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A CONVERSE TO OCNEANU'S THEOREM

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Ocneanu's theorem ([2]) states, grosso modo, that for any countable amenable discrete group G, there is only one action by outer automorphisms on the hyperfinite type II_1 factor R up to outer conjugacy. Here we exhibit, for any countable *non-amenable* discrete group G, two outer actions of G on R which are not outer conjugate. The invariant we shall use is the fixed point algebra for the action on the algebra of central sequences.

The proof will be based on the following elementary Hilbert space lemma, obtained by K. Schmidt and the author. The setting is as follows. Let $\mathscr{H}$ be a separable Hilbert space with orthonormal basis $\{e_i \mid i \in I \cup \{0\}\}$. If G is a countable group we form the infinite tensor product $\mathscr{X} = \bigotimes_{g \in G} \mathscr{H}_g$ (where each $\mathscr{H}_g$ is a copy of $\mathscr{H}$), taken along the vector $\bigotimes_{g \in G} (e_0)_g$. Let $g \to u_g$ be the unitary representation corresponding to the Bernoulli shift, i.e. $u_g(\bigotimes_{h \in G} e_{i(h)}) = \bigotimes_{g \in G} e_{i(hg)}$.

LEMMA 1. *The representation $\{u_g\}$ is a direct sum of the trivial representation and a subrepresentation of a direct sum of copies of the regular representation.*

Proof. An orthonormal basis for $\mathscr{X}$ is given by the collection F of all functions $i: G \to I \cup \{0\}$ such that $i(g) = 0$ except on a finite subset of G. Moreover the action of G on $\mathscr{X}$ comes from permutations of the basis: $(gi)(h) = i(hg)$. If Θ denotes the space of orbits of F under the action of G, then the representation $g \to u_g$ is $\bigoplus_{\theta \in \Theta} \pi_\theta$ where π_θ is the representation coming from right translation on ℓ^2 of the orbit θ. Such a representation π_θ is a subrepresentation of the regular representation whenever the stabilizer subgroup of a point $i \in \theta$ is finite. It thus suffices to show that the stabilizer of every point $i \in F$ is finite, except when $i(g) = 0$, $\forall g \in G$. But if $T = \{g \in G \mid i(g) \neq 0\}$ then T is finite and if h is in the stabilizer of i then $Th = T$. This is only possible for finitely many h. Q.E.D.

Now let τ be the trace on R, $\tau(1) = 1$, and let us say that an action α of the group G on R is *ergodic at* ∞ if there is no sequence p_n of projections in R, $\sigma(p_n) = 1/2$, with $\|p_n - \alpha_g(p_n)\|_2 \to 0$ and $\|[p_n, y]\|_2 \to 0$ as $n \to \infty$, $\forall g \in G$, $y \in R$.

If G is a countable group, represent R as $\bigotimes_{g \in G} M_2(\mathbf{C})$, the tensor product taken with respect to the trace. Then G has an outer action on R by the Bernoulli shift. Call this action α.

PROPOSITION 2. *If G is not amenable, the action α is ergodic at ∞.*

Proof. The Hilbert space $L^2(R, \tau)$ is nothing but the space $\bigotimes_{g \in G} L^2(M_2(\mathbf{C}), \tau)$ taken along the vector $\bigotimes_{g \in G} 1$, on which G acts exactly as in Lemma 1. If p_n were a sequence of projections as in the definition of ergodicity at ∞, then $\{2p_n - 1\}$ would be a sequence of vectors in the orthocomplement of $1 \in L^2(R, \tau)$ exhibiting the weak containment of the trivial representation. By the characterization (3.6 of [3]) of amenability and Lemma 1, α is ergodic at ∞. Q.E.D.

PROPOSITION 3. *Ergodicity at ∞ is an invariant of outer conjugacy.*

Proof. By definition, two actions α and β are outer conjugate if there is an automorphism θ of R with $\theta\alpha_g\theta^{-1} = \mathrm{Ad}\, u_g \beta_g$ where $\{u_g\}$ are some unitaries in R. If p_n s a sequence for β, $\theta(p_n)$ is one for α. Q.E.D.

THEOREM 4. *If G is not amenable, there are two actions of G on R which are not outer conjugate.*

Proof. By Propositions 2 and 3, it suffices to give an outer action which is not ergodic at ∞. But since $R \cong R \overline{\otimes} R$, we may begin with the action α as in Proposition 3 and let $\beta = \alpha \otimes \mathrm{id}$. Then any central sequence of projections $1 \otimes p_n$, $\tau(p_n) = 1/2$ will suffice to show that β is not ergodic at ∞. Q.E.D.

REMARKS. The reader will notice the similarity with the method of [1]. One can also use Lemma 1 to embed any II_1 factor M in a full II_1 factor N. One would take, say, G to be the free group on 2 generators and form the infinite tensor product $\bigotimes_{g \in G} M$. Then let N be the crossed product of $\bigotimes_{g \in G} M$ by the Bernoulli shift action of G so that $M \subset \overline{\bigotimes}_G M \subset N$.

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