2(G))$ so we may define a trace τ on $R\otimes\mathscr{L}(L^2(G))$ with $\tau(p) = 1$. Since $R\times_\alpha G$ is a factor and $p \in R\times_\alpha G$, the restriction of τ to $R\times_\alpha G$ is a normal faithful semifinite trace. We see that there are three possibilities for $R\times_\alpha G$:

(1) $R\times_\alpha G$ is of type I_∞ when R^α is of type I_n.

(2) $R\times_\alpha G$ is of type II_1 when G is finite.

(3) $R\times_\alpha G$ is of type II_∞ when $R^\alpha \cong R$, and $|G| = \infty$.

In case (1), the trace of a minimal projection in $R\times_\alpha G$ is $1/n$ since reduction by p gives a type I_n factor. In cases (2) and (3) $R\times_\alpha G$ is necessarily approximately finite dimensional since $p(R\times_\alpha G)p$ is.

This shows in particular that conditions a) and b) are necessary for α and β to be conjugate. Let us show that they are also sufficient.

The above comments also hold for β so in all cases we may choose an isomorphism $\sigma: R\times_\alpha G \to R\times_\beta G$ preserving the restrictions of τ.

If $R\times_\alpha G$ is I_∞, then since $\Lambda(\hat{\alpha}) = \Lambda(\hat{\beta})$, the actions $\hat{\alpha}$ and $\sigma\hat{\beta}\sigma^{-1}$ differ by a cocycle. To apply Ocneanu's theorem in the other cases we need to know that $\hat{\alpha}$ and $\hat{\beta}$ are approximately inner. This is true iff they preserve τ. But this is clear (e.g. from their effect on p). Hence in general we may suppose that there is an $\hat{\alpha}$-cocycle $\gamma \mapsto v_\gamma$ in $R\times_\alpha G$ with $\sigma\hat{\beta}_\gamma\sigma^{-1} = \mathrm{Ad}\, v_\gamma\hat{\alpha}_\gamma$.

Thus there is a trace-preserving isomorphism

$$\varphi: (R\times_\alpha G)\times_{\hat{\alpha}}\hat{G} \to (R\times_\beta G)\times_{\hat{\beta}}\hat{G}$$

defined by

$$\varphi\Big(\sum_{\gamma\in\hat{G}} a_\gamma u_\gamma\Big) = \sum_{\gamma\in\hat{G}} v_\gamma^*\sigma(a_\gamma)u_\gamma\,,$$

where a_γ are in $R\times_\alpha G$. Clearly $\varphi\hat{\hat{\beta}}\varphi^{-1} = \hat{\hat{\alpha}}$. By Takesaki duality we deduce that $\alpha\otimes\rho$ and $\beta\otimes\rho$ are conjugate. Even better, since the isomorphism of Takesaki duality sends p onto p, we deduce the existence of an automorphism ψ of $R\overline{\otimes}\mathscr{L}(L^2(G))$, which preserves τ and such that $\psi\beta\otimes\rho(g)\psi^{-1} = \alpha\otimes\rho(g)$. But then $\psi(p)$ has the same finite trace as p and p, and $\psi(p)$ are both in $R\overline{\otimes}\mathscr{L}(L^2(G))^{\alpha\otimes\rho}$. Hence there is a unitary x in this algebra with $\operatorname{Ad} x(\psi(p)) = p$. But $\beta\otimes\rho|_p$ is conjugate to β and $\alpha\otimes\rho|_p = \operatorname{Ad} x\psi(\beta\otimes\rho)\psi^{-1}\operatorname{Ad} x^*|_{\operatorname{Ad} x\psi(p)}$ is conjugate to both α and β. Q.E.D.

§ 3. EXISTENCE OF ACTIONS

We now answer the question: which elements of $\Lambda(\hat{G}, N)$ arise as $\Lambda(\hat{\alpha})$ for a prime action of G on R ? Let us begin by showing that our classification agrees with that of [8] in the case of ergodic actions.

In [8] it is shown that all ergodic actions of G on R may be constructed as follows: choose a 2-cocycle $\mu: \hat{G}\times\hat{G}\to\mathbf{T}$ representing an element of $H^2(\hat{G}, T)$ and form the twisted group algebra $\mathbf{C}_\mu\hat{G}$ ([5]). Complete with respect to the obvious trace. Then G acts ergodically on $\mathbf{C}_\mu\hat{G}$ by the dual action. To see when $\mathbf{C}_\mu\hat{G}$ is a factor, we use the fact that $H^2(\hat{G}, T)$ is isomorphic to the group $\Lambda(\hat{G}, \hat{G})$ of antisymmetric bilinear $\mathbf{T}$-valued functions. The isomorphism is given by the antisymmetrization map $\mu\mapsto\lambda$ where $\lambda(\gamma, \delta) = \mu(\gamma, \delta) - \mu(\delta, \gamma)$, on normalized cocycles (see [5], p. 245). For $\mathbf{C}_\mu\hat{G}$ to be a factor it is necessary and sufficient that λ be non-degenerate, i.e. the map from $\hat{G}$ to G defined by $\gamma\mapsto\lambda(\gamma,\cdot)$ is injective. For $\mathbf{C}_\mu\hat{G}$ to be R one also needs that $\hat{G}$ be infinite.

We claim that if an action α of G is constructed as above from μ and λ, then $\Lambda(\hat{\alpha}) = \lambda$.

It suffices to exhibit v_γ's for $\gamma\in G$ with $\operatorname{Ad} v_\gamma = \hat{\alpha}_\gamma$ on $R\times_\alpha G$ and $\operatorname{Ad} v_\gamma(v_\delta) = \lambda(\gamma, \delta)v_\delta$. Let $\{z_\gamma\}$ be the unitary basis of $\mathbf{C}_\mu\hat{G}$ (so that $z_\gamma z_\delta = \mu(\gamma, \delta)z_{\gamma\delta}$) and $g\mapsto u_g$ be the unitary representation of the crossed product. Put $v_\gamma = z_\gamma u^*_{\sigma(\gamma)}$ where $\sigma(\gamma)$ is the unique element of G with $\delta(\sigma(\gamma)) = \lambda(\gamma, \delta)$ for all $\delta\in\hat{G}$. Then one checks that $\operatorname{Ad} v_\gamma(z_\delta) = z_\delta$ and $\operatorname{Ad} v_\gamma(u_g) = \gamma(g)u_g$ so that $\operatorname{Ad} v_\gamma$ is $\hat{\alpha}_\gamma$. Also $\operatorname{Ad} v_\gamma(v_\delta) = \lambda(\gamma, \delta)v_\delta$.

It follows that, given a type I_n factor M and a non-degenerate element λ of $\Lambda(\hat{G}, \hat{G})$ one may construct an action α of G on R with $R^\alpha\cong M$ and $\Lambda(\hat{\alpha}) = \lambda$ by constructing an ergodic one β as above on R and putting $\alpha = \beta\otimes\mathrm{id}$ on $R\otimes M\cong R$. Any action of G with fixed point algebra M arises in this way as R splits as $R\otimes M$.

We see that there are in fact restrictions on the characteristic invariant of the dual action. Indeed for any action α, $(R\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$ is a factor and one may calculate the centre of this crossed product in terms of $\Lambda(\hat{\alpha}) = \lambda$. For a prime action a necessary and sufficient condition for $(R\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$ to be a factor is that the subgroup $F_\lambda = \{\delta\in N : \lambda(\gamma, \delta) = 1 \quad \forall\,\gamma\in\hat{G}\}$ be reduced to the indentity.

Given a λ with F_λ reduced to the indentity, one may construct an action α of G on R with $R^\alpha\cong R$ and $\Lambda(\hat{\alpha}) = \lambda$ as follows: by Theorem 2.1 of [5], choose an action β of $\hat{G}$ on R with $\Lambda(\beta)=\lambda$. Since F_λ is trivial, the crossed product $R\times_\beta\hat{G}$ is a factor isomorphic to R by [3]. The action α of G is defined as $\hat{\beta}$, and by Takesaki duality $\Lambda(\hat{\alpha})=\lambda$.

It follows from the above constructions and the theorem that every prime action α of G on R with $R^\alpha\cong R$ is dual to some action of $\hat{G}$ on R^α. This could have been deduced directly from Ocneanu's nonabelian cohomology result and stability of prime actions. Then the theorem could have been proved directly from the classification of actions on R. We prefer to give the proof that figures above as it illustrates a general method for classifying compact group actions: First classify up to stable conjugacy, i.e. classify actions of the form $\alpha\otimes\rho$ on $R\otimes\mathscr{L}(L^2(G))$ then, to compare two stably conjugate actions α and β, calculate the space of all projections in $R\times_\alpha G$ which are conjugate to p under the automorphism group of $R\otimes\mathscr{L}(L^2(G))$. The group of automorphisms of $R\otimes\mathscr{L}(L^2(G))$ which commute with $\alpha\otimes\rho$ acts on this space so let Q be the space of orbits. Since α and β are stably conjugate, there is θ with $\theta(\beta\otimes\rho)\theta^{-1}=\alpha\otimes\rho$ on $R\otimes\mathscr{L}(L^2(G))$. The image of $\theta(p)$ in Q is an invariant of the action β with respect to α. The actions α and β are conjugate iff the images in Q of p and $\theta(p)$ are the same. In general calculating Q will involve the use of some model action $\alpha\otimes\rho$.

§ 4. PRIME TORUS ACTIONS

The n-torus $\mathbf{T}^n$ is the dual of $\mathbf{Z}^n$ and by the structure theorem for finitely generated abelian groups, if N is a subgroup of $\mathbf{Z}^n$ there is a basis $e_1,\ldots,e_n$ of $\mathbf{Z}^n$ and numbers $m_1,\ldots,m_k$, $k\leqslant n$, such that $m_1e_1,\ldots,m_ke_k$ is a basis for N. An element $\lambda\in\Lambda(\mathbf{Z}^n,N)$ is bilinear so it may be specified by its values on the pairs (e_i, m_je_j).

Let σ_{ij} be elements of $\mathbf{T}$ satisfying (in additive notation) $(i,j\leqslant k)$ $m_i\sigma_{ij}=-m_j\sigma_{ji}$ and $m_i\sigma_{ii}=0$. Then setting $\lambda(e_i,m_je_j)=\sigma_{ij}$ defines an element of $\Lambda(\mathbf{R}^n,N)$, and all elements occur in this way. In order for F_λ to be zero we require that $\sum_{j=1}^{k} r_j\sigma_{ij}$ for all i implies that $r_j=0$ for all $j=1,2,\ldots,k$.

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