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ORTHOGONAL PAIRS OF $*$-SUBALGEBRAS IN FINITE VON NEUMANN ALGEBRAS

SORIN POPA

1. INTRODUCTION

Let M be a finite von Neumann algebra with a finite faithful normal trace τ, $\tau(1) = 1$. Let $B_1, B_2 \subset M$ be von Neumann subalgebras of M. B_1 is orthogonal to B_2 if $\tau(b_1 b_2) = 0$ for all $b_1 \in B_1$, $b_2 \in B_2$, $\tau(b_1) = \tau(b_2) = 0$.

Starting from this notion we present in this paper a method of computing the von Neumann algebra generated by the normalizer of certain subalgebras in some type II_1 factors. Let $B \subset M$ be a von Neumann subalgebra, $\mathcal{N}(B)$ its normalizer in M. In order to compute $\mathcal{N}(B)''$ it is enough to compute its orthogonal in M (that is the set of all $x \in M$ such that $\tau(x^* b) = 0$ for $b \in B$). To this end we first show the following technical result: if B has no atoms and u is a unitary element in M such that uBu^* is orthogonal to B then u is orthogonal to $\mathcal{N}(B)''$. Consequently, to determine $\mathcal{N}(B)''$ it is sufficient to find enough such unitary elements in M.

This method is particularly useful in the case when M is the von Neumann algebra $L(G)$ associated to the left regular representation of a discrete group G.

Most of the paper contains applications of this method.

We show that the hyperfinite II_1 factor R has a proper subfactor R_0 such that any maximal abelian $*$-subalgebra (abbreviated as MASA) of R_0 is maximal abelian in R. This gives a negative answer to the type II_1 case of a problem of Kadison (cf. [8], Problem 12; for the case when M is of type III, see [16]).

Let S be an arbitrary nonempty set and $S_0 \subset S$ a nonempty subset of S. Denote by $\mathbf{F}_S$ the free group with generators in S. Then $L(\mathbf{F}_{S_0})$ is naturally embedded as a subalgebra in $L(\mathbf{F}_S)$. We show that if B is an arbitrary completely nonatomic subalgebra of $L(\mathbf{F}_{S_0})$ then the normalizer of B in $L(\mathbf{F}_S)$ is contained in $L(\mathbf{F}_{S_0})$. In particular, this shows that any MASA of $L(\mathbf{F}_{S_0})$ is maximal abelian in $L(\mathbf{F}_S)$. It also shows that the commutant in $L(\mathbf{F}_S)$ of any completely nonatomic von Neumann subalgebra is separable. Thus, if S is uncountable then $L(\mathbf{F}_S)$ is not the tensor product of two type II_1 factors. Moreover, in this case, $L(\mathbf{F}_S)$ has no regular MASA's.

Let M be a type II_1 factor, ω a free ultrafilter on $\mathbf{N}$ and denote by M^ω the type II_1 factor defined as in [10], [4]. We show in the last part of the paper that for any M, M^ω has no regular MASA's.

We mention that in Section 3 of the paper we include a discussion on orthogonal pairs of MASA's in the algebra M_n of n by n complex matrices. More precisely, we show that for any n there exists such a pair of MASA's in M_n and that for n prime there exist $n+1$ mutually orthogonal MASA's in M_n. We also present a conjecture on the structure of orthogonal pairs of MASA's in M_n.

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2. DEFINITIONS AND TECHNICAL RESULTS

Let M be a finite von Neumann algebra of countable type and τ a fixed normal finite faithful trace on M, $\tau(1) = 1$. If $B \subset M$ is a von Neumann subalgebra then we denote by E_B the unique normal τ-preserving conditional expectation of M onto B (see [15], Chapter 10). Note that if $A_0 \subset M$ is finite dimensional and abelian, with minimal projections $e_1, \ldots, e_n$, then

$$E_{A_0}(x) = \sum_{i=1}^{n} \frac{\tau(e_i x e_i)}{\tau(e_i)} e_i, \quad E_{A_0' \cap M}(x) = \sum_{i=1}^{n} e_i x e_i, \quad x \in M.$$

We also denote by $\|x\|_2 = \tau(x^* x)^{1/2}$ the Hilbert norm on M given by τ. If $\mathscr{H}_\tau$ is the Hilbert space obtained by completing M in the norm $\|\ \|_2$, then $E_B(x)$ may be regarded as the orthogonal projection of the vector $x \in M \subset \mathscr{H}_\tau$ on the subspace $B \subset \mathscr{H}_\tau$. It follows that if B_0, $B \subset M$ are von Neumann subalgebras then $B_0 \subset B$ iff $\|E_{B_0}(x)\|_2 \leqslant \|E_B(x)\|_2$, for all $x \in M$. In particular, for $B_0 = \mathbf{C}$ we get $|\tau(x)| \leqslant \|E_B(x)\|_2$, $x \in M$. Moreover, if $S^\perp = \{x \in M \mid \tau(x^* s) = 0$ for all $s \in S\}$ denotes the orthogonal in M of a subset $\emptyset \neq S \subset M$, then we have $(B^\perp)^\perp = B$ and $B_0 \subset B$ iff $B_0^\perp \supset B^\perp$.

2.1. LEMMA. *Let B_1, B_2 be von Neumann subalgebras of M. The following conditions are equivalent:*

(i) $\tau(b_1 b_2) = 0$, *for* $b_1 \in B_1$, $b_2 \in B_2$, $\tau(b_1) = \tau(b_2) = 0$;

(ii) $\tau(b_1 b_2) = \tau(b_1)\tau(b_2)$, *for all* $b_1 \in B_1$, $b_2 \in B_2$;

(iii) $\|b_1 b_2\|_2 = \|b_1\|_2 \|b_2\|_2$, *for all* $b_1 \in B_1$, $b_2 \in B_2$;

(iv) $E_{B_1} E_{B_2}(x) = \tau(x) 1_M$, $x \in M$;

(v) $E_{B_1}(B_2) \subset \mathbf{C} \cdot 1_M$.

Moreover (i) — (v) *are equivalent with the analogue conditions obtained by interchanging* B_1 *with* B_2.

Proof. Take $b_1 \in B_1$, $b_2 \in B_2$ to be arbitrary elements. Then

$$\tau((b_1 - \tau(b_1))(b_2 - \tau(b_2))) = \tau(b_1 b_2) - \tau(b_1)\tau(b_2)$$

so that (i) is equivalent to (ii). Since $\|b_1 b_2\|_2^2 = \tau(b_1^* b_1 b_2 b_2^*)$, (ii) implies (iii). Conversely, if (iii) holds then $\tau(b_1 b_2) = \tau(b_1)\tau(b_2)$, for all positive elements $b_1 \in B_1$, $b_2 \in B_2$, and by linearity it follows that (iii) implies (ii). The equivalence between (i) and (iv) follows by the equality

$$E_{B_1}(E_{B_2}(x) - \tau(x)1_M) = E_{B_1}E_{B_2}(x) - \tau(x)1_M ,$$

and by the interpretation of the conditional expectation E_{B_1} as a Hilbert space orthogonal projection. Clearly (iv) implies (v). Conversely, if (v) holds then $\tau(x) = \tau(E_{B_1}E_{B_2}(x))$ (since E_{B_1} and E_{B_2} preserve the trace τ) so that $E_{B_1}E_{B_2}(x) = \tau(x)1_M$. Since (ii) is symmetric in B_1 and B_2 it follows that in all the conditions (i) —(v) we may replace B_1 with B_2. Q.E.D.

2.2. DEFINITION. The von Neumann subalgebras B_1 and B_2 of M are *mutually orthogonal* ($B_1 \perp B_2$) if one of the conditions (i) — (v) of Lemma 2.1 is satisfied.

2.3. REMARKS. 1) Since the orthogonality of two von Neumann subalgebras $B_1, B_2 \subset M$ depends on τ, one should call it τ-orthogonality and write $B_1 \perp_\tau B_2$. However, as for the notation of the norm $\| \ \|_2$ (instead of $\| \ \|_\tau$) and of the τ-preserving conditional expectation E_B (instead of E_B^τ) we shall not mention τ. We do this in order not to complicate the notations. There will be no confusions, since the trace is always assumed fixed from the begining. In fact, in most of the applications the algebra M is a factor, so that the normalized trace τ is unique.

2) By a result of A. Connes (see [15], 10.18) if $B \subset M$ is a von Neumann subalgebra such that $B' \cap M \subset B$ then there exist a unique normal conditional expectation of M onto B and so, in particular, this conditional expectation preserves all traces on M. Suppose $B_1, B_2 \subset M$ are von Neumann subalgebras of M and $B_1' \cap M \subset B_1$. It follows that if $E_{B_1}(B_2) \subset \mathbf{C} \cdot 1_M$ then B_1 is orthogonal to B_2 with respect to any trace on M.

3) Suppose M is commutative and let $(X, \mathcal{X}, \mu)$ be a probability space such that (M, τ) is isomorphic to $(L^\infty(X), \mu)$. If $B_1, B_2 \subset M$ are von Neumann subalgebras then $B_1 \perp B_2$ if and only if the corresponding σ-subalgebras of $\mathcal{X}$ are independent with respect to the measure μ. But in the case M is not commutative it is possible that $B_1 \perp B_2$ although B_1 does not commute with B_2 (as it is the case with all the examples in Sections 3—7). Thus, in the noncommutative case it is not appropriate to use the term "independence" instead of "orthogonality".

From Definition 2.2 we easily deduce:

2.4. LEMMA. *Let $(M_n, \tau_n)_n$ be a sequence of finite von Neumann algebras with normal finite faithful traces, $\tau_n(1) = 1$, $n \geq 1$. Let B_1^n, $B_2^n \subset M_n$ be von Neumann subalgebras such that $B_1^n \perp B_2^n$, $n \geq 1$. Denote by $(M, \tau) = \bigotimes_n (M_n, \tau_n)$, $B_1 = \bigotimes_n (B_1^n, \tau_n | B_1^n)$, $B_2 = \bigotimes_n (B_2^n, \tau_n | B_2^n)$. Then $B_1 \perp B_2$ in (M, τ).*

Proof. It is enough to show that $\tau(xy) = \tau(x)\,\tau(y)$ for x in a total subset of B_1 and y in a total subset of B_2. So, let $x = x_1 \otimes \ldots \otimes x_n \otimes 1$, $y = y_1 \otimes \ldots \otimes y_n \otimes 1$, where $x_k \in B_1^k$, $y_k \in B_2^k$. Then

$$\tau(xy) = \tau((x_1 \otimes \ldots \otimes x_n \otimes 1)(y_1 \otimes \ldots \otimes y_n \otimes 1)) =$$

$$= \tau(x_1y_1 \otimes \ldots \otimes x_ny_n \otimes 1) = \tau_1(x_1y_1) \ldots \tau_n(x_ny_n) =$$

$$= \tau_1(x_1) \ldots \tau_n(x_n)\,\tau_1(y_1) \ldots \tau_n(y_n) = \tau(x_1 \otimes \ldots \otimes x_n \otimes 1) \cdot$$

$$\cdot\, \tau(y_1 \otimes \ldots \otimes y_n \otimes 1) = \tau(x)\tau(y). \qquad \text{Q.E.D.}$$

If $B \subset M$ is a von Neumann subalgebra then we denote by $\mathcal{N}(B) = \{u$ unitary element in $M \mid uBu^* = B\}$ the *normalizer* of B in M. Note that $\mathcal{N}(B)'' = \overline{\text{span}}^{w}\mathcal{N}(B)$ and $\mathcal{N}(B)'' \supset B$, $\mathcal{N}(B)'' \supset B' \cap M$. The subalgebra B is called *regular* if $\mathcal{N}(B)'' = M$. It is called *singular* if $\mathcal{N}(B)'' = B$ (see [5]).

The next lemma is the basic tool in the proof of the main results of the paper.

2.5. LEMMA. *Let $B \subset M$ be a von Neumann subalgebra and $u \in M$ a unitary element. Suppose that for any $\varepsilon > 0$ there exists a finite dimensional abelian von Neumann subalgebra A_ε in B such that $\tau(e) < \varepsilon$ for any minimal projection e in A_ε and $uA_\varepsilon u^* \perp B$. Then u is orthogonal to the algebra $\mathcal{N}(B)''$.*

Proof. Let $v \in \mathcal{N}(B)$, $\varepsilon > 0$ and denote by $e_1, \ldots, e_n$ the minimal projections of A_ε. Since $uA_\varepsilon u^* \perp B$ it follows that $vuA_\varepsilon u^*v^* \perp vBv^* = B$ and so

$$\tau(vue_iu^*v^*e_i) = \tau(vue_iu^*v^*)\,\tau(e_i) = \tau(e_i)^2.$$

Summing up over i we get

$$|\tau(vu)|^2 \leq \|E_{A_\varepsilon' \cap M}(vu)\|_2^2 = \|\sum_i e_ivue_i\|_2^2 =$$

$$= \sum_i \|e_ivue_i\|_2^2 = \sum_i \tau(vue_iu^*v^*e_i) = \sum_i \tau(e_i)^2 \leq$$

$$\leq (\max_j \tau(e_j)) \sum_i \tau(e_i) < \varepsilon.$$

Tending with ε to zero we obtain $\tau(vu) = 0$. Thus $\tau(xu) = 0$ for all $x \in \overline{\text{span}}^{w}\mathcal{N}(B) = \mathcal{N}(B)''$ and u is orthogonal to $\mathcal{N}(B)''$. Q.E.D.

2.6. COROLLARY. *Let B be a von Neumann subalgebra of M and $u \in M$ a unitary element. If there exists a completely nonatomic von Neumann subalgebra $B_0 \subset B$ such that $uB_0u^* \perp B$ then u is orthogonal to $\mathcal{N}(B)''$.*

Proof. Since B_0 has no atoms, given any $\varepsilon > 0$, there exist projections $e_1, \ldots, e_n \in B_0$ such that $\sum_i e_i = 1$, $\tau(e_i) < \varepsilon$. By 2.5 the statement follows. Q.E.D.

Before proceeding with applications of 2.5 and 2.6 we construct some examples of mutually orthogonal MASA's in finite dimensional factors.

3. ORTHOGONAL PAIRS OF MASA'S IN MATRIX ALGEBRAS

In this section M_n will denote the type I_n factor, that is the algebra of n by n complex matrices, and τ will be the unique normalized trace on M_n.

3.1. LEMMA. *Let u_0, $u_1 \in M_n$ be unitary elements satisfying $u_1u_0 = \lambda u_0u_1$, for some primitive root of the identity $\lambda \in \mathbf{C}$. Denote by A_0, A_1 the $*$-algebras generated by u_0 and u_1 respectively. Then A_0 and A_1 are mutually orthogonal MASA's in M_n.*

Proof. Note first that A_0, A_1 are abelian von Neumann subalgebras of M_n, u_0, u_1 being unitary elements. Since $u_1u_0u_1^{-1} = \lambda u_0$, it follows that the spectrum of u_0 is invariant to rotations by λ. Thus $\operatorname{spec} u_0 = \{\mu_0, \mu_0\lambda, \ldots, \mu_0\lambda^{n-1}\}$, for some scalar $\mu_0 \in \mathbf{C}$, $|\mu_0| = 1$. Similary $\operatorname{spec} u_1 = \{\mu_1, \mu_1\lambda, \ldots, \mu_1\lambda^{n-1}\}$, $\mu_1 \in \mathbf{C}$, $|\mu_1| = 1$. It follows that A_0, A_1 have linear dimension n, so that they are maximal abelian in M_n.

Taking $\mu_0^{-1}u_0$, $\mu_1^{-1}u_1$ instead of u_0, u_1 we may suppose $\operatorname{spec} u_0 = \operatorname{spec} u_1 = \{1, \lambda, \ldots, \lambda^{n-1}\}$. Let $e_1^0, \ldots, e_n^0$ be the spectral projections of u_0 corresponding to the spectral sets $\{1\}, \{\lambda\}, \ldots, \{\lambda^{n-1}\}$. Then $u_1e_k^0u_1^* = e_{k-1}^0$, $2 \leqslant k \leqslant n$, $u_1e_1^0u_1^* = e_n^0$. Thus, we may represent M_n as the algebra of all bounded operators on a Hilbert space $\mathscr{H}_n$ with orthonormal basis $\xi_1, \ldots, \xi_n$ such that the projection onto $\mathbf{C}\xi_k$ is e_k, $1 \leqslant k \leqslant n$, and $u_1\xi_k = \xi_{k-1}$, $2 \leqslant k \leqslant n$, $u_1\xi_1 = \xi_n$. It follows that, with respect to he basis $\xi_1, \ldots, \xi_n$, u_0 and u_1 have the matrix form:

$$u_0 = \begin{pmatrix} 1 & & & \\ & \lambda & & 0 \\ & & \ddots & \\ & 0 & & \lambda^{n-1} \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

A straightforward computation shows that $\tau(u_0^k u_1^l) = 0$ whenever $0 \leqslant k, l \leqslant n-1$ and either k or l are not zero. Since any element in A_0 (or A_1) is a linear combination of powers of u_0 (resp. u_1), we get $A_0 \perp A_1$. Q.E.D.

3.2. THEOREM. *For any $n \geqslant 2$ there exist two mutually orthogonal MASA's in M_n. Moreover if n is prime then there exist $n+1$ mutually orthogonal MASA's in M_n.*

Proof. Let u_0, u_1 be two unitaries in M_n, satisfying $u_1 u_0 = \lambda u_0 u_1$, where $\lambda = \exp 2\pi i/n$; for instance

$$u_0 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \lambda^{n-1} \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

By Lemma 3.1 it follows that the $*$-algebras A_0 and A_1, generated by u_0 and respectively u_1, are maximal abelian and mutually orthogonal.

If in addition n is prime then let $u_k = u_0^{k-1} u_1$, $2 \leqslant k \leqslant n$, and denote by A_k the algebra generated by u_k. An easy calculation shows that any pair of unitaries u_i, u_j, $0 \leqslant i < j \leqslant n$, satisfies the conditions of Lemma 3.1. Thus $\{A_i\}_{0 \leqslant i \leqslant n}$ are maximal abelian and mutually orthogonal in M_n. Q.E.D.

3.3. REMARK. Let K_n be the maximal number of mutually orthogonal MASA's in M_n. Since each MASA in M_n has linear dimension n and M_n has linear dimension n^2, it follows that $K_n(n-1)+1 \leqslant n^2$. Thus $K_n \leqslant n+1$, so that for n prime $K_n = n+1$. In the case n is not prime we know two possibilities of constructing sets of mutually orthogonal MASA's: either as in the proof of the case n prime in 3.2 (i.e. consider a set $I \subset \{0, 1, \dots, n\}$ of maximal cardinal such that u_i, u_j satisfy 3.1 for any $i, j \in I$, $i \neq j$) or by decomposing M_n in a tensor product of algebras M_p, with p prime, and then applying 3.2 and 2.6. In both cases one gets $q_n + 1$ mutually orthogonal MASA's in M_n, q_n being the minimal prime divizor of n. Hence $K_n \geqslant q_n + 1$. In particular $K_n \geqslant 3$, for $n \geqslant 2$. In the case n is prime it seems reasonable to believe that the construction in Theorem 3.2 is the only possible one. More precisely we have the following:

3.4. CONJECTURE. *If n is prime and $A_0, A_1 \subset M_n$ are mutually orthogonal MASA's then there exist unitaries $u_0 \in A_0$, $u_1 \in A_1$, such that $u_1 u_0 = \lambda u_0 u_1$, $\lambda = \exp 2\pi i/n$.*

Let us mention that 3.4 is equivalent with the following statement:

If n is prime then there exists a unique unitary matrix of the form $n^{-1/2}(\lambda_{ij})_{1 \leqslant i, j \leqslant n}$, with $\lambda_{i1} = \lambda_{1j} = 1$, $|\lambda_{ij}| = 1$, $1 \leqslant i, j \leqslant n$ (the uniqueness is understood modulo permutations of rows and columns).

This unitary matrix is the one that conjugates the MASA of the main diagonal on a MASA orthogonal to it, and this statement says that it should be equal to $n^{-1/2}(\lambda^{(i-1)(j-1)})_{n\geqslant i,j\geqslant 1}$, where $\lambda = \exp 2\pi i/n$.

We end this section with a useful technical result:

3.4. LEMMA. *Let M be a finite von Neumann algebra of countable type and $M_0 \subset M$ a finite dimensional subfactor of M. If $A_1, A_2 \subset M_0$ are mutually orthogonal MASA's of M_0 then A_1 is orthogonal to $A_2' \cap M$ with respect to any trace on M.*

Proof. Since $(A_2' \cap M)' \cap M = A_2 \subset A_2' \cap M$ it follows that there is a unique normal conditional expectation $E_{A_2' \cap M}$ of M onto $A_2' \cap M$. If $e_1, \ldots, e_n$ are the minimal projections of A_2 then $E_{A_2' \cap M}(x) = \sum_i e_i x e_i$, $x \in M$, so that if $x \in M_0$ then $E_{A_2' \cap M}(x) \in M_0$. But $E_{A_2' \cap M}(x) \in A_2' \cap M$ and so $E_{A_2' \cap M}(x) \in (A_2' \cap M) \cap M_0 = A_2' \cap M_0 = A_2$ for all $x \in M_0$. It follows that if E_0 denotes the conditional expectation of M_0 onto A_1 (which is unique since $A_2' \cap M_0 = A_2$) then E_0 is the restriction of $E_{A_2' \cap M}$ to M_0, so that $E_{A_2' \cap M}(A_1) = E_0(A_1) \subset \mathbf{C}\cdot 1_{M_0} = \mathbf{C}\cdot 1_M$. By 2.1 and 2.3, 2) the statement follows. Q.E.D

4. SOME CONSEQUENCES FOR GROUP ALGEBRAS

We shall consider now the case when M is a group algebra (cf. [11]; see also [15, Chapter 22]). So, let G be a discrete group and λ the left regular representation of G on the Hilbert space $\ell^2(G)$. Denote by $L(G)$ the von Neumann algebra generated by $\lambda(G)$ in $\mathscr{B}(\ell^2(G))$. An operator $T_0 \in \mathscr{B}(\ell^2(G))$ is in $L(G)$ if and only if there exists an element $f_0 \in \ell^2(G)$ such that $T_0(f) = f_0 * f$ for any $f \in \ell^2(G)$. Such an f_0 is called a convolver on $\ell^2(G)$. Hence we may identify $L(G)$ with a subset of $\ell^2(G)$, i.e. with the set of convolvers. Denote by τ the normal finite faithful trace on $L(G)$ given by evaluation in $e \in G$, the unit of G. So, $L(G)$ is a finite von Neumann algebra with the normal finite faithful trace τ, $\tau(1) = 1$. $L(G)$ is a factor iff G is an I.C.C. group, and in this case $L(G)$ is of type II_1. $L(G)$ is completely nonatomic iff G is an infinite group.

It is easy to see that, with the above identification of $L(G)$ as a subset of $\ell^2(G)$, the completion of $L(G)$ in the Hilbert norm $\| \ \|_2$ given by τ is $\ell^2(G)$.

If $G_0 \subset G$ is a subgroup of G then the natural inclusion $\ell^2(G_0) \subset \ell^2(G)$ induces the inclusion of $L(G_0)$ as a von Neumann subalgebra of $L(G)$. The τ-preserving conditional expectation of $L(G)$ onto $L(G_0)$ is induced by the orthogonal projection of $\ell^2(G)$ onto $\ell^2(G_0)$: if $x \in L(G)$ then

$$E_{L(G_0)}(x)(g) = \begin{cases} x(g), & \text{if } g \in G_0 \\ 0 & \text{elsewhere.} \end{cases}$$

If $G_1, G_2 \subset G$ are subgroups then $L(G_1)$ is orthogonal to $L(G_2)$ iff $G_1 \cap \cap G_2 = \{e\}$.

As a consequence of 2.6 we get:

4.1. PROPOSITION. *Let $G_0 \subset G_1 \subset G$ be infinite discrete groups. Suppose that for any $g \in G \setminus G_1, gG_0g^{-1} \cap G_0 = \{e\}$. If $B \subset L(G_0)$ is a completely nonatomic von Neumann subalgebra then $\mathcal{N}(B)'' \subset L(G_1)$. If in addition G_0 is a normal subgroup of G_1 then $\mathcal{N}(L(G_0))'' = L(G_1)$ (all normalizers are considered in $L(G)$).*

Proof. In order to prove $\mathcal{N}(B)'' \subset L(G_1)$ it is enough to prove $\mathcal{N}(B)^{\perp} \supset \supset L(G_1)^{\perp}$. Since $\{\lambda(g) \mid g \in G \setminus G_1\}$ is a total subset of $L(G_1)^{\perp}$, it is enough to show that $\lambda(g) \perp \mathcal{N}(B)''$ for any $g \in G \setminus G_1$. But $gG_0g^{-1} \cap G_0 = \{e\}$, so that $\lambda(g)L(G_0)\lambda(g^{-1})$ is orthogonal to $L(G_0)$. In particular $\lambda(g)B\lambda(g^{-1})$ is orthogonal to B. By 2.6 we get that $\lambda(g) \perp \mathcal{N}(B)''$.

The second part of the statement follows now easily, since the normality of G_0 in G_1 entails $\mathcal{N}(L(G_0))'' \supset L(G_1)$. Q.E.D.

4.2. REMARKS. 1) The preceding proposition can be used instead of Lemma 3 in [5] to obtain all the examples of MASA's in [2], [5], [14].

2) Recall that if (M, τ) is a type II_1 factor then a *central sequence* of M is a sequence of elements $(x_n)_n \subset M$ bounded in the uniform norm and commuting asymptotically with all elements in M, $\|xx_n - x_nx\|_2 \to 0$, $x \in M$. A central sequence is called *trivial* if $\|x_n - \tau(x_n)\|_2 \to 0$. Suppose $B \subset M$ is a von Neumann subalgebra of M and $u \in M$ is a unitary element such that $uBu^* \perp$ B. Then B contains no nontrivial central sequence of M. Indeed, if $(x_n)_n \subset B$ is central for M then $(x_n - \tau(x_n))_n$ is also central, so that we may suppose $\tau(x_n) = 0$, $n \geqslant 1$. It follows that $2\|x_n\|_2^2 = = \|ux_nu^* - x_n\|_2^2 \to 0$ and thus $(x_n)_n$ is trivial.

In particular if G is an I.C.C. group and $G_0 \subset G$ is a subgroup such that $gG_0g^{-1} \cap G_0 = \{e\}$ for some $g \in G$, then the subalgebra $L(G_0)$ contains no nontrivial central sequence of $L(G)$.

From 2.6 we also obtain a criterion for two von Neumann subalgebras of a group algebra to be not unitarily conjugated:

4.3. COROLLARY. *Let G be a discrete group and $G_1, G_2 \subset G$ two infinite subgroups. Suppose that $gG_2g^{-1} \cap G_1 = \{e\}$ for all $g \in G$. If $B_1 \subset L(G_1)$, $B_2 \subset L(G_2)$ are completely nonatomic von Neumann subalgebras then they are not unitarily conjugated in $L(G)$.*

Proof. Suppose that $uB_1u^* = B_2$ for some unitary element $u \in L(G)$. Since $gG_2g^{-1} \cap G_1 = \{e\}$ it follows that

$$\lambda(g)uB_1u^*\lambda(g)^* = \lambda(g)B_2\,\lambda(g)^* \subset L(gG_2g^{-1}) \perp L(G_1).$$

Thus $\lambda(g)uB_1u^*\lambda(g)^* \perp B_1$. By 2.6 we get that $\lambda(g)u$ is orthogonal to $\mathcal{N}(B_1)''$. In particular $\tau(\lambda(g)u) = 0$ for all $g \in G$, so that $\tau(xu) = 0$ for all $x \in L(G) = \overline{\text{span}}^{\text{w}}\{\lambda(g) \mid g \in G\}$. Hence $1 = \tau(1) = \tau(u^*u) = 0$, which is a contradiction.

Q.E.D.

5. APPLICATIONS TO THE HYPERFINITE II_1 FACTOR

In connection with the strong form of the Stone-Weierstrass theorem for C^*-algebras, R. V. Kadison asked at the Baton Rouge Conference in 1967 the following question: if all MASA's of a subfactor $N \subset M$ are maximal abelian in M does it follow that $N = M$? This problem was answered in the negative by Takesaki ([16]), who gave a conterexample for the case when M is of type III. The next theorem give another counterexample, for the case when M is of type II_1.

5.1. THEOREM. *If R is the separable hyperfinite II_1 factor then there exists a proper subfactor R_0 in R such that for any nonatomic von Neumann subalgebra $B \subset R_0, \mathcal{N}(B)'' \subset R_0$. In particular any MASA of R_0 is maximal abelian in R and its normalizer in R is contained in R_0.*

Proof. Using Section 3 and Connes' results in [4] all we have to do is to construct a countable I.C.C. amenable group G with a proper I.C.C. subgroup G_0 such that $gG_0g^{-1} \cap G_0 = \{e\}$ for all $g \in G \setminus G_0$.

Let G_0 be the group of affine transformations over the rationals. Then G_0 is amenable and I.C.C.. Take $P(G_0)$ to be the group of all the permutations of G_0 and denote by $S \subset P(G_0)$ the normal subgroup of finite permutations. G_0 may be identified with the subgroup of translations in $P(G_0)$. Denote by G the group generated by S and G_0 in $P(G_0)$. It is easy to check that G/S is isomorphic to G_0. Since S is also amenable, G is amenable. Moreover $gG_0g^{-1} \cap G_0 = \{e\}$ for all $g \in G \setminus G_0$. This implies that G is I.C.C., since G_0 is I.C.C.. Q.E.D.

5.2. REMARKS. 1) The subfactor $R_0 \subset R$ constructed in Theorem 2.1 satisfies in particular $\mathcal{N}(R_0)'' = R_0$, i.e. R_0 is singular in R. In fact, if $R_0 \subset R$ is a subfactor such that any maximal abelian $*$-subalgebra of R_0 is maximal abelian in R then R_0 is singular in R. Suppose on the contrary that $R_1 = \mathcal{N}(R_0)'' \neq R_0$. Using Connes' theorem ([14]) and Ocneanu's results ([12]), as in [7] it follows that there exist an amenable group G acting freely on the hyperfinite factor R_0, such that $R_1 = R_0 \times G$ and such that the inclusion $R_0 \subset R_1$ becomes the natural inclusion of R_0 in $R_0 \times G$. Using again Ocneanu's results we can choose any free action of G on R_0. For instance we can take $R_0 = L^\infty(\mathbf{T}^G) \times \alpha$, where $\mathbf{T}^G$ is the probability space of all sequences $(t_g)_{g \in G}$ in the thorus $\mathbf{T}$, α is the action on $\mathbf{T}^G$ given by the same irrational rotation over each $\mathbf{T}$, and G to act on R_0 as the Bernoulli shift on $L^\infty(\mathbf{T}^G)$ and trivially on α. Thus, the unitary elements u_α and u_g canonically implementing the actions α and G commutes in R_1, although u_α generates a maximal abelian subalgebra in R_0. It follows that R_0 has a maximal abelian subalgebra which is not maximal abelian in R_1 and thus it is not maximal abelian in R.

2) In [17] R. J. Tauer constructed the following examples of MASA's in R. Let $\{M_n\}_{n\geq 1}$ be an increasing sequence of matrix subalgebras such that $\overline{\bigcup_n M_n}^{\mathrm{w}} = R$. Take $A_n \subset M_n$ to be maximal abelian in M_n, such that $A_n \subset A_{n+1}$. Then $A = \overline{\bigcup_n A_n}^{\mathrm{w}} \subset$ $\subset R$ is maximal abelian. Indeed, let $x \in M$, $[x, A] = 0$. Then $[x, A_n] = 0$ so that $[E_{M_n}(x), A_n] = 0$, which implies $E_{M_n}(x) \in A_n$. Thus $x = \lim_n E_{M_n}(x) \in \overline{\bigcup_n A_n}^{\mathrm{w}} = A$.

Take now $\mathscr{U}_n \subset M_n$ to be the set of nilpotent minimal partial isometries normalizing A_n (i.e. $v \in M_n$, $vA_nv^* \subset A_n$, vv^*, v^*v minimal projections in M_n and $(v^*v)(vv^*) = 0$); let $\mathscr{U} = \bigcup_n \mathscr{U}_n$. Then $A^\perp = \overline{\mathrm{sp}}^{\mathrm{w}}\, \mathscr{U}$. Suppose that for a certain A the set $\mathscr{U}$ may be splitted in two sets, $\mathscr{U} = \mathscr{U}^1 \cup \mathscr{U}^2$, where $\mathscr{U}^1$ is included in the groupoid associated to $\mathscr{N}(A)$ and $\mathscr{U}^2$ consists of elements v in $\mathscr{U}$ for which there exists a completely nonatomic abelian subalgebra $B \subset A$ (B depends on v) such that vBv^* is orthogonal to Avv^* in Rvv^*. Using a slightly different version of 2.6 we obtain that every $v \in \mathscr{U}^2$ is orthogonal to $\mathscr{N}(A)''$, so that $\mathscr{U}^2$ is orthogonal to $\mathscr{N}(A)''$ and $\mathscr{N}(A)'' = (A \cup \mathscr{U}^1)''$. Thus for such an algebra A, $\mathscr{N}(A)''$ is completely determined. Note that if $\mathscr{U}^1 = \emptyset$, $\mathscr{U}^2 = \mathscr{U}$, then A is singular (see [17], Section 4); if $\mathscr{U}^1$ is big enough and $\mathscr{U}^2 \neq \emptyset$, then A is semiregular but not regular ([17], Section 5). Using Section 3, the algebras A_n can be constructed recursively such that $\mathscr{U}$ satisfies the above condition and such that $\mathscr{U}^1$ and $\mathscr{U}^2$ have certain properties (for instance $\mathscr{U}^1 = \emptyset$, $\mathscr{U}^2 = \mathscr{U}$).

The next result give a counterexample to a conjecture of V. Jones and the author (cf. 3.2 in [7]):

5.3. THEOREM. *Let $A \subset R$ be a regular MASA in the hyperfinite* II_1 *factor. There exists a proper subfactor $R_1 \subset R$ containing A such that for any subfactor $A \subset N \subset R_1$, the normalizer of N in R is contained in R_1. In particular, there are no regular subfactors of R between A and R_1.*

Proof. Let G be the group of affine transformations over $\mathbf{Q}$. Denote by T, $H \subset G$ the subgroups of translations and respectively homotheties over $\mathbf{Q}$. Note that $gHg^{-1} \cap H = \{e\}$ for all $g \in G \setminus H$ ($e = \mathrm{id}_{\mathbf{Q}}$ is the unit of G).

Let (X_0, μ_0) be a two point probability space and denote by $(X, \mu) = \prod_{g\in G} (X_0, \mu_0)_g$. Take $A = L^\infty(X)$ and denote by α the Bernoulli shift action of G on A. Since G is amenable $R = A \times_\alpha G$ is the hyperfinite II_1 factor (cf. [4]). Let $R_1 = A \times_\alpha H$. Also, denote by $\{u_g\}_{g\in G}$ the unitary elements of $R = A \times_\alpha G$ which canonically implement the action α. A and R_1 will be identified with their canonical images in R. We infer that $A \subset R_1 \subset R$ satisfy the requirements. Indeed, let $A \subset N \subset R_1$ be a subfactor. Given $\varepsilon > 0$ let $e_1, \ldots, e_n$ be projections in A such that $\sum_i e_i = 1, \tau(e_i) = n^{-1}$, $1 \leqslant i \leqslant n$, where τ is the unique normalized trace on R. Thus $e_1, \ldots e_n$ are equivalent projections in R and also in N. So, we may choose a matrix algebra $M_n \subset N$ such that $e_1, \ldots, e_n$ are in M_n and generate a MASA in M_n. Let $A_0 = \text{span}\{e_i\}_i$ and choose A_1 a MASA in M_n such that $A_1 \perp A_0$ (cf. Section 3). By 3.5 we get $A_1 \perp A_0' \cap R \supset A$, so that $A_1 \perp A$. We show that $u_g A_1 u_g^* \perp N$ for every $g \in G \setminus H$. To do this let $x \in A_1, \tau(x) = 0$. Then $x \in R_1$ (since $A_1 \subset M_n \subset N \subset R_1$) and x is orthogonal to A. Thus $x = \sum_{\substack{h \in H \\ h \neq e}} a_h u_h$, with $a_h \in A$. Consequently

$$u_g x u_g^* = \sum_{\substack{h \in H \\ h \neq e}} (u_g a_h u_g^*) u_{ghg^{-1}} = \sum_{g' \in G \setminus H} b_{g'} u_{g'},$$

with $b_{g'} \in A$. It follows that $u_g x u_g^*$ is orthogonal to R_1 and hence $u_g A_1 u_g^* \perp N$. Now Lemma 2.5 applies and we get that u_g is orthogonal to $\mathcal{N}(N)''$. Since $A \subset \mathcal{N}(N)''$ this implies that au_g is orthogonal to $\mathcal{N}(N)''$ for all $a \in A$. Thus $R_1^{\perp} = \overline{\text{span}}^{\text{w}} \{Au_g \mid g \in G \setminus H\} \subset (\mathcal{N}(N)'')^{\perp}$. Q.E.D.

5.4. REMARKS. 1) The subfactors R_0, R_1 of R constructed in 5.1, 5.3 have trivial relative commutant in R, since both contain MASA's of R.

2) Using 4.2. 2) one can compute the set of central sequences of R contained in R_0 (resp. R_1). So, let $(x_n)_n \subset R_0$, $(y_n)_n \subset R_1$ be central sequences of R. By the proof of 4.1 there exist unitary elements $u \in R$ such that $uR_0u^* \perp R_0$. It follows that $(x_n)_n$ is trivial. Next we show that $\|y_n - E_A(y_n)\|_2 \to 0$, where $A \subset R_1 \subset R$ is the regular MASA in the proof of 5.3. To do this, note first that $(E_A(y_n))_n$ is also a central sequence of R, since for any $v \in \mathcal{N}(A)$ we have $vE_A(v^* y_n v)v^* = E_A(y_n)$ (by the uniqueness of the conditional expectation E_A), so that

$$\|vE_A(y_n) - E_A(y_n)v\|_2 = \|vE_A(y_n)v^* - E_A(y_n)\|_2 =$$

$$= \|E_A(vy_nv^*) - E_A(y_n)\|_2 \leqslant \|vy_nv^* - y_n\|_2 \to 0.$$

Hence $\|xE_A(y_n) - E_A(y_n)x\|_2 \to 0$ for any $x \in \overline{\text{span}}^{\text{w}} \mathcal{N}(A) = R$. It follows that $(y_n - E_A(y_n))_n \subset R_1$ is also a central sequence of R. By the proof of 5.3 there exists a unitary element $u \in R$ such that uxu^* is orthogonal to x for any $x \in R_1$, $x \perp A$. We get that $u(y_n - E_A(y_n))u^*$ is orthogonal to $y_n - E_A(y_n)$, so that u cannot commute asymptotically with $(y_n - E_A(y_n))_n$, unless $\|y_n - E_A(y_n)\|_2 \to 0$. We have thus proved that if a central sequence of R is contained in R_1 then it is contained in A (of course, asimptotically).

3) By 2) and [10] it follows that there exist no type II_1 subfactors $N \subset R_0$ (or $N \subset R_1$) such that R splits as $N \otimes M$ for some factor M. So, the inclusion $R_1 \subset R$

give a counterexample to a conjecture of V. Jones ([6]), on whether any subfactor of R containing a regular MASA of R has a subfactor that splits R in a tensor product.

6. APPLICATIONS TO ALGEBRAS ASSOCIATED WITH FREE GROUPS

Consider an arbitrary set S with at least two elements, and denote by $\mathbf{F}_S$ the free group with generators in S. If $S_0 \subset S$ is a nonempty subset then we have a natural inclusion $\mathbf{F}_{S_0} \subset \mathbf{F}_S$ which implements the inclusion $L(\mathbf{F}_{S_0}) \subset L(\mathbf{F}_S)$.

6.1. THEOREM. *If $B \subset L(\mathbf{F}_{S_0})$ is a completely nonatomic von Neumann subalgebra then the normalizer of B in $L(\mathbf{F}_S)$ is contained in $L(\mathbf{F}_{S_0})$.*

Proof. If $g \in \mathbf{F}_S \setminus \mathbf{F}_{S_0}$ then clearly $g\mathbf{F}_{S_0}g^{-1} \cap \mathbf{F}_{S_0} = \{e\}$. By 3.1 it follows that $\mathcal{N}(B)'' \subset L(\mathbf{F}_{S_0})$. Q.E.D.

6.2. COROLLARY. *Every MASA in $L(\mathbf{F}_{S_0})$ is a MASA in $L(\mathbf{F}_S)$.*

6.3. REMARKS. 1) Take $S_0 \subset S$ to be a subset with one element and denote by $u \in L(\mathbf{F}_S)$ the unitary element that corresponds to the image of S_0 by λ. Denote $L(\mathbf{F}_{S_0}) = A_u$. Then A_u is itself a completely nonatomic abelian von Neumann subalgebra so that by 6.1 it is a singular MASA of $L(\mathbf{F}_S)$. Moreover if $A \subset A_u$ is a completely nonatomic proper von Neumann subalgebra of A_u, such as for instance $\{u^2\}''$, then by 6.1, $A' \cap L(\mathbf{F}_S) \subset A_u$, so that $A' \cap L(\mathbf{F}_S) = A_u$. Suppose now that $S_1 \subset S$ is another subset with one element, $S_0 \neq S_1$, and denote by v and A_v the analogues of u and A_u corresponding to S_1. It is easy to see that $g\mathbf{F}_{S_1}g^{-1} \cap \mathbf{F}_{S_0} = \{e\}$ for any $g \in \mathbf{F}_S$, so that by 4.3 A_u and A_v are not unitarily conjugated in $L(\mathbf{F}_S)$. We reobtained in this way some well-known results of R. V. Kadison, part of which can be found in [8].

2) Let N_1, N_2 be finite von Neumann algebras with normal finite faithful traces τ_1 and τ_2 respectively. Let $(N_1 * N_2, \tau_1 * \tau_2)$ be the free product of (N_1, τ_1) with (N_2, τ_2), as in [3]. Then N_1, N_2 are naturally included in $N_1 * N_2$. If we suppose N_1 is completely nonatomic then the same results as 6.1, 6.2 and 1) hold for the pair $N_1 \subset N_1 * N_2$, instead of $L(\mathbf{F}_{S_0}) \subset L(\mathbf{F}_S)$.

6.4. COROLLARY. *The relative commutant of any completely nonatomic von Neumann subalgebra of $L(\mathbf{F}_S)$ is separable (in the weak topology). In particular any abelian von Neumann subalgebra of $L(\mathbf{F}_S)$ is separable.*

Proof. Let $B \subset L(\mathbf{F}_S)$ be completely nonatomic. It follows that there exist projections $\{e_n^k\}_{1 \leq k \leq 2^n, n \geq 0} \subset B$ such that $\tau(e_n^k) = 2^{-n}$, $e_1^0 = 1$, $e_{n+1}^{2k-1} + e_{n+1}^{2k} = e_n^k$, $n \geq 0$, $2^n \geq k \geq 1$. Thus $\{e_n^k\}_{k,n}$ generate a completely nonatomic von Neumann subalgebra B_0 in B. If we regard e_n^k as a convolver, $e_n^k \in \ell^2(\mathbf{F}_S)$, then it is supported on a countable subset of $\mathbf{F}_S$. Consequently there exists a countable subset $S_N \subset S$ such that every e_n^k, $n \geqslant 0$, $2^n \geqslant k \geqslant 1$, is supported in $\mathbf{F}_{S_N}$ and thus $B_0 \subset L(\mathbf{F}_{S_N})$. Since $B' \cap L(\mathbf{F}_S) \subset \subset B_0' \cap L(\mathbf{F}_S) \subset \mathcal{N}(B_0)'' \subset L(\mathbf{F}_{S_N})$, it follows that $B' \cap L(\mathbf{F}_S)$ is separable, because $L(\mathbf{F}_{S_N})$ is separable. Q.E.D.

6.5. COROLLARY. *If S is uncountable then $L(\mathbf{F}_S)$ has no regular MASA's.*

Proof. If $A \subset L(\mathbf{F}_S)$ is a MASA then, by 6.4, $A \subset L(\mathbf{F}_{S_N})$ for some countable $S_N \subset S$. It follows by 6.1 that $\mathcal{N}(A)'' \subset L(\mathbf{F}_{S_N}) \neq L(\mathbf{F}_S)$ and so A is not regular in $L(\mathbf{F}_S)$. Q.E.D.

6.6. COROLLARY. *If S is uncountable then $L(\mathbf{F}_S)$ is not a tensor product of two II_1 factors.*

Proof. If $L(\mathbf{F}_S) = M_1 \otimes M_2$ then M_1 or M_2 is not separable. Suppose M_2 is not separable. If M_1 is of type II_1 then it is completely nonatomic and by 6.4, $(M_1 \otimes 1)' \cap L(\mathbf{F}_S) = 1 \otimes M_2$ is separable, contradiction. Q.E.D.

6.7. COROLLARY. *If S is uncountable then the reduced C^*-algebra of the group $\mathbf{F}_S$, $C_r^*(\mathbf{F}_S)$, is not separable in the uniform norm but all its abelian $*$-subalgebras are norm separable.*

Proof. Consider $C_r^*(\mathbf{F}_S)$ as the norm closed $*$-subalgebra generated by $\lambda(\mathbf{F}_S)$ in $L(\mathbf{F}_S)$ and let $A_0 \subset C_r^*(\mathbf{F}_S)$ be an abelian $*$-subalgebra. Take $A \subset L(\mathbf{F}_S)$ to be a MASA such that $A_0 \subset A$. By 6.4 there exists a countable set $S_N \subset S$ such that $A \subset L(\mathbf{F}_{S_N})$. Thus $A_0 \subset C_r^*(\mathbf{F}_S) \cap L(\mathbf{F}_{S_N}) = C_r^*(\mathbf{F}_{S_N})$, so that A_0 is norm separable. Q.E.D.

The problem of the existence of a nonseparable C^*-algebra with only norm separable abelian $*$-subalgebras was raised by J. Dixmier at the Nice Congress in 1970, in connection with Naimark's problem. In [1] C. Akemann and J. Doner construct another example using the continuum hypothesis.

7. M^ω HAS NO REGULAR MASA'S

Let ω be a free ultrafilter on $\mathbf{N}$ and M a type II_1 factor with normalized trace τ. Denote by M^ω the quotient of the von Neumann algebra $\ell^\infty(\mathbf{N}, M)$ by the zero ideal of the trace τ_ω, defined by $\tau_\omega((x_n)_n) = \lim_{n \to \omega} \tau(x_n)$ (see [10], [4]). Then M^ω is a type II_1 factor.

7.1. LEMMA. *Let $A_1, A_2 \subset M^\omega$ be a pair of separable completely nonatomic commutative von Neumann subalgebras of M^ω. Then there exists a unitary element u in M^ω such that $uA_1u^* = A_2$.*

Proof. Let $\{e_k^{n,i}\}_{\substack{2^n \geqslant k \geqslant 1 \\ n \geqslant 0}}$, be a set of projections in A_i, $i = 1,2$, generating A_i and such that:

1) $\sum_{2^n \geqslant k \geqslant 1} e_k^{n,i} = 1, \quad n \geqslant 0, \quad i = 1,2;$

2) $\tau(e_k^{n,i}) = 2^{-n}, \quad 2^n \geqslant k \geqslant 1, \quad n \geqslant 0, \quad i = 1,2;$

3) $e_{2k-1}^{n+1,i} + e_{2k}^{n+1,i} = e_k^{n,i}, \quad 2^n \geqslant k \geqslant 1, \quad n \geqslant 0, \quad i = 1,2.$

Choose by induction over k and n, sequences of projections $(e_{k,m}^{n,i})_{m \geqslant 0}$ in M, representing $e_k^{n,i}$ and such that:

1) $\sum_{2^n \geqslant k \geqslant 1} e_{k,m}^{n,i} = 1, \quad n \geqslant 0, \quad m \geqslant 0, \quad i = 1,2;$

2) $\tau(e_{k,m}^{n,i}) = 2^{-n}, \quad 2^n \geqslant k \geqslant 1, \quad n \geqslant 0, \quad m \geqslant 0, \quad i = 1,2;$

3) $e_{2k-1,m}^{n+1,i} + e_{2k,m}^{n+1,i} = e_{k,m}^{n,i}, \quad 2^n \geqslant k \geqslant 1, \quad n \geqslant 0, \quad m \geqslant 0.$

For each $m \geqslant 0$ choose a unitary element u_m in M such that $u_m e_{k,m}^{m,1} u_m^* = e_{k,m}^{m,2}$, $2^m \geqslant k \geqslant 1$. It follows that the unitary element $u = (u_m)_{m \geqslant 0} \in M^\omega$ satisfies $u e_k^{n,1} u^* = e_k^{n,2}$ for all $2^n \geqslant k \geqslant 1$, $n \geqslant 0$, so that $uA_1u^* = A_2$. Q.E.D.

7.2. LEMMA. *Let M be an arbitrary type* II_1 *factor and let $R \subset M$ be a subfactor of M isomorphic to the hyperfinite* II_1 *factor. There exist two MASA's of R, A and B, such that $A \perp B' \cap M$.*

Proof. Let N_0 be the algebra of two by two matrices. Let $A_0, B_0 \subset N_0$ be two mutually orthogonal MASA's of N_0 (cf. Section 3).

Let $\{N_m\}_{m \geqslant 1}$, $\{A_n^0\}_{n \geqslant 1}$ and $\{B_n^0\}_{n \geqslant 1}$ be sequences of algebras isomorphic to N_0, A_0 and respectively B_0. Then $\bigotimes_{m \geqslant 1} N_m$ is isomorphic to R and $A = \bigotimes_{m \geqslant 1} A_m^0$, $B = \bigotimes_{m \geqslant 1} B_m^0$ are maximal abelian subalgebras of R. Moreover if $M_n = \bigotimes_{n \geqslant m \geqslant 1} N_m$, $A_n = \bigotimes_{n \geqslant m \geqslant 1} A_m^0$, $B_n = \bigotimes_{n \geqslant m \geqslant 1} B_m^0$, then A_n, B_n are maximal abelian subalgebras in M_n such that $A_n \perp B_n$. By Lemma 3.5, $A_n \perp B_n' \cap M$. Since $B_n' \cap M \supset B' \cap M$ it follows that $A_n \perp B' \cap M$ and thus $A = \overline{\bigcup_n A_n}^{\mathrm{w}} \perp B' \cap M$. Q.E.D.

7.3. THEOREM. *M^ω has no regular MASA's.*

Proof. Let $A \subset M$ be a maximal abelian $*$-subalgebra in M. Let $B \subset A$ be a separable completely nonatomic von Neumann subalgebra of A. By Lemma 7.1 and Lemma 7.2 there exist a hyperfinite subfactor R in M and a maximal abelian subalgebra A_1 of R, such that $B \subset R$ and such that $A_1 \perp B' \cap M$. Since $B \subset A$

it follows that $B' \cap M \supset A' \cap M = A$ so that $A_1 \perp A$. Moreover by Lemma 7.1 there exists a unitary element u in M^ω such that $uBu^* = A_1 \perp A$. Applying Lemma 1.3 it follows that u is orthogonal to $\mathcal{N}(A)''$ so that $\mathcal{N}(A)'' \neq M$. Q.E.D.

7.4. REMARKS. 1) The same result as 7.3 holds if instead of M^ω we take the ultraproduct algebra $\prod_n M_n$ of a sequence of type II_1 factors M_n, as in [4]. The proof is obviously the same.

2) We have thus provided in 6.5 and 7.3 two classes of examples of type II_1 factors without regular MASA's. All these factors are not separable, but there is a big difference between 6.5 and 7.3: while all MASA's of $L(\mathbf{F}_S)$ are separable, the MASA's of M^ω are all nonseparable (cf. 4.3 in [13]).

3) If $(M_n)_n$ is a sequence of II_1 factors and $A_n \subset M_n$ are semiregular MASA's, $n \geqslant 1$, then $A = \prod A_n \subset \prod M_n$ is a semiregular MASA (recall that a MASA in a factor is semiregular if its normalizer generates a subfactor). This follows easily by 2.3 in [13]. Thus $N = \mathcal{N}(A)'' \subset M_n$ is a nonseparable II_1 factor (since A is nonseparable), with regular MASA, A.

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SORIN POPA
Department of Mathematics,
INCREST,
Bdul Păcii 220, 79622 Bucharest,
Romania.

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