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EXTENDED SPECTRAL OPERATORS

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0. INTRODUCTION

The spectral theory of non-selfadjoint operators has been developed by N. Dunford and others. A comprehensive treatment can be found in the monographs of Dunford and Schwartz, [4]. The central concept is that of a spectral operator. For a spectral operator there exists a decomposition of the space into a discrete or continuous direct sum of reducing subspaces, in each of which the operator acts in a particularly simple way. This decomposition of the space is realised by a projection valued measure called the resolution of the identity for that operator. However, as pointed out by N. Dunford in the survey [2], many interesting and important operators of analysis may fail to be spectral in this sense. For this reason it is desirable to have a more general concept of spectral operator.

This note is devoted to extending the notion of spectral operator in the sense of Dunford. For an operator T in a space X, we shall seek a space Y containing a copy of X, such that T has a natural extension to a spectral operator on Y. That is, the space X is continuously included in a space Y which accomodates the resolution of the identity for T.

For example, let T be the operator on the space $X = L^\infty([0,1])$ given by $T(f) = g$ where $g(t) = tf(t)$, $t \in [0,1]$. A prospective resolution of the identity for T is defined by $P(E)(f) = \chi_E f$, $f \in X$, for each Borel subset E of $[0,1]$. However, the map $E \mapsto P(E)$ is not σ-additive and so T is not a spectral operator in X. All the projections needed to form the resolution of the identity are available but the topology of X is too strong to allow P to be countably additive. This can be simply remedied by equipping X with a topology weaker than its norm topology.

Many differential operators T arising from boundary value problems are "almost" spectral operators (see Example 2.14 below). They have associated with them a large family of projection operators which are in a certain sense dense in the prospective resolution of the identity. However, the space X on which T acts is too small to accomodate the additional projections needed to form a resolution of the identity for T. In this case, merely changing the topology of X cannot make T a spectral operator. However, there may exist a space containing X as a proper subspace, in which T does have enough reducing subspaces to form a resolution of the identity.

There also exist operators which have very few reducing subspaces. For example, the operator T on the space $X = C([0,1])$ specified by $T(f) = g$ where $g(t) = tf(t)$, $t \in [0,1]$, has no non-trivial reducing subspaces. However, if X is interpreted as a part of the space $Y = L^1([0,1])$, then the natural extension of T to the space Y is a spectral operator. The projections which form the resolution of the identity for T in Y, are multiplication by characteristic functions of Borel subsets of $[0,1]$. These projections are not available in the space X.

The idea of extending the space so as to make an operator spectral is not new. It is recurring often in mathematical physics. For example, the (unbounded) operator of differentiation does not have any eigenfunctions in $L^2(\mathbf{R})$. However, $L^2(\mathbf{R})$ can be considered as part of a larger space, such as $L^2_{\text{loc}}(\mathbf{R})$, which accomodates the complete set of eigenfunctions, $x \mapsto \exp(i\lambda x)$, of the differentiation operator.

It is interesting to note that the theory of Banach algebras or locally convex algebras does not provide the most natural framework for this type of problem. It does not allow for the use of the inner structure of a specific operator. The theory of integration provides a more effective means for studying individual operators and is therefore the approach used in this note.

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1. SPECTRAL OPERATORS IN LOCALLY CONVEX SPACES

Let X be a locally convex Hausdorff space, X' its continuous dual and $L(X)$ the space of all continuous linear operators on X. The space X is always assumed to be quasi-complete. The space $L(X)$ will always have the topology of pointwise convergence. If we wish to indicate the topology τ of a space X, possibly different from its original topology, we will denote the space by (X, τ). In particular, if W is a total subspace of functionals in X', then $\sigma(X, W)$ denotes the weakest topology on X for which each member of W is continuous. The identity operator is denoted by I. The adjoint of an operator T in X is denoted by T'.

Let $\mathbf{C}$ denote the complex plane and $\mathbf{C}^*$ the extended complex plane. Let $T \in L(X)$. If $\lambda \in \mathbf{C}$ is such that $\mathrm{R}(\lambda;T) = (\lambda I - T)^{-1}$ exists in $L(X)$, then $\mathrm{R}(\lambda;T)$ is called the *resolvent operator* of T at λ. Define $\mathrm{R}(\infty;T)$ to be the zero operator. If it is clear which operator T is being considered, then $\mathrm{R}(\lambda;T)$ is denoted simply by $\mathrm{R}(\lambda)$. The resolvent set of T, which is denoted by $\rho(T)$, consists of those points $\lambda \in \mathbf{C}^*$ for which the *resolvent map* $\mathrm{R}(\cdot) = \mathrm{R}(\cdot\,;T)$ is defined and bounded (in $L(X)$) in a neighbourhood of λ (see [19]). The complement of $\rho(T)$ in $\mathbf{C}^*$ is denoted by

$\sigma(T)$, and is called the *spectrum* of T. It is always non-empty and closed, but may be unbounded.

Let Ω be a set and $\mathcal{M}$ a σ-algebra of subsets of Ω. A map $P\colon \mathcal{M} \to L(X)$ is called a *spectral measure* if,

(i) $P(\emptyset) = 0$ and $P(\Omega) = I$;

(ii) $P(E \cap F) = P(E)P(F)$ for each $E, F \in \mathcal{M}$ and;

(iii) $E \mapsto P(E)(x)$, $E \in \mathcal{M}$, is σ-additive for each $x \in X$.

A spectral measure $P\colon \mathcal{M} \to L(X)$ is said to be *equicontinuous* if its range, that is, $\{P(E)\,;\ E \in \mathcal{M}\}$, is an equicontinuous part of $L(X)$. If the space X is barrelled, then any spectral measure with values in $L(X)$ is equicontinuous.

An $\mathcal{M}$-measurable function $f\colon \Omega \to \mathbf{C}$ is said to be *P-integrable* if,

(i) f is integrable with respect to the measure $\langle P(\cdot)(x), x'\rangle$ for each $x \in X$ and $x' \in X'$, and

(ii) for each $E \in \mathcal{M}$ there is an operator $\int_E f\mathrm{d}P \in L(X)$ such that

$$\left\langle \left(\int_E f\mathrm{d}P \right)(x), x' \right\rangle = \int_E f(\omega)\mathrm{d}\langle P(\omega)(x), x'\rangle$$

for each $x \in X$ and $x' \in X'$; (see [13]).

A measurable function $f\colon \Omega \to \mathbf{C}$ is said to be *P-essentially bounded* if,

$$|f|_P = \inf\{\sup\{|f(\omega)|\,;\ \omega \in E\}\,;\ E \in \mathcal{M},\ P(E) = I\} < \infty.$$

The space of P-essentially bounded functions is denoted by $B_{ess}(P)$ and is a Banach algebra with respect to the norm $|\cdot|_P$.

PROPOSITION 1.1. *Let $P\colon \mathcal{M} \to L(X)$ be an equicontinuous spectral measure. Then every function $f \in B_{ess}(P)$ is P-integrable and the integration map*

$$f \mapsto \int_\Omega f\mathrm{d}P, \quad f \in B_{ess}(P),$$

is a continuous homomorphism.

Proof. Let W be a convex, balanced neighbourhood of zero. It is shown that there exists a convex, balanced neighbourhood V of zero, such that for every simple function f with $\|f\|_\infty \leqslant 1$ and for every $E \in \mathcal{M}$,

$$\text{(1)} \qquad \left(\int_E f\mathrm{d}P \right)(V) \subseteq W.$$

It suffices to assume that $0 \leqslant f \leqslant 1$. By equicontinuity of P there is a convex, balanced neighbourhood V of zero satisfying $P(E)(V) \subseteq W$ for all $E \in \mathcal{M}$.

Fix $v \in V$. Then $E \mapsto P(E)(v)$, $E \in \mathcal{M}$, is an X-valued measure. If

$$f = \sum_{i=1}^{m} \alpha_i \chi_{E_i}$$

with the E_i pairwise disjoint and $0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_m \leqslant 1$, then it follows from Abel's summation formula that $\int_E f \mathrm{d}P(v)$ is a convex combination of points from the range of $P(\cdot)(v)$. The inclusion (1) follows.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of simple functions converging uniformly on Ω to a function f. It suffices to consider the case when $\|f\|_{\infty} \leqslant 1$ and $\|f_n\|_{\infty} \leqslant 1$, $n = 1,2, \ldots$. If $x \in X$, then $P(\cdot)(x)$ is an X-valued measure. By II, Lemma 3.1 and II, Theorem 4.2 of [13], f is $P(\cdot)(x)$-integrable and $\left\{\int_E f_n \mathrm{d}P(x)\right\}_{n=1}^{\infty}$ tends to $\int_E f \mathrm{d}P(x)$ uniformly with respect to E in $\mathcal{M}$. For each $E \in \mathcal{M}$, denote by $\int_E f \mathrm{d}P$ the linear map,

$$x \mapsto \int_E f \mathrm{d}P(x), \quad x \in X.$$

Since (1) implies that $\left\{\int_E f_n \mathrm{d}P\,;\ E \in \mathcal{M},\ n = 1,2, \ldots\right\}$ is an equicontinuous part of $L(X)$, it follows from Chapter 3, Theorem 4.3 of [21], that for each $E \in \mathcal{M}$ the operator $\int_E f \mathrm{d}P \in L(X)$ can be defined (uniquely) as the limit of the sequence $\left\{\int_E f_n \mathrm{d}P\right\}_{n=1}^{\infty}$.

That the integration map is a continuous homomorphism follows from the multiplicativity of spectral measure and II, Lemma 3.1 of [13].

If X is a Banach space, then a function f is P-integrable if and only if it belongs to $B_{\mathrm{ess}}(P)$. For locally convex spaces this is no longer the case. If P is an equicontinuous spectral measure, then the space of P-integrable functions may be substantially larger than $B_{\mathrm{ess}}(P)$. In fact, a function $f\colon \Omega \to \mathbf{C}$ is P-integrable if and only if there is an increasing sequence $E_n \in \mathcal{M}$, $n = 1,2, \ldots$, with $E_n \uparrow \Omega$ such that $f_n = f\chi_{E_n}$ is a member of $B_{\mathrm{ess}}(P)$ for each $n = 1,2, \ldots$, and the limits,

$$\lim_{n \to \infty} \int_E f_n \mathrm{d}P, \tag{2}$$

exist in $L(X)$ uniformly with respect to E in $\mathcal{M}$. In this case, for each $E \in \mathcal{M}$, the operator $\int_E f \mathrm{d}P$ is given by (2). It is important to note that if f is P-integrable and $g \in B_{ess}(P)$, then fg is P-integrable and for each $E \in \mathcal{M}$,

$$\int_E fg \, dP = \left(\int_E f \, dP\right)\left(\int_E g \, dP\right).$$

An operator $T \in L(X)$ is called a *scalar-type operator*, briefly, a *scalar operator*, if there exists a spectral measure $P: \mathcal{M} \to L(X)$ and a P-integrable function f such that

$$T = \int_\Omega f \, dP.$$

If $B(\mathbf{C})$ denotes the σ-algebra of Borel sets in $\mathbf{C}$, then the $L(X)$-valued map given by

$$P_f: E \mapsto P(f^{-1}(E)), \quad E \in B(\mathbf{C}),$$

is a spectral measure for which the identity function of $\mathbf{C}$ is integrable and

$$T = \int_{\mathbf{C}} z \, dP_f(z).$$

If $\rho(T) \neq \emptyset$, then the measure P_f is unique and is called the resolution of the identity for T; (proof is similar to [20], Theorem 3). The *support*, $S(P_f)$, of P_f, is the complement of the union of all open sets $U \subseteq \mathbf{C}$ for which $P_f(U) = 0$.

A proof similar to that of [20], Proposition 12, shows that any operator $S \in L(X)$ which commutes with a scalar operator T also commutes with its resolution of the identity.

PROPOSITION 1.2. *Let $T \in L(X)$ be a scalar-type operator with $\sigma(T) \neq \mathbf{C}$ such that its resolution of the identity, P, is equicontinuous. Then $S(P) = \sigma(T)$.*

Proof. See [3], Theorem 16, for example.

Scalar operators $T \in L(X)$ for which $\sigma(T) \subseteq \mathbf{T} = \{z \in \mathbf{C}; |z| = 1\}$ are called *pseudo-unitary*. Those for which $\sigma(T) \subseteq \mathbf{R}$ are called *pseudo-hermitean*.

If X is a Hilbert space and $T \in L(X)$ is a self-adjoint operator with resolution of the identity P, then there is a well known formula which gives the projection operators, $P((a, b))$, for $a < b$, in terms of the resolvent map of T, (Chapter X, Theorem 6.1, [4]). Such a result remains valid for operators in more general spaces. The following proof is included only to highlight the fact that the compactness of $\sigma(T)$ is not necessary.

PROPOSITION 1.3. *Let the space X be quasi-complete and $T \in L(X)$ be a pseudo-hermitean operator with equicontinuous resolution of the identity P. If $E = (a, b)$, $a < b$, is an open interval, then*

$$P(E) = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (\mathrm{R}(\mu - i\varepsilon) - \mathrm{R}(\mu + i\varepsilon)) \, d\mu$$

in $L(X)$, where $\mathrm{R}(\cdot)$ is the resolvent map of T.

Proof. Let $\varepsilon > 0$ and $0 < \delta < \frac{1}{2}(b-a)$. Since $\sigma \mapsto (\mu - \sigma \pm i\varepsilon)^{-1}$, $\sigma \in \mathbf{R}$, is P-essentially bounded, it follows that

$$\mathrm{R}(\mu \pm i\varepsilon) = \int_{\mathbf{R}} (\mu - \sigma \pm i\varepsilon)^{-1} \, dP(\sigma), \quad \mu \in \mathbf{R}.$$

If $f(\mu, \sigma) = (\mu - \sigma - i\varepsilon)^{-1} - (\mu - \sigma + i\varepsilon)^{-1}$ for $\mu \in (a+\delta, b-\delta)$ and $\sigma \in \mathbf{R}$, then

$$(3) \qquad \frac{1}{2\pi i} \int_{\delta}^{b-\delta} (\mathrm{R}(\mu - i\varepsilon) - \mathrm{R}(\mu + i\varepsilon)) d\mu = \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} f(\mu, \sigma) \, dP(\sigma) \, d\mu.$$

Fix $x \in X$. The equality $f(\mu, \sigma) = 2i\varepsilon(\varepsilon^2 + (\mu - \sigma)^2)^{-1}$ together with

$$\frac{1}{2} \int_{a+\delta}^{b-\delta} |f(\mu, \sigma)| \, d\mu = \arctan(\varepsilon^{-1}(b - \delta - \sigma)) - \arctan(\varepsilon^{-1}(a + \delta - \sigma)) = h_{\delta,\varepsilon}(\sigma),$$

shows that $\sigma \mapsto \int_{a+\delta}^{b-\delta} |f(\mu, \sigma)| \, d\mu$ is $P(\cdot)(x)$-integrable and hence, that f is $P(\cdot)(x) \otimes d\mu$-integrable in X. An application of the Fubini theorem in (3) gives

$$\frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (\mathrm{R}(\mu - i\varepsilon) - \mathrm{R}(\mu + i\varepsilon))(x) \, d\mu = \frac{1}{\pi} \int_{\mathbf{R}} h_{\delta,\varepsilon}(\sigma) \, dP(\sigma)(x).$$

The result follows from II, Theorem 4.2 of [13] since

$$\lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} h_{\delta,\varepsilon}(\sigma) = \pi \, \chi_{(a,b)}(\sigma), \quad \sigma \in \mathbf{R}.$$

An operator $N \in L(X)$ is said to be *quasi-nilpotent* if

$$(4) \qquad \lim_{n \to \infty} |\langle N^n(x), x' \rangle|^{1/n} = 0,$$

for each $x \in X$ and $x' \in X'$.

An operator $T \in L(X)$ will be called a *spectral operator* if there exists a scalar operator $S \in L(X)$ and a quasi-nilpotent operator $N \in L(X)$ which commutes with S, such that $T = S + N$.

2. EXTENDED SCALAR OPERATORS

N. Dunford noted in [2] that many operators $T \in L(X)$ which are natural candidates to be scalar operators, fail to be such. The operator does not have a countably additive resolution of the identity. An alternative point of view is not that the operator is deficient, but rather that the space on which it acts is deficient, in that it is not "large enough" to accomodate a spectral decomposition of T. In this section we shall seek criteria which imply the existence of a space Y containing X, such that T has an extension to a scalar operator on Y. Adopting this point of view, many unbounded operators can also be extended to continuous, everywhere defined scalar operators.

It is assumed throughout this section that X is a quasi-complete, Hausdorff locally convex space. Let T be a densely defined linear operator in X. A locally convex, Hausdorff space Y is said to be *admissible* for T if there is a continuous, linear injection $\imath: X \to Y$ such that Y is the completion or quasi-completion of $\imath(X)$, and an operator $T_Y \in L(Y)$ such that for each x in the domain of T,

$$T_Y(\imath(x)) = \imath(T(x)). \tag{5}$$

Let Y be an admissible space for T and $\imath$ the imbedding of X into Y. The equality (5) is written simply as $T_Y(x) = T(x)$ for each x in the domain of T. Each $y' \in Y'$ is a member of X' when it is identified with the element $y' \circ \imath$. Therefore, we write $Y' \subseteq X'$. If T' denotes the adjoint of T in X, then Y' is necessarily an invariant subspace of T', that is, $T'(Y') \subseteq Y'$. The subspace Y' of X' separates the points of X. If A is a subset of X, then we write $A \subseteq Y$ rather than $\imath(A) \subseteq Y$. Sets which are bounded in X are also bounded in Y.

It is important to note that if Y is an admissible space for the operator T in X, then $\sigma(T_Y)$ can be vastly different from $\sigma(T)$. For example, if $X = \ell^1(\mathbf{N})$ and $T \in L(X)$ is given by

$$T(x) = (0, x_1, x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in X,$$

then $\sigma(T) = \{\lambda \in \mathbf{C}\,;\, |\lambda| \leqslant 1\}$. Let $Y = \mathbf{C}^{\mathbf{N}}$ with the topology of pointwise convergence. Then Y is admissible for T. If T_Y is the natural extension of T to Y, then $\sigma(T_Y) = \{0\}$.

PROPOSITION 2.1. (i) *Let $T \in L(X)$. Let W be a total subspace of X' which is invariant for T'. If Y denotes the completion of $(X, \sigma(X, W))$, then Y is admissible for T.*

(ii) *Let the space X be barrelled and X' have a locally convex Hausdorff topology for which $X \subseteq (X')'$. Let $T \in L(X')$. If X is an invariant subspace for T', then X' equipped with the weak-star topology, $\sigma(X', X)$, is an admissible space for T.*

Proof. If $x \in X$ and $\langle x, \cdot\rangle$ denotes the linear functional on W defined by $w \mapsto \langle x, w\rangle$, $w \in W$, then the map $\imath: X \to Y$ given by $\imath(x) = \langle x, \cdot\rangle$, $x \in X$, is a continuous linear injection. Since W is invariant for T', it follows that T is continuous from $(X, \sigma(X, W))$ into Y and hence, has a (unique) extension $T_Y \in L(Y)$.

Statement (ii) follows from the fact that the dual of a barrelled space is quasi--complete for the weak-star topology.

The following series of examples illustrate that the phenomenon of an operator being scalar in an admissible space is in no way pathological.

EXAMPLE 2.2. Let $X = \ell^p(\mathbf{Z})$ for some p satisfying $1 \leqslant p < 2$ and $T \in L(X)$ be the bilateral shift operator. That is, $T(x) = y$, $x \in X$, where $y_n = x_{n-1}$ for each $n \in \mathbf{Z}$. Then T is not a scalar operator in X (Theorem 5.7, [6]). However, the space $Y := \ell^2(\mathbf{Z})$ is admissible for T and the natural extension of T to Y is a scalar operator in Y.

EXAMPLE 2.3. Let $X = \ell^\infty(\mathbf{Z})$. Define $\lambda_n = e^{i/n}$ for $n \neq 0$, $n \in \mathbf{Z}$, and $\lambda_0 = 1$. The operator $T \in L(X)$ specified by $T(x) = y$, $x \in X$, where $y_n = \lambda_n x_n$, $n \in \mathbf{Z}$, is an isometry of X onto X with $\sigma(T) \subseteq \mathbf{T}$. However, T is not a scalar operator. If $Y = \mathbf{C}^{\mathbf{Z}}$ with the topology of pointwise convergence, then Y is an admissible space and T has a natural extension $T_Y \in L(Y)$. Let $B(\mathbf{T})$ denote the Borel sets of $\mathbf{T}$. Then

$$T_Y = \int_{\mathbf{T}} z \, dP(z), \tag{6}$$

where P is the $L(Y)$-valued spectral measure given by

$$P(E) = \sum_{\lambda_n \in E} P_n, \quad E \in B(\mathbf{T}),$$

and $P_n \in L(Y)$ is the operator of pointwise multiplication by $\chi_{\{n\}}$ on Y.

EXAMPLE 2.4. Let $X = C([0,1])$ with the uniform norm and let $T \in L(X)$ be the operator defined by $T(f) = g$, $f \in X$, where $g(t) = tf(t)$, $t \in [0,1]$. Then T is not scalar in X (Example 2.2, [6]). If $Y := L^p([0,1])$ for any p satisfying $1 \leqslant p < \infty$, then Y is admissible and T has a natural extension $T_Y \in L(Y)$ which is a scalar operator. In fact

$$T_Y = \int_{[0,1]} \sigma \, dP(\sigma),$$

where P is the spectral measure of multiplication by characteristic functions of Borel subsets of $[0, 1]$.

EXAMPLE 2.5. Let $X = L^2([0,2\pi])$ and let T be the unbounded operator of differentiation defined by $T(f) = -if'$. The domain of T consists of those functions f for which f' exists and is a member of X. Let $W \subset X'$ be the subspace of trigonometric polynomials and Y the completion of $(X, \sigma(X, W))$. Then Y can be identified as the space of formal expressions

$$\left\{ f = \sum_{n=-\infty}^{\infty} a_n e^{inx};\ \{a_n\}_{n=-\infty}^{\infty} \in \mathbf{C}^{\mathbf{Z}} \right\},$$

with the natural linear operations. If $w = \sum_{n=-\infty}^{\infty} w_n e^{inx} \in W$, then the duality between Y and W is given by the (finite) sum,

$$\langle f, w\rangle = \sum_{n=-\infty}^{\infty} a_n w_n .$$

The topology of Y is determined by the semi-norms p_r, $r = 1,2, \ldots,$ where

$$p_r\left(\sum_{n=-\infty}^{\infty} a_n e^{inx}\right) = \max\{|a_n| \,;\, |n| \leqslant r\}.$$

The operator T has a natural extension $T_Y \in L(Y)$, given by

$$T_Y\left(\sum_{n=-\infty}^{\infty} a_n e^{inx}\right) = \sum_{n=-\infty}^{\infty} n a_n e^{inx}.$$

Furthermore, T_Y is scalar. In fact, if $B(\mathbf{R})$ denotes the Borel subsets of $\mathbf{R}$, then

$$T_Y = \int_{\mathbf{R}} \sigma \, dP(\sigma), \tag{7}$$

where P is the $L(Y)$-valued spectral measure given by $P(E) = \sum_{m \in E} P_m$, $E \in B(\mathbf{R})$, and $P_m\left(\sum_{n=-\infty}^{\infty} a_n e^{inx}\right) = a_m e^{imx}$ for each $m \in \mathbf{Z}$.

EXAMPLE 2.6. Let $X = L^2(\mathbf{R})$ and T be the unbounded operator specified by $T(f) = g$ where $g(t) = tf(t)$, $t \in \mathbf{R}$. The domain of T consists of those functions $f \in X$ for which $t \mapsto tf(t)$, $t \in \mathbf{R}$, belongs to X. If $Y = L^2_{\mathrm{loc}}(\mathbf{R})$ is the space of locally square integrable functions on $\mathbf{R}$, then Y is admissible and T has a natural extension $T_Y \in L(Y)$. Let P be the $L(Y)$-valued spectral measure of multiplication by characteristic functions of Borel subsets of $\mathbf{R}$. Then T_Y is a scalar operator given as in (7).

A collection, $\mathcal{M}$, of subsets of a non-empty set, Ω, is said to be a δ-ring if it is a ring of sets and is closed under countable intersections. The σ-ring generated by $\mathcal{M}$ is denoted by $R_\sigma(\mathcal{M})$.

Let $\mathcal{M}$ be a ring of sets. An operator valued map $Q : \mathcal{M} \to L(X)$ is said to be σ-additive if

$$Q\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} Q(E_n)$$

in the topology of $L(X)$, whenever E_n, $n = 1,2, \ldots,$ are pairwise disjoint members of $\mathcal{M}$ whose union belongs to $\mathcal{M}$. The map Q is said to be *multiplicative* if $Q(E_1 \cap E_2) = Q(E_1)Q(E_2)$ for all $E_1, E_2 \in \mathcal{M}$. If $Q(E)(x) = 0$ for all $E \in \mathcal{M}$ implies that $x = 0$, and $Q(E)'(x') = 0$ for all $E \in \mathcal{M}$ implies that $x' = 0$, then Q is said to be determining.

Let $T \in L(X)$. Suppose that associated with T is an additive, equicontinuous, projection valued map P defined on a ring or δ-ring, $\mathcal{M}$, such that P commutes with T and $P(E)$ is a reducing subspace of T for each $E \in \mathcal{M}$. The extension of P to an $L(X)$-valued, σ-additive measure on the σ-ring generated by $\mathcal{M}$ is not always possible. Even though the space X may be large enough to accomodate the projections needed to extend P from $\mathcal{M}$ to $R_\sigma(\mathcal{M})$, thereby forming a prospective resolution of the identity for T, its topology may be too strong to allow countable additivity of the extended map P. This problem can often be overcome by simply weakening the topology of X. A proper, total subspace W of X' is declared to be the dual space of X, and with respect to the topology, $\sigma(X, W)$, the operator T is a scalar operator. A scalar operator of this type is referred to as a prespectral operator of class W (see [6]).

A more serious difficulty to overcome occurs when P is unbounded on the ring $\mathcal{M}$. In this case P cannot be extended to an $L(X)$-valued measure by merely weakening the topology of X. However, there may exist a space Y, admissible for each operator $P(E)$, $E \in \mathcal{M}$, such that the set function $E \mapsto P_Y(E)$, $E \in \mathcal{M}$, has an $L(Y)$-valued extension to a σ-additive measure on $R_\sigma(\mathcal{M})$. Accordingly, if Y is admissible for T, then P_Y is the resolution of the identity for T_Y.

EXAMPLE 2.7. Let $X = \ell^\infty(\mathbf{N})$. Define $\lambda_n = (n-1)/n$ for $n \geq 2$, $n \in \mathbf{N}$, and $\lambda_1 = 1$. The spectrum of the operator $T \in L(X)$ specified by $T(x) = y$, $x \in X$, where $y_n = \lambda_n x_n$, $n \in \mathbf{N}$, is the set $\{\lambda_n : n \in \mathbf{N}\}$. For $\mu \notin \sigma(T)$, the resolvent operator $\mathrm{R}(\mu) = \mathrm{R}(\mu; T) \in L(X)$ is given by

$$\mathrm{R}(\mu)(x) = ((\lambda_1 - \mu)^{-1} x_1,\ (\lambda_2 - \mu)^{-1} x_2, \ldots), \quad x \in X.$$

Carrying out the calculations of Proposition 1.3 formally, gives

$$P((a, b))(x) = (x_1 \chi_{(a,b)}(\lambda_1), x_2 \chi_{(a,b)}(\lambda_2), \ldots),\ x \in X, \tag{8}$$

for each $a < b$. If $\mathcal{M}$ denotes the ring generated by the family of intervals (a,b) with $a < b$, then it follows from (8) that P is an additive, uniformly bounded, projection valued map on $\mathcal{M}$ which commutes with T. However, P cannot be extended to a σ-additive, $L(X)$-valued measure on $B(\mathbf{R})$.

If Y denotes the space X equipped with the weak-star topology, then Y is admissible for T and for each operator $P(E)$, $E \in \mathcal{M}$. Let $P_n \in L(Y)$ be the operator of pointwise multiplication by $\chi_{\{n\}}$ for each $n \in \mathbf{N}$. Then the natural extension, T_Y, of T is a scalar operator given by (7), where P is the $L(Y)$-valued measure defined by $P(E) = \sum_{\lambda_n \in E} P_n$, $E \in B(\mathbf{R})$. In this case T is a prespectral operator of class $\ell^1(\mathbf{N})$.

EXAMPLE 2.8. Let $1 < p < 2$ and T denote the bilateral shift operator on $X = \ell^p(\mathbf{Z})$ (cf. Example 2.2). Let

$$Z = \{f \in L^q(\mathbf{T})\ ;\ \hat{f} \in X\}.$$

where q satisfies $p^{-1} + q^{-1} = 1$ and $\hat{f}$ denotes the Fourier transform of f. With respect to the norm specified by

$$\|f\| = \|f\|_q + \|\hat{f}\|_X, \quad f \in Z,$$

the space Z is a Banach space which is isomorphic to X. The isomorphism is the Fourier transform map $F: Z \to X$. Let $S \in L(Z)$ denote the map $F^{-1}TF$. Then S is the operator defined by $S(f) = g$, $f \in Z$, where $g(z) = zf(z)$, $z \in \mathbf{T}$. Furthermore, S is scalar if and only if T is scalar.

By an arc in $\mathbf{T}$ we mean a subset of the form $\{e^{it};\ t \in I\}$, where I is an interval in $\mathbf{R}$. Let $\mathscr{A}$ denote the collection of all arcs in $\mathbf{T}$ and $\mathscr{M}$ the ring generated by $\mathscr{A}$. The map $P: \mathscr{M} \to L(Z)$ defined by

$$P(E)(f) = \chi_E f, \quad f \in Z$$

for each $E \in \mathscr{M}$, is additive and multiplicative. The theorem of M. Riesz (Theorem 6.4.2, [5]) implies that P is uniformly bounded on $\mathscr{A}$. However, P is not uniformly bounded on $\mathscr{M}$ and so cannot be extended to a measure on $B(\mathbf{T})$. This can be seen for example, by the existence of sets $E \subseteq \mathbf{T}$, of positive measure, for which $\chi_E \notin Z$, [9].

However, the space $Y = L^2(\mathbf{T})$ is admissible for each $P(E)$, $E \in \mathscr{M}$, and for S, and the natural extension, S_Y, of S, is a pseudo-unitary operator in Y. In fact, S_Y is given by (6) where P is the spectral measure of multiplication by characteristic functions of Borel subsets of $\mathbf{T}$. Using the identification of $\ell^2(\mathbf{Z})$ with Y we deduce, as in Example 2.2, that the natural extension of T from X to $\ell^2(\mathbf{Z})$ is pseudo-unitary.

The preceding examples illustrate the need for criteria which guarantee that an additive, projection valued map in $L(X)$ defined on a ring (algebra) of sets, can be extended to an $L(Y)$-valued measure (spectral measure) with values in a suitable space Y containing X.

Let $\mathscr{M}$ be a ring of subsets of a set Ω. Let $P: \mathscr{M} \to L(X)$ be an additive, multiplicative map. A locally convex space Y is said to be admissible for P if Y is admissible for each operator $P(E)$, $E \in \mathscr{M}$, and if the $L(Y)$-valued set function $E \mapsto P_Y(E)$, $E \in \mathscr{M}$, has equicontinuous range in $L(Y)$.

A family $\mathscr{C}$ of subsets of Ω is said to be compact if every countable subfamily of $\mathscr{C}$, which has the finite intersection property, has a non-empty intersection; (see [18]).

Let $\mathscr{M}$ be a ring of subsets of a set Ω and $\mathscr{C}$ a compact family in Ω. An additive, $\mathbf{C}$-valued function μ defined on $\mathscr{M}$ is said to be *$\mathscr{C}$-regular* if, for every $A \in \mathscr{M}$ and $\varepsilon > 0$, there exist sets $B \in \mathscr{M}$ and $C \in \mathscr{C}$ such that $B \subseteq C \subseteq A$ and $|\mu(E)| < \varepsilon$ for every $E \in \mathscr{M}$ such that $E \subseteq A \setminus B$; (see [18]). An additivemap P: $\mathscr{M} \to L(X)$ is said to be *$\mathscr{C}$-regular* if, for every $x \in X$ and $x' \in X'$, the additive function

$$E \mapsto \langle P(E)(x), x' \rangle, \quad E \in \mathscr{M},$$

is $\mathscr{C}$-regular. An additive, C-valued function μ on $\mathscr{M}$ or an additive map $P: \mathscr{M} \to L(X)$ is called *regular*, if it is $\mathscr{C}$-regular for some compact family $\mathscr{C}$ of subsets of Ω.

LEMMA 2.9. *Let μ be a bounded, regular, additive* C*-valued function defined on a ring $\mathscr{M}$. Then μ is σ-additive on $\mathscr{M}$.*

Proof. Since the variation of μ is again bounded and regular the result follows from 4(i) of [18].

PROPOSITION 2.10. *Let $\mathscr{M}$ be a ring of sets and $P: \mathscr{M} \to L(X)$ a multiplicative additive map. Let Y be a barrelled space which is admissible for P. If the map $P_Y : \mathscr{M} \to L(Y)$ is regular, and for each $y \in Y$ the set $\{P_Y(E)(y)\ ;\ E \in \mathscr{M}\}$ is relatively weakly compact in Y, then there exists a unique σ-additive, multiplicative map $\mathscr{K}: R_\sigma(\mathscr{M}) \to L(Y)$, such that $\mathscr{K}(E) = P_Y(E)$ for every $E \in \mathscr{M}$.*

Proof. The equicontinuity of $\{P_Y(E)\ ;\ E \in \mathscr{M}\}$ and Lemma 2.9 imply that the map

$$E \mapsto \langle P_Y(E)(y), y' \rangle, \quad E \in \mathscr{M}, \tag{9}$$

is σ-additive for each $y \in Y$ and $y' \in Y'$. Hence, $P_Y(\cdot)(y)$ is weakly σ-additive on $\mathscr{M}$ for each $y \in Y$. By the Theorem of Extension in [10], there exists a unique σ-additive measure $\mathscr{K}(\cdot)(y): R_\sigma(\mathscr{M}) \to Y$ such that $\mathscr{K}(E)(y) = P_Y(E)(y)$ for every $E \in \mathscr{M}$.

If $B = \{E \in R_\sigma(\mathscr{M})\ ;\ \mathscr{K}(E) \in L(Y)\}$, then B contains $\mathscr{M}$. It follows from the Banach-Steinhaus theorem that B is a monotone class. Hence, $B = R_\sigma(\mathscr{M})$ and $\mathscr{K}$ is σ-additive with values in $L(Y)$.

It remains to show that if $E, F \in R_\sigma(\mathscr{M})$, then

$$\mathscr{K}(E \cap F) = \mathscr{K}(E)\mathscr{K}(F). \tag{10}$$

The multiplicativity of P_Y shows that (10) holds whenever $E, F \in \mathscr{M}$. Let $E \in \mathscr{M}$. Denote by B_1 the system of all sets $F \in R_\sigma(\mathscr{M})$ such that (10) is valid. Clearly $\mathscr{M} \subseteq B_1$. Due to the σ-additivity of $\mathscr{K}$ the collection B_1 is a monotone class. Consequently $R_\sigma(\mathscr{M}) \subseteq B_1$. Now let F be an arbitrary element of $R_\sigma(\mathscr{M})$. Denote by B_2 the system of all sets $E \in R_\sigma(\mathscr{M})$ such that (10) is valid. Since $\mathscr{M} \subseteq B_2$ and B_2 is a monotone class it follows that $R_\sigma(\mathscr{M}) \subseteq B_2$.

For certain types of spaces Y, the criterion of Proposition 2.10 can be simplified.

A locally convex space X is said to be *weakly Σ-complete* if every sequence $\{x_n\}_{n=1}^\infty$ of its elements, such that $\{\langle x_n, x' \rangle\}_{n=1}^\infty$ is absolutely summable for each $x' \in X'$, is itself summable with the sum belonging to X. Weakly sequentially complete, in particular reflexive, spaces are weakly Σ-complete.

PROPOSITION 2.11. *Let $\mathscr{M}$ be a ring of sets and $P: \mathscr{M} \to L(X)$ a multiplicative, additive map. Let Y be a barrelled, weakly Σ-complete space which is admissible for P. If the map $P_Y: \mathscr{M} \to L(Y)$ is regular, then there exists a unique σ-additive, multiplicative map $\mathscr{K}: R_\sigma(\mathscr{M}) \to L(Y)$, such that $\mathscr{K}(E) = P_Y(E)$ for each $E \in \mathscr{M}$.*

Proof. For each $y \in Y$ and $y' \in Y'$ the map given by (9) is bounded and additive, hence, is σ-additive. That is, the map $m : \mathscr{M} \to Y$ defined by $m(E) = P_Y(E)(y)$, is weakly σ-additive for each $y \in Y$. Furthermore, as m is bounded and its range is contained in the weakly Σ-complete space Y, by the Theorem of Extension in [10], there exists a σ-additive map $\mathscr{K}(\cdot)(y) : R_\sigma(\mathscr{M}) \to Y$ such that $\mathscr{K}(E)(y) = P_Y(E)(y)$ for each $E \in \mathscr{M}$.

That $\mathscr{K}$ is a σ-additive, multiplicative map with values in $L(Y)$ can be shown as in the proof of Proposition 2.10.

The following statement — which is a particular case of Proposition 2.11 — provides a method for constructing admissible spaces.

PROPOSITION 2.12. *Let $\mathscr{M}$ be a ring of sets and $P: \mathscr{M} \to L(X)$ an additive, multiplicative map. Let W be a total subspace of X' which is invariant for each operator $P(E)'$, $E \in \mathscr{M}$, and such that the range of P is an equicontinuous part of $L((X, \sigma(X, W)))$. If Y is the completion of $(X, \sigma(X, W))$ and the map $P_Y: \mathscr{M} \to L(Y)$ is regular, then Y is admissible for P and there exists a unique σ-additive, multiplicative map $\mathscr{K} : R_\sigma(\mathscr{M}) \to L(Y)$, such that $\mathscr{K}(E) = P_Y(E)$ for each $E \in \mathscr{M}$.*

Proof. By Proposition 2.1, for each $E \in \mathscr{M}$ there exists an operator $P_Y(E) \in L(Y)$ which is an extension of $P(E)$. From the hypothesis it follows that $\{P_Y(E); E \in \mathscr{M}\}$ is an equicontinuous part of $L(Y)$. Since the space Y is barrelled and weakly sequentially complete the result follows from Proposition 2.11.

If the ring of sets, $\mathscr{M}$, in Propositions 2.10, 2.11 and 2.12 is actually an algebra, and $P(\Omega) = I$, then the measure $\mathscr{K}$ on the generated σ-algebra is a spectral measure.

The following result (Theorem 2, [15]) gives another method of constructing admissible spaces.

PROPOSITION 2.13. *Let X be a Hilbert space and $\mathscr{M}$ a δ-ring of Borel subsets of $\mathbf{C}$ such that $\mathscr{M}$ contains every Borel subset of each of its members. Let $P: \mathscr{M} \to L(X)$ be a σ-additive, multiplicative map which is determining. If Y denotes the projective limit of the system*

$$\{(P(E)(X),\ P(E))\ ;\ E \in \mathscr{M}\},$$

then Y is complete and each operator $P(E)$, $E \in \mathscr{M}$, has a unique extension $P_Y(E) \in L(Y)$. The map $E \mapsto P_Y(E)$, $E \in \mathscr{M}$, has an extension to the Borel sets of $\mathbf{C}$ which is an $L(Y)$-valued spectral measure.

Examples such as 2.7 and 2.8 illustrate the importance of extendibility of additive, projection valued maps to spectral measures in larger spaces. There are many differential operators which give rise to such operator valued set functions on rings and δ-rings (see for example [16] and [22]). These operators can be treated as everywhere defined spectral operators in extended spaces.

EXAMPLE 2.14. Under suitable hypotheses an operator T determined by a singular second order formal differential operator

$$-\frac{d^2}{dt^2}+p(t), \quad 0 \leqslant t < \infty, \tag{11}$$

is a spectral operator (see Chapter XX, [4]). The steps for calculating the spectral resolution of T, which are known to be valid by the Weyl-Kodaira theorem if T is self-adjoint, can in any case be carried out formally. This gives rise to a family of operators $P(E)$ which are logical candidates for the spectral resolution of T, if T has in fact any spectral resolution. If the operator $P(E)$ form a uniformly bounded family it follows that T is spectral. However, if this is not the case, then T may still be spectral but in a larger space.

Let p be a C-valued function satisfying

$$\int_0^\infty e^{\varepsilon t} |p(t)| \, dt < \infty,$$

for some $\varepsilon > 0$, and γ an arbitrary complex number. Consider the operator T given by (11) together with the boundary condition

$$f'(0) - \gamma f(0) = 0. \tag{12}$$

The domain of T consists of those functions $f \in X = L^2([0, \infty))$, having derivatives f' absolutely continuous in bounded intervals of $[0, \infty)$, satisfying (12) and such that $T(f) \in X$.

The spectrum of T consists of the continuous spectrum $[0, \infty)$ and of a finite number of eigenvalues $\lambda_k = \mu_k^2$, $1 \leqslant k \leqslant r$, with $\operatorname{Im} \mu_k > 0$, which are zeros of some function ψ holomorphic in the half-plane $\operatorname{Im} z > -\varepsilon/2$. These eigenvalues are of finite multiplicity and the eigenfunctions corresponding to them belong to X, (see [16]). It can happen that ψ also has real zeros. They too can only be finite in number. If these real zeros are denoted by $\sigma_1, \ldots, \sigma_p$, then the positive numbers given by $\tilde{\lambda}_i = \sigma_i^2$, $1 \leqslant i \leqslant p$, are called the spectral singularities of the operator T. The "eigenfunctions" corresponding to the spectral singularities are not elements of the space X.

Assume that p and γ are such, that $r = 0$. Perhaps the simplest case of such an operator occurs when p is identically zero and $\gamma = -i$ (see [22]). In this case T has precisely one spectral singularity $\tilde{\lambda}_1 = 1$. The corresponding "eigenfunction" is $2e^{-it}$, $t \in [0, \infty)$, which is not an element of X.

Denote by $\mathscr{M}$ the δ-ring of all Borel sets in $\sigma(T)$ which are a positive distance from the set $\Lambda = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_p\}$. Then there exists a projection valued map $P: \mathscr{M} \to L(X)$ which is σ-additive, multiplicative and determining, [16]. If E_n, $n = 1, 2, \ldots$, is a sequence of sets from $\mathcal{M}$ whose distance from A tends to zero as $n \to \infty$, then the sequence $\{\|P(E_n)\|, n = 1,2, \ldots\}$ is unbounded. Accordingly, P cannot be extended to an $L(X)$-valued measure in $B(\mathbf{C})$.

A suitable admissible space Y is suggested by Proposition 2.13. If Y denotes the projective limit of the system $\{(P(E)(X), P(E))\ ;\ E \in \mathcal{M}\}$, then there is an $L(Y)$-valued spectral measure in $B(\mathbf{C})$ such that $P_Y(E)$ is the unique extension of $P(E)$ for each $E \in \mathcal{M}$. The operator T is scalar in Y with spectral resolution P_Y (Theorem 5.7, [16]).

3. EXTENDED QUASI-NILPOTENT OPERATORS

In the previous section it was shown that an operator T in X may not be scalar, but that there may exist an admissible space Y for T in which the extended operator T_Y is scalar. In this section it is shown that an operator N in a space X may have an admissible space in which the extension of N is quasi-nilpotent.

An operator $N \in L(X)$ is quasi-nilpotent if and only if (4) is valid for each $x \in X$ and $x' \in X'$. If the space X is quasi-complete and barrelled, then this is equivalent to $\sigma(N) = \{0\}$. In particular, if X is a Banach space, then an operator N is quasi-nilpotent if and only if

$$\lim_{n\to\infty} \|N^n\|^{1/n} = 0.$$

Given an operator $N \in L(X)$, it may happen that there exists a total subspace W of X', such that

$$\lim_{n\to\infty} |\langle N^n(x),\ x'\rangle|^{1/n} = 0,$$

for all $x \in X$ and $x' \in W$. If W is an invariant subspace of N', then it follows that $N \in L((X,\ \sigma(X, W)))$ is quasi-nilpotent and the completion, Y, of $(X, \sigma(X,\ W))$, is an admissible space for N. However, the extension $N_Y \in L(Y)$, of N, may not be quasi-nilpotent.

EXAMPLE 3.1. Let X denote the space of all $\mathbf{C}$-valued, C^∞-functions, $(x, y) \mapsto$ $\mapsto f(x, y)$ on $E = [0,1] \times \mathbf{R}$, such that

(i) $(\partial^k f/\partial x^k)(0,y) = 0$ for $k = 0,1,2, \ldots$, and $y \in \mathbf{R}$, and

(ii) $y \mapsto f(\cdot, y)$, $y \in \mathbf{R}$, is rapidly decreasing as $|y| \to \infty$.

Convergence in X is determined by the semi-norms $p_{k,m,q}$ for $k, m, q \in \{0,1,2, \ldots\}$ where

$$p_{k,m,q}(f) = \sup\{|y^k(\partial^{m+q}f/\partial x^m\partial y^q)(x,y)|\ ;\ (x,y) \in E\}, \quad f \in X.$$

If the operator $N \in L(X)$ is defined by

$$N(f)(x, y) = y\int_0^x f(t, y)\,\mathrm{d}t, \quad (x, y) \in E, f \in X,$$

then N is not quasi-nilpotent in X (see [17]).

Let W denote the space of all regular complex Borel measures μ on E of the form $\mu = \nu \otimes m$, where ν is a regular Borel measure on [0,1] and m is a regular Borel measure on $\mathbf{R}$ such that $y \mapsto y^n$, $y \in \mathbf{R}$, is integrable for each $n = 1,2, \ldots$, and the sequence

$$\left(\int_{\mathbf{R}} |y|^n d|m|(y) \right)^{1/n}, \quad n = 1,2, \ldots,$$

is bounded. Then W is a total subspace of functionals in X' which is invariant for N'. Furthermore, N is a member of $L((X, \sigma(X, W)))$ and is a quasi-nilpotent operator. However, the extension of N to the completion of $(X, \sigma(X, W))$ is not quasi-nilpotent.

From the point of view of extending quasi-nilpotent operators to admissible spaces, the definition (4) is too weak. This is partially so because, if a subset B in an arbitrary locally convex space X is a barrel, then the closure, $\bar{B}$, of B, in the completion, $\tilde{X}$, of X, need not be a barrel in $\tilde{X}$. The set $\bar{B}$ may fail to be absorbing in $\tilde{X}$.

An operator $N \in L(X)$ is said to be *uniforomly quasi-nilpotent* in X if, for every $x' \in X'$ there is a barrel B in X such that,

(i) the closure of B in the completion, $\tilde{X}$, is a barrel in $\tilde{X}$, and

(ii) the equality (4) is valid uniformly with respect to x in B.

An operator which is uniformly quasi-nilpotent is also quasi-nilpotent. It is worth noting that if $B \subseteq X$ is actually a closed, convex, balanced neighbourhood of zero satisfying (ii) of the previous definition, then it necessarily satisfies (i).

PROPOSITION 3.2. *Let X be a locally convex space and $N \in L(X)$ a uniformly quasi-nilpotent operator. If $\tilde{X}$ is the completion of X and $\tilde{N} \in L(\tilde{X})$ is the extension of N from X to $\tilde{X}$, then $\tilde{N}$ is uniformly quasi-nilpotent in $\tilde{X}$.*

Proof. Let $w \in \tilde{X}'$. Then $w \in X'$ and there is a barrel, B, in X, whose closure in $\tilde{X}$ is again a barrel and such that

$$\lim_{n \to \infty} |\langle N^n(x), w \rangle|^{1/n} = 0,$$

uniformly for $x \in B$. If $\varepsilon > 0$, then there is a positive integer l such that

$$|\langle \tilde{N}^n(x), w \rangle| \leqslant (\varepsilon/2)^n, \quad x \in B, \ n \geqslant l. \tag{13}$$

It follows from the continuity of $\tilde{N}$ that (13) is valid for all $x \in \bar{B}$, which is the required result.

Let X be a locally convex space and W a total subspace of X'. Then the barrels of X are the same for any locally convex topology consistent with the duality of X and W. However, whether the closure of a barrel in the completion of X with respect to such a topology is again a barrel, depends on the particular topology. In this way duality can be used to construct admissible spaces for uniformly quasi-nil potent operators.

The proof of the following result is a consequence of the preceding discussion and Proposition 3.2. Accordingly, the proof is omitted.

PROPOSITION 3.3. *Let X be a locally convex space and $N \in L(X)$. Let W be a total subspace of X'. Let τ be a locally convex topology on X consistent with the duality of X and W, such that $N \in L((X, \tau))$ and N is uniformly quasi-nilpotent in (X, τ). Then the completion, Y, of (X, τ), is an admissible space for N and the extension, N_Y, of N, is uniformly quasi-nilpotent.*

EXAMPLE 3.4. Let $X = \ell^1(\mathbf{N})$ and $N \in L(X)$ be the operator given by

$$N(x) = (0, x_1, x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in X.$$

Since $\|N^n\| = 1$ for each $n = 1,2, \ldots$, it follows that N is not quasi-nilpotent. Yet N is a natural candidate for being a quasi-nilpotent operator. It is an infinite dimensional version of the nilpotent operator

$$N(x_1, x_2, \ldots, x_k) = (0, x_1, \ldots, x_{k-1})$$

in finite dimensional $\mathbf{C}^k$-space. If W is the subspace of X' consisting of sequences of finite support, then W is invariant for N' and $N \in L((X, \sigma(X, W)))$. Furthermore, N is uniformly quasi-nilpotent in $(X, \sigma(X, W))$. Hence, the extension of N to the com pletion, $Y = \mathbf{C}^{\mathbf{N}}$, of $(X, \sigma(X, W))$, is a quasi-nilpotent operator.

The previous example is a particular case of a large class of operators on certain types of spaces.

Let X be a quasi-complete space. A set of vectors $\{e_j; j \in \mathscr{J}\}$ in X, where $\mathscr{J}$ is an interval of ordinal numbers, is said to be a *basis* for X if

(i) every element $x \in X$ has a unique expansion

$$x = \sum_{j \in \mathscr{J}} \alpha_j e_j = \lim_j \sum_{0 \leq k \leq j} \alpha_k e_k, \tag{14}$$

and

(ii) the associated coefficient functionals $e'_j : X \to \mathbf{C}$ such that

$$\langle x, e'_j \rangle = \alpha_j, \quad j \in \mathscr{J},$$

for every x given by (14), belong to X'.

It is clear that the above definition is an extension to locally convex spaces of the notion of a Schauder base in Banach space.

Let X be a space with basis $\{e_n ; n = 1,2, \ldots\}$. Let k be a positive integer and T a densely defined linear operator in X. Then T will be called a *weighted shift ope-*

rator of order k (with respect to this basis), if there exists a function $\xi : \mathbf{N} \to \mathbf{C}$ such that

$$T(x) = \sum_{n=1}^{\infty} \xi_n x_n e_{n+k},$$

for each x given by (14) which belongs to the domain of T; (see § 10 of [7] for example). The function ξ is called the *weight function*.

PROPOSITION 3.5. *Let X be a quasi-complete space with basis $\{e_n ; n = 1,2, \ldots\}$ and let W be the linear span of the associated coefficient functionals. Let N be a weighted shift operator of order k in X with weight function ξ, satisfying $N'(W) \subseteq W$. Then the completion, Y, of $(X, \sigma(X, W))$, is an admissible space for N in which the natural extension $N_Y \in L(Y)$ of N, is a quasi-nilpotent operator.*

Proof. The space Y is $\mathbf{C}^{\mathbf{N}}$ with the topology of pointwise convergence. It is clear that Y is an admissible space for N and that N_Y is the operator defined by $N_Y(y) = z$ where $z_n = 0$ for $1 \leqslant n \leqslant k$ and $z_n = \xi_{n-k} y_{n-k}$ for $k < n$. If $w = \sum_{n=1}^{l} \beta_n e'_n$ is a member of W and $y \in Y$, then the duality of Y and W is specified by

$$\langle y, w \rangle = \sum_{n=1}^{l} \beta_n y_n .$$

It follows from the equation $\langle N_Y^n(y), w \rangle = 0$ for all $y \in Y$ and $n > l - k$, that N_Y is quasi-nilpotent in Y.

Let N be a quasi-nilpotent operator in a space X. If Y is an admissible space for N, then it does not follow that N_Y is quasi-nilpotent.

EXAMPLE 3.6. Let X be the strict inductive limit of the spaces $\ell^1(I_n)$, $n = 1,2, \ldots$, where $I_n = \{1,2, \ldots, n\}$ for each $n \in \mathbf{N}$. The operator $N \in L(X)$ defined by

$$N(x) = (x_2, x_3, \ldots), \quad x = (x_1, x_2, \ldots) \in X,$$

is quasi-nilpotent. If $Y = \ell^1(\mathbf{N})$, then Y is admissible for N, but the natural extension of N to Y is not quasi-nilpotent in Y.

Similarly, if P is an $L(X)$-valued, equicontinuous spectral measure, f is a P-integrable function and Y is an admissible space for P, then it does not follow that f is P_Y-integrable.

EXAMPLE 3.7. Let X be the strict inductive limit of the Banach spaces $L^1([-n, n])$, $n = 1,2, \ldots$. The operator $S \in L(X)$ given by $S(f) = g$, $f \in X$, where $g(t) = tf(t)$, $t \in \mathbf{R}$, is a scalar operator in X. In fact, S is given by (7) where P is the $L(X)$-valued spectral measure of multiplication by characteristic functions of Borel subsets of $\mathbf{R}$. If $Y = L^1(\mathbf{R})$, then Y is admissible for P, but the identity function on $\mathbf{R}$ is not P_Y-integrable.

Let S and N be commuting operators in a space X. Suppose that S is a candidate for being an extended scalar operator and N is a candidate for being an extended quasi-nilpotent operator. The previous two examples show the difficulties in choosing an admissible space in which $S+N$ has an extension to a spectral operator. Such a space Y must be chosen so that S_Y is a scalar operator and, simultaneously, N_Y is a quasi-nilpotent operator.

It is worth noting that an operator T on X which is not spectral in $(X, \sigma(X, W))$ for any separating subspace $W \subseteq X'$, can still have an extension, T_Y, to an admissible space, Y, for T, such T_Y is spectral; (cf. Example 2.4. and [1], pp. 174).

4. GROUP REPRESENTATIONS

If X is a Banach space and $T \in L(X)$, then $f(T)$ exists as a member of $L(X)$ for every function f analytic in a neighbourhood of $\sigma(T)$. Hence, as noted by various authors (see for example [8] and [11]) the group

$$\{e^{isT};\ s \in \mathbf{R}\}, \tag{15}$$

exists for any such T. This group can then be effectively used for analysing the operator T.

For locally convex spaces this is no longer the case. Even if f is an entire function, $f(T)$ may not exist. Hence, the group (15) may not be available for the study of the operator T.

In this section, we give sufficient conditions, in terms of the group (15) or the group $\{T^n;\ n \in \mathbf{Z}\}$, for the existence of an admissible space Y for T such that the extension T_Y of T, is pseudo-hermitean or pseudo-unitary in Y.

Let G be a locally compact Hausdorff space. Let $B_c(G)$ and $B(G)$ be the σ-ring generated by all compact and all open sets, respectively. Regularity of a measure is meant with respect to the family of all compact subsets of G. Let $C_0(G)$ denote the Banach space of continuous functions vanishing at infinity with the uniform norm. The symbol $M(G)$ denotes the space of all finite regular complex measures on $B(G)$. The set of those measures in $M(G)$ which have finite support is denoted by $M_d(G)$.

If G is a locally compact Abelian group, then Γ denotes the group dual to G. The value of $\gamma \in \Gamma$ at the point $g \in G$ is denoted by $\langle g, \gamma \rangle$. The symbols dg and $d\gamma$ denote (a choice of) Haar measure on G and Γ, respectively.

Let X be a quasi-complete space and $m\colon B_c(\Gamma) \to X$ a regular vector measure. The function $\hat{m}\colon G \to X$, defined by

$$\hat{m}(g) = \int_\Gamma \overline{\langle g, \gamma \rangle}\, dm(\gamma), \quad g \in G,$$

is called the *Fourier-Stieltjes transform* of the measure m. A function $f: G \to X$ is a Fourier-Stieltjes transform in X if there exists a regular vector measure $m: B_c(\Gamma) \to X$ such that $f = \hat{m}$.

The proof of the following result is omitted as it is only a slight modification of that in [11].

PROPOSITION 4.1. *A weakly continuous function $f: G \to X$ is a Fourier-Stieltjes transform in X if and only if one (hence, both) of the following sets is relatively weakly compact in X:*

$$\left\{ \int_G h(g)f(g)\mathrm{d}g\,;\ \|\hat{h}\|_\infty \leqslant 1,\ h \in L^1(G) \right\}, \tag{16}$$

$$\left\{ \int_G f(g)\mathrm{d}\mu(g)\,;\ \|\hat{\mu}\|_\infty \leqslant 1,\ \mu \in M_d(G) \right\}. \tag{17}$$

Let I be a directed index set. Let the functions u_i, $i \in I$, form an approximate unit for $L^1(\Gamma)$ and let ω_i, $i \in I$, be their Fourier transforms. It is assumed that the functions ω_i, $i \in I$, have compact supports, are nonnegative and tend to 1 uniformly on compact subsets of G, [12]. A system of functions u_i, $i \in I$, on Γ, and ω_i, $i \in I$, on G, which are so related, is said to be a summation kernel for Γ and is denoted by $[u_i; \Gamma]$, $i \in I$.

Let $f: G \to X$ be a bounded, weakly continuous function. Define $F_i: \Gamma \to X$, $i \in I$, by

$$F_i(\gamma) = \int_G \langle g, \gamma \rangle \omega_i(g) f(g)\mathrm{d}g, \qquad \gamma \in \Gamma. \tag{18}$$

The integral in (18) is in the sense of Pettis, [12]. If each F_i, $i \in I$, is integrable with respect to the Haar measure, then define linear maps $\Phi_i: C_0(\Gamma) \to X$, $i \in I$, by

$$\Phi_i(\psi) = \int_\Gamma \psi(\gamma)F_i(\gamma)\mathrm{d}\gamma, \qquad \psi \in C_0(\Gamma). \tag{19}$$

The family of maps $\{\Phi_i\,;\ i \in I\}$ is said to be weakly equicompact if there is a weakly compact set C in X such that $\Phi_i(\psi) \in C$ for every ψ with $\|\psi\|_\infty \leqslant 1$ and every $i \in I$.

The following result is from [12].

PROPOSITION 4.2. *A bounded, weakly continuous function $f: G \to X$ is a Fourier-Stieltjes transform in X if and only if each function F_i, $i \in I$, is integrable and the maps Φ_i, $i \in I$, given by* (19) *are weakly equicompact.*

If X is weakly sequentially complete, then Propositions 4.1 and 4.2 remain valid if relative weak compactness of the sets (16) and (17), and weak equicompactness of the maps (19), is replaced by boundedness and equiboundedness, respectively.

A weakly continuous representation of the group G in the locally convex space X is a homomorphism $U\colon G \to L(X)$ such that

$$g \mapsto \langle U(g)(x), x' \rangle, \quad g \in G,$$

is continuous for each $x \in X$ and $x' \in X'$. A representation U is said to be *equicontinuous* if $\{U(g)\ ;\ g \in G\}$ is an equicontinuous part of $L(X)$.

If $P\colon B(\Gamma) \to L(X)$ is any equicontinuous spectral measure, it follows from Proposition 1.1 that the map $U\colon G \to L(X)$ given by

$$U(g) = \int_\Gamma \overline{\langle g, \gamma \rangle} \mathrm{d}P(\gamma), \quad g \in G, \tag{20}$$

is an equicontinuous representation. Conversely, we may ask which representations $U\colon G \to L(X)$ can be written in the form (20) for some spectral measure P. If such a measure P exists, then it is uniquely determined by U.

PROPOSITION 4.3. *Let the space X be quasi-complete and barrelled, $U\colon G \to L(X)$ a weakly continuous representation of G and $[u_i; \Gamma]$, $i \in I$, a summation kernel for Γ. Suppose that one of the following conditions is satisfied:*

(i) *The set $A(x)$ is relatively weakly compact in X for each $x \in X$, where*

$$A(x) = \left\{ \int_G h(g) U(g)(x) \mathrm{d}g\ ;\ \|\hat{h}\|_\infty \leqslant 1,\ h \in L^1(G) \right\}.$$

(ii) *The set $A(x)$ is relatively weakly compact in X for each $x \in X$, where*

$$A(x) = \left\{ \int_G U(g)(x) \mathrm{d}\mu(g)\ ;\ \|\hat{\mu}\|_\infty \leqslant 1,\ \ \mu \in M_d(G) \right\}.$$

(iii) *The representation U is bounded, the functions $F_i\colon \Gamma \to X$, $i \in I$, given by*

$$F_i(\gamma) = \int_G \langle g,\gamma \rangle \omega_i(g) U(g)(x) \mathrm{d}g, \quad \gamma \in \Gamma,$$

are integrable and the corresponding maps given by (19) *are weakly equicompact, for every $x \in X$.*

Then there exists a regular, equicontinuous spectral measure $P\colon B(\Gamma) \to L(X)$ such that (20) *holds for every $g \in G$.*

Proof. It suffices to verify the result using any one of the conditions. If $x \in X$, then by Proposition 4.1 or 4.2 there exists a regular measure $m_x\colon B(\Gamma) \to X$ such that $U(\cdot)(x) = \hat{m}_x$. It follows that the map $g \mapsto U(g)(x)$, $g \in G$, is bounded and continuous. Since X is barrelled, U is equicontinuous.

For every $h \in L^1(G)$, define a linear map $T_h: X \to X$ by

$$T_h(x) = \int_G h(g)U(g)(x)\mathrm{d}g, \quad x \in X.$$

If $M \subseteq X'$ is equicontinuous, then the continuous semi-norm p given by

$$p(x) = \sup\{|\langle x, x'\rangle| \; ; \; x' \in M\},$$

satisfies the inequality

$$p(T_h(x)) \leqslant \|h\|_1 \sup\{|\langle x, U(g)'(x')\rangle| \; ; \; g \in G, x' \in M\}.$$

Let $K = \{U(g)'(x') \; ; \; g \in G, \; x' \in M\}$. Then K is equicontinuous and

$$p(T_h(x)) \leqslant \|h\|_1 q(x), \tag{21}$$

where q is the continuous semi-norm, $q(x) = \sup\{|\langle x, x'\rangle|; \; x' \in K\}$. Inequality (21) implies that $T_h \in L(X)$ for each $h \in L^1(G)$. The relative weak compactness of $A(x)$, $x \in X$, implies that the family $\{T_h \; ; \; \|\hat{h}\|_\infty \leqslant 1, \; h \in L^1(G)\}$ is equicontinuous.

Hence, given p, there exist continuous semi-norms $q_1, \ldots, q_k$ and $\alpha > 0$ such that for every $x \in X$ and $h \in L^1(G)$,

$$p\left(\int_G h(g)U(g)(x)\mathrm{d}g\right) \leqslant \alpha \|\hat{h}\|_\infty \max\{q_i(x) \; ; \; 1 \leqslant i \leqslant k\}. \tag{22}$$

For $x \in X$ and $h \in L^1(G)$, the equality

$$\int_G h(g)U(g)(x)\mathrm{d}g = \int_\Gamma \hat{h}(\gamma)\mathrm{d}m_x(\gamma),$$

together with (22) shows that,

$$p(m_x(E)) \leqslant \alpha \max\{q_i(x) \; ; \; 1 \leqslant i \leqslant k\}, \quad x \in X,$$

for all $E \in B(\Gamma)$. For each $E \in B(\Gamma)$, define a map $P(E): X \to X$ by $P(E)(x) = m_x(E)$, $x \in X$. Then the proof can be completed as in Theorem 4 of [11].

The following result was proved for reflexive Banach spaces in [8] and for arbitrary Banach spaces in [11]. In both cases the proofs use certain Banach algebra techniques. The following proof uses only the theory of integration.

PROPOSITION 4.4. *Let the space X be quasi-complete and barrelled and $T \in L(X)$. Suppose that the representation $U(s) = \mathrm{e}^{isT}$, $s \in \mathbf{R}$, exists, is weakly continuous and*

$$\lim_{s \to 0} \frac{1}{is}(I - \mathrm{e}^{-isT}) = T \tag{23}$$

in $L(X)$. If U satisfies any of the conditions of Proposition 4.3, *then there exists a regular spectral measure $P\colon B(\mathbf{R}) \to L(X)$ such that T is pseudo-hermitean and is given by* (7).

Conversely, if T is pseudo-hermitean, then the equicontinuous representation $U\colon \mathbf{R} \to L(X)$ given by $U(s) = e^{isT}$, $s \in \mathbf{R}$, exists, is weakly continuous and (23) *is valid.*

Proof. By Proposition 4.3 there is a regular measure $P\colon B(\mathbf{R}) \to L(X)$ such that

$$(24) \qquad e^{isT} = \int_{\mathbf{R}} e^{is\sigma} dP(\sigma), \quad s \in \mathbf{R}.$$

Let $\{s_n\}_{n=1}^{\infty}$ be any real sequence converging to zero. Define bounded functions $f_n \in C(\mathbf{R})$, $n = 1,2,\ldots$, by $f_n(\sigma) = (is_n)^{-1}(1 - e^{-is_n\sigma})$, $\sigma \in \mathbf{R}$. Then

$$(25) \qquad \lim_{n\to\infty} f_n(\sigma) = \sigma = f(\sigma), \quad \sigma \in \mathbf{R}.$$

It follows from (23), (24) and (25) that for every $E \in B(\mathbf{R})$,

$$\lim_{n \leftarrow \infty} \int_E f_n(\sigma)\, dP(\sigma) = TP(E) = P(E)T.$$

Since $\left\{\int_E f_n(\sigma) d\langle P(\sigma)(x),x'\rangle\right\}_{n=1}^{\infty}$ is a Cauchy sequence for each $E \in B(\mathbf{R})$, it follows that f is $\langle P(\cdot)(x), x'\rangle$-integrable for each $x \in X$ and $x' \in X'$ (Lemma 2.3, [14]). Hence, f is P-integrable and (7) holds.

If $x \in X$, then

$$(26) \qquad \sup\{|\langle e^{isT}(x), x'\rangle| \,;\, s \in \mathbf{R}\} \leqslant |\langle P(\cdot)(x), x'\rangle| < \infty$$

for each $x' \in X'$. This shows that U is necessarily equicontinuous.

Conversely, suppose that T is pseudo-hermitean. Then $\sigma(T) \subseteq \mathbf{R}$ and there is a spectral measure $Q\colon \mathscr{M} \to L(X)$ together with a Q-integrable function h such that $T = \int_{\Omega} h(\omega) dQ(\omega)$. Define a spectral measure $P\colon B(\mathbf{C}) \to L(X)$ by $P(E) = Q(h^{-1}(E))$, $E \in B(\mathbf{C})$. Then Proposition 1.2 implies that $S(P) = \sigma(T) \subseteq \mathbf{R}$ and (7) is valid. The operator given by (24) exists in $L(X)$ for each $s \in \mathbf{R}$ by Proposition 1.1. Since f is P-integrable, it follows from (25) that (23) is valid (Theorem 2, p. 30, [13]). Fix $x \in X$, $x' \in X'$ and $t \in \mathbf{R}$. The inequality

$$|\langle e^{isT}(x),x'\rangle - \langle e^{itT}(x),x'\rangle| \leqslant |s - t| \int_{\mathbf{R}} |\sigma| d|\langle P(\sigma)(x), x'\rangle|$$

shows that U is weakly continuous. Equicontinuity of U is clear from (26).

It was noted that the group (15) may not be defined if X is an arbitrary locally convex space. Suppose that the space X is quasi-complete and barrelled and $T \in L(X)$ has compact spectrum in $\mathbf{C}$. Then

$$\sup \left\{ \limsup_{n \to \infty} (p(T^n))^{1/n} \; ; \; p \text{ a continuous semi-norm on } L(X) \right\} < \infty$$

(see [19]). It follows that the group (15) exists, is continuous and (23) holds. However, there are also many operators with unbounded spectrum which satisfy these conditions.

PROPOSITION 4.5. *Let the space X be quasi-complete and barrelled and $T \in L(X)$. Suppose that for each continuous semi-norm p determining the topology of X, there is $\beta > 0$ and a function $\alpha\colon X \to (0,\infty)$ such that*

$$p(T^n(x)) \leqslant \alpha(x)\beta^n, \quad x \in X, \; n = 0,1,2, \ldots .$$

Then e^{isT} exists in $L(X)$ for each $s \in \mathbf{R}$, the representation $s \mapsto e^{isT}$, $s \in \mathbf{R}$, is continuous and (23) *is valid.*

Proof. Let p be a semi-norm determining the topology of X and

$$Q_n = \sum_{j=0}^{n} \frac{1}{j!} (isT)^j, \quad n = 0,1,2, \ldots .$$

The estimate

$$p(Q_n(x) - Q_m(x)) \leqslant \alpha(x) \sum_{j=m+1}^{n} \frac{1}{j!} s^j \beta^j, \quad x \in X,$$

shows that $\{Q_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence for each $x \in X$. Hence, $\{Q_n\}_{n=1}^{\infty}$ has a limit in $L(X)$ denoted by e^{isT}. If $t \in \mathbf{R}$, then the inequality

$$p(e^{-itT}(e^{isT} - e^{itT})(x)) \leqslant \alpha(x)\,(e^{\beta|s-t|} - 1),$$

shows that $\lim_{s \to 0} e^{-itT}(e^{isT} - e^{itT})(x) = 0$ for each $x \in X$. The continuity of $s \mapsto e^{isT}$, $s \in \mathbf{R}$, follows. The equality (23) is a consequence of the estimate

$$p\left(\left(\frac{1}{is}(I - e^{-isT})(x)\right) - T(x)\right) \leqslant k\alpha(x)|s|, \quad s \in [-1, 1],$$

where $k = \sum_{j=2}^{\infty} (j!)^{-1}\beta^j$.

If the constants β in Proposition 4.5 can be chosen independent of the semi-norms p, then $\sigma(T)$ is compact in $\mathbf{C}$. In particular, this is the case if $\{T^n; n=1,2, \ldots\}$ is equicontinuous.

The following result follows from Proposition 4.3 (ii) applied to the group $G = \mathbf{Z}$.

PROPOSITION 4.6. *Let the space X be quasi-complete and barrelled and the operator $T \in L(X)$ have an inverse in $L(X)$. Let $\mathcal{N}$ denote the system of finite subsets of* $\mathbf{Z}$. *If the set*

$$A(x) = \left\{ \sum_{n\in\delta} a_n T^n(x) \,;\, \left\| \sum_{n\in\delta} a_n z^n \right\|_\infty \leqslant 1,\ \delta \in \mathcal{N} \right\},$$

is relatively weakly compact in X for each $x \in X$, where the $a_n \in \mathbf{C}$ are arbitrary, then there exists a regular spectral measure $P: B(\mathbf{T}) \to L(X)$ such that

$$T^n = \int_{\mathbf{T}} z^n \mathrm{d}P(z), \tag{27}$$

for each $n \in \mathbf{Z}$. In particular, $\sigma(T) \subseteq \mathbf{T}$ and T is pseudo-unitary.

Conversely, if T is a pseudo-unitary operator then (27) *holds for every $n \in \mathbf{Z}$.*

Let X be a quasi-complete space and G a locally compact Abelian group. Let $U: G \to L(X)$ be a representation of G in X. A locally convex space Y is said to be U-admissible, if Y is an admissible space for each operator $U(g)$, $g \in G$, and the representation U_Y is equicontinuous in $L(Y)$. The equicontinuity of the extended representation U_Y in $L(Y)$ does not necessarily follow from the equicontinuity of U in $L(X)$. If $\imath$ is the imbedding of X into Y, then a sufficient condition for U_Y to be equicontinuous is that $\imath(X)$ be of second category in Y.

If U is not a Fourier-Stieltjes transform in $L(X)$, then there may exist a U-admissible space Y such that U_Y is a Fourier-Stieltjes transform in $L(Y)$. There is no general procedure for finding such a space Y. A first step towards constructing a space Y is to find a total subspace W of X' such that each $w \in W$ is bounded on the sets $A(x)$ of Proposition 4.3 or on the union of the sets $\Phi_i(D_1)$, $i \in I$, for every $x \in X$, where D_1 is the unit ball of $C_0(\Gamma)$ and the maps Φ_i, $i \in I$, are given by (19) for each $x \in X$. Then the problem reduces to finding a space Y which has W as its dual, contains a copy of X and such that each operator $U(g)$, $g \in G$, has an extension to Y.

PROPOSITION 4.7. *Let $U: G \to L(X)$ be a representation. Let W be a subspace of X' which separates the points of X and is invariant for each operator $U(g)'$, $g \in G$. Suppose that for each $x \in X$ and $w \in W$ the function*

$$g \mapsto \langle U(g)(x), w\rangle, \quad g \in G, \tag{28}$$

is continuous, and for each $w \in W$ there is $\alpha > 0$ such that

$$\sup\{ \langle U(g)(x), w\rangle| \,;\, g \in G\} \leqslant \alpha |\langle x, w\rangle|, \quad x \in X. \tag{29}$$

If Y is the completion of $(X, \sigma(X, W))$, then the space Y is U-admissible for X and U_Y is a weakly continuous representation.

Proof. The space Y is barrelled and it follows from Proposition 2.1 that Y is admissible for each operator $U(g)$, $g \in G$. The equicontinuity of the representation U_Y follows from (29).

Fix $y \in Y$ and $y' \in W$. Then it must be shown that the function

$$\psi(g) = \langle U_Y(g)(y), y' \rangle, \qquad g \in G,$$

is continuous. Let $h \in G$ and $\varepsilon > 0$. By equicontinuity of U_Y there is an element $x \in X$ such that, for each $g \in G$,

$$|\langle U_Y(g)(y - x), y' \rangle| < \frac{\varepsilon}{3}. \tag{30}$$

It follows from (28) that there is a neighbourhood V of h such that

$$|\langle U(g)(x) - U(h)(x), y' \rangle| < \frac{\varepsilon}{3}, \tag{31}$$

for every $g \in V$. The continuity of ψ follows from (30) and (31).

For spaces with a basis, the subspace W in the previous proposition can often be taken as the linear span of the associated coefficient functionals.

PROPOSITION 4.8. *Let G be a locally compact Abelian group and $[u_i; \Gamma]$, $i \in I$, a summation kernel for Γ. Let $\{e_j;\ j \in \mathscr{J}\}$ be a basis for the locally convex space X and W the linear span of the associated coefficient functionals. Let $U: G \to L(X)$ be a representation such that W is an invariant subspace for each operator $U(g)'$, $g \in G$. For each $x \in X$ and $j \in \mathscr{J}$, let $\alpha_j(x, \cdot): G \to \mathbf{C}$ be the function satisfying*

$$U(g)(x) = \sum_{j \in \mathscr{J}} \alpha_j(x,g) e_j, \quad g \in G. \tag{32}$$

Suppose that the function $\alpha_j(x, \cdot)$ is continuous for each $x \in X$ and $j \in \mathscr{J}$. If for each $j \in \mathscr{J}$, there is $\beta_j > 0$, such that

$$|\alpha_j(x, g)| \leqslant \beta_j |\langle x, e_j' \rangle|, \tag{33}$$

for each $x \in X$ and $g \in G$, then the completion, Y, of $(X, \sigma(X, W))$, is a U-admissible space and the representation U_Y is weakly continuous.

For each $j \in \mathscr{J}$ and $g \in G$, let $\tilde{\alpha}_j(\cdot, g)$ denote the continuous extension of $\alpha_j(\cdot, g)$ to Y. If

$$\sup\left\{\left|\int_\Gamma \psi(\gamma)(\hat{u}_i \tilde{\alpha}_j(y, \cdot))\hat{\ }(-\gamma) d\gamma\right| ;\ i \in I,\ \psi \in D_1\right\} < \infty, \tag{34}$$

for each $y \in Y$ and $j \in \mathscr{J}$, then U_Y is a Fourier-Stieltjes transform.

Proof. As $\langle U(g)(x),e_j'\rangle = \alpha_j(x,g)$, it follows from the continuity of $\alpha_j(x,\cdot)$ and (33) that the conditions of Proposition 4.7 are satisfied. Hence, the space Y is U-admissible and the representation U_Y is weakly continuous.

It also follows from (33) that for each $j \in \mathcal{J}$ and $g \in G$, the function $x \mapsto \alpha_j(x,g)$, $x \in X$, has a (unique) continuous extension, $\tilde{\alpha}_j(\cdot, g)$, to Y. Furthermore, U_Y is still given by (32) for the extended functions $\tilde{\alpha}_j(\cdot, g)$. The weak continuity of the representation U_Y implies that $\tilde{\alpha}_j(y,\cdot)$ is continuous on G for each $y \in Y$ and $j \in \mathcal{J}$.

Fix $y \in Y$. Since each function $\hat{u}_i$, $i \in I$, is continuous with compact support and

$$\langle F_i(\gamma), e_j'\rangle = \int_G \langle g,\gamma\rangle \hat{u}_i(g)\tilde{\alpha}_j(y,g)\mathrm{d}g, \quad \gamma \in \Gamma,$$

for each $i \in I$ and $j \in \mathcal{J}$, it follows that each F_i, $i \in I$, as given by (18), is bounded and scalarly $\mathrm{d}\gamma$-measurable. Consequently, since Y is reflexive, the functions F_i, $i \in I$, are integrable in Y for each $y \in Y$.

For each $i \in I$, $j \in \mathcal{J}$ and $\psi \in C_0(\Gamma)$, the equation

$$\langle \Phi_i(\psi), e_j'\rangle = \int_\Gamma \psi(\gamma)\,(\hat{u}_i\tilde{\alpha}_j(y,\cdot))^\wedge(-\gamma)\mathrm{d}\gamma,$$

is valid for every $y \in Y$. It follows from (34) that each $w \in W$ is bounded on the union of the sets $\Phi_i(D_1)$, $i \in I$, for each $y \in Y$. Since the space Y is weakly sequentially complete, Proposition 4.3 (iii) implies that U_Y is a Fourier-Stieltjes transform.

The classical theorem of Stone asserts that if $U\colon G \to L(X)$ is a weakly continuous representation of the locally compact Abelian group by unitary operators $U(g)$, $g \in G$, on a Hilbert space X, then there is a regular spectral measure $P\colon B(\Gamma) \to L(X)$ such that (20) is valid.

For Banach spaces (even reflexive) this is no longer the case. For example, the group $U = \{T^n;\, n \in \mathbf{Z}\}$ of surjective isometries generated by the bilateral shift, T, on $X = \ell^p(\mathbf{Z})$, $1 \leqslant p < 2$, is not a Fourier-Stieltjes transform in $L(X)$. Consequently, T is not pseudo-unitary in X. However, if $Y = \ell^2(\mathbf{Z})$, then Y is U-admissible and it follows from Stone's theorem (or Proposition 4.6) that T_Y is pseudo-unitary (cf. Example 2.2).

If an operator T in X is not scalar, the results of this section applied to the groups $U = \{T_Y^n;\, n \in \mathbf{Z}\}$ or $U = \{e^{isT_Y};\, s \in \mathbf{R}\}$, can often be used to deduce that T is pseudo-unitary or pseudo-hermitean in a suitable U-admissible space Y. In fact, all of the operators in Examples 2.2 to 2.8 can be shown to be extended scalar operators using the results of this section. In particular, Examples 2.3 and 2.5 illustrate the applicability of Proposition 4.8.

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