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STABILITY OF OPERATOR INEQUALITIES

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1. INTRODUCTION

Let $A_0, A_1, \ldots, A_n$ be a finite sequence of bounded linear operators on a Hilbert space $\mathscr{H}$. It often occurs that the mutual relationship among these operators is governed by a finite (or infinite) number of inequalities;

$$(*) \qquad F_i(A_0, A_1, \ldots, A_n) \geqslant 0 \quad (i = 1, 2, \ldots, N),$$

where $X \geqslant 0$ means that X is a positive semidefinite operator on $\mathscr{H}$. One might be interested in when the system of inequalities is stable under Cesáro averaging. More precisely, let $\{B_k\}$ be the *Cesáro average* of $\{A_k\}$;

$$B_k = (A_0 + A_1 + \ldots + A_k)/(k + 1) \quad (k = 0, 1, \ldots, n).$$

Does the system of inequalities $(*)$ imply that

$$F_i(B_0, B_1, \ldots, B_n) \geqslant 0 \quad (i = 1, 2, \ldots, N)?$$

In this paper, these stability properties are discussed. The first example concerns the convex sequences of operators. Together with this, the second section discusses the stability of log-convexity of operator sequences. The following two sections discuss the stability of the relations governed by truncated Hankel and Toeplitz operator matrices. The last section gives an application and a continuous analogue of the stability.

In concluding this introduction, the author would like to express his heartly thanks to Prof. T. Ando for many valuable suggestions and discussions.

2. CONVEX SEQUENCES

An operator sequence $\{A_k\}$ is *convex* if

$$(A_{k-1} + A_{k+1})/2 - A_k \geqslant 0, \quad (k = 1, 2, \ldots, n - 1).$$

After some routine calculations, it follows that

$$F_k \geqslant [(k-1)/(k+2)]F_{k-1} \quad (k=2, 3, \ldots, n-1),$$

where

$$F_k = (B_{k-1}+B_{k+1})/2 - B_k \quad (k=1, 2, \ldots, n-1),$$

and hence the Cesáro average $\{B_k\}$ is also convex.

A stronger condition on an operator sequence $\{A_k\}$ is given by

$$\begin{bmatrix} A_{k-1} & A_k \\ A_k & A_{k+1} \end{bmatrix} \geqslant 0 \quad (k=1, 2, \ldots, n-1),$$

and called *log-convexity*.

REMARK. From this definition, it is implied that all the terms of the sequence $\{A_k\}$ are positive semidefinite. If the terms A_k commute with each other, the stability of the log-convexity under Cesáro average is already known in essence through the spectral theory, cf. [4] and [5]. But their proof does not work for this general case.

Note that if the operators A, B and C are positive semidefinite and A has the inverse, then the positive semidefiniteness of the 2×2 operator matrix

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

is equivalent to

$$C \geqslant BA^{-1}B.$$

THEOREM 1. *If $\{A_k\}$ is log-convex, then so is its Cesáro average $\{B_k\}$.*

Proof. By the induction argument, it is sufficient to show that

$$\begin{bmatrix} B_{n-3} & B_{n-2} \\ B_{n-2} & B_{n-1} \end{bmatrix} \geqslant 0 \quad \text{and} \quad \begin{bmatrix} A_{n-2} & A_{n-1} \\ A_{n-1} & A_n \end{bmatrix} \geqslant 0$$

together imply that

$$\begin{bmatrix} B_{n-2} & B_{n-1} \\ B_{n-1} & B_n \end{bmatrix} \geqslant 0.$$

To this end, it may be assumed that each A_k is positive definite and that

$$A_0 + A_1 + \ldots + A_{n-2} = I \quad \text{(the identity operator).}$$

In view of the preceding remarks, the assumptio ns yield that

$$A_n \geqslant A_{n-1} A_{n-2}^{-1} A_{n-1}$$

and

$$(n-1)^2(I + A_{n-1}) \geqslant n(n-2)(I - A_{n-2})^{-1}.$$

From the last inequality, one gets

$$A_{n-2}^{-1} \geqslant (n-1)^2 (I + A_{n-1}) [I + (n-1)^2 A_{n-1}]^{-1}.$$

This inequality further implies that

$$n^2(I + A_{n-1}) - (n^2 - 1)(I + A_{n-1})^2 + n^2 A_{n-1} A_{n-2}^{-1} A_{n-1} \geqslant$$

$$\geqslant [I + (n-1)^2 A_{n-1}]^{-1} (I + A_{n-1}) [I - (n-1) A_{n-1}]^2,$$

whose right hand side is shown to be positive definite by a routine. Again by using he preceding remark, it follows that

$$\begin{bmatrix} I/(n-1) & (I + A_{n-1})/n \\ (I + A_{n-1})/n & (I + A_{n-1} + A_{n-1} A_{n-2}^{-1} A_{n-1})/(n+1) \end{bmatrix} \geqslant 0,$$

and hence the requirement. ▨

3. OPERATOR HANKEL MATRICES

Each matrix $\begin{bmatrix} A_{k-1} & A_k \\ A_k & A_{k+1} \end{bmatrix}$ in the preceding section is a submatrix of the *operator Hankel matrix* $\mathbf{H}(A_0, A_1, \ldots, A_n)$ associated with the sequence $\{A_k\}$;

$$\mathbf{H}(A_0, A_1, \ldots, A_n) = \begin{bmatrix} A_0 & A_1 & \ldots & A_m \\ A_1 & A_2 & \ldots & A_{m+1} \\ \vdots & \vdots & & \vdots \\ A_m & A_{m+1} & \ldots & A_n \end{bmatrix} \qquad (n = 2m).$$

If the operator sequence consists of odd terms, we have the following stability theorem.

THEOREM 2. *Let n be an even positive integer. If the operator Hankel matrix* $\mathbf{H}(A_0, A_1, \ldots, A_n)$ *is positive semidefinite, then so is the operator Hankel matrix* $\mathbf{H}(B_0, B_1, \ldots, B_n)$, *where* $\{B_k\}$ *is the Cesáro average of the sequence* $\{A_k\}$.

Proof. It may be assumed that $A_0 = I$. An operator version of a truncated Hamburger moment problem (see [2]) and the Naimark dilation theorem (see [6], p. 32) show that there is a bounded Hermitian operator A on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

$$A_k = PA^k|_{\mathcal{H}} \quad (k = 0, 1, \ldots, n-1),$$

and

$$A_n \geqslant PA^n|_{\mathcal{H}},$$

where P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. Since

$$(A^0 + A^1 + \ldots + A^k)/(k+1) = \int_0^1 B(t)^k \mathrm{d}t,$$

where $B(t) = tA + (1-t)I$, it follows that

$$\mathbf{H}(B_0, B_1, \ldots, B_n) \geqslant \int_0^1 \mathbf{H}(C_0(t), C_1(t), \ldots, C_n(t))\mathrm{d}t$$

where

$$C_k(t) = PB(t)^k|_{\mathcal{H}} \quad (k = 0, 1, \ldots, n).$$

The integrand of the right hand side of this inequality is positive semidefinite at every t. In fact, it is factorized as follows;

$$\mathbf{H}(C_0(t), C_1(t), \ldots, C_n(t)) =$$

$$= \begin{bmatrix} PB(t)^0 \\ PB(t)^1 \\ \vdots \\ PB(t)^{n/2} \end{bmatrix} [B(t)^0|_{\mathcal{H}},\ B(t)^1|_{\mathcal{H}}, \ldots, B(t)^{n/2}|_{\mathcal{H}}].$$

Now the positive semidefiniteness of $\mathbf{H}(B_0, B_1, \ldots, B_n)$ follows. ∎

If the sequence of operators consists of even terms, the stability theorem is stated as follows.

THEOREM 3. *Let n be an odd positive integer. If the operator Hankel matrices $\mathbf{H}(A_0, A_1, \ldots, A_{n-1})$ and $\mathbf{H}(A_1, A_2, \ldots, A_n)$ are positive semidefinite, then so are the operator Hankel matrices $\mathbf{H}(B_0, B_1, \ldots, B_{n-1})$ and $\mathbf{H}(B_1, B_2, \ldots, B_n)$, where $\{B_k\}$ is the Cesáro average of the sequence $\{A_k\}$.*

The proof of this theorem is based on the truncated Stieltjes moment problem for operator sequence just as the preceding theorem on Hamburger's type and will be omitted. The moment problem will be found in [2].

Another example of stable inequality associated with operator Hankel matrices is given in the following. The proof is based on the truncated Hausdorff moment problem for an operator sequence which can be found in [2].

THEOREM 4. *Let n be an even positive integer. If the operator Hankel matrices associated with* $\{A_k\}$ *satisfy*

$$\mathbf{H}(A_0, A_1, \ldots, A_n) \geqslant 0$$

and

$$\mathbf{H}(A_1, A_2, \ldots, A_{n-1}) \geqslant \mathbf{H}(A_2, A_3, \ldots, A_n) \geqslant 0,$$

then so do the operator Hankel matrices associated with $\{B_k\}$;

$$\mathbf{H}(B_0, B_1, \ldots, B_n) \geqslant 0$$

and

$$\mathbf{H}(B_1, B_2, \ldots, B_{n-1}) \geqslant \mathbf{H}(B_2, B_3, \ldots, B_n) \geqslant 0,$$

where $\{B_k\}$ *is the Cesáro average of the sequence* $\{A_k\}$.

THEOREM 5. *Let n be an odd positive integer. If the operator Hankel matrices associated with* $\{A_k\}$ *satisfy*

$$\mathbf{H}(A_0, A_1, \ldots, A_{n-1}) \geqslant \mathbf{H}(A_1, A_2, \ldots, A_n) \geqslant 0,$$

then so do the operator Hankel matrices associated with $\{B_k\}$;

$$\mathbf{H}(B_0, B_1, \ldots, B_{n-1}) \geqslant \mathbf{H}(B_1, B_2, \ldots, B_n) \geqslant 0,$$

where $\{B_k\}$ *is the Cesáro average of the operator sequence* $\{A_k\}$.

4. OPERATOR TOEPLITZ MATRICES

In connection with the power moment problems in the preceding section, it will be of interest to consider Toeplitz matrices. The *operator Toeplitz matrix* $\mathbf{T}(A_0, A_1, \ldots, A_n)$ associated with the operator sequence $\{A_k\}$ is by definition

$$\mathbf{T}(A_0, A_1, \ldots, A_n) = \begin{bmatrix} A_0 & A_1 & \ldots & A_{n-1} & A_n \\ A_1^* & A_0 & \ldots & A_{n-2} & A_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ A_n^* & A_{n-1}^* & \ldots & A_1^* & A_0 \end{bmatrix}.$$

THEOREM 6. *If the operator Toeplitz matrix* $\mathbf{T}(A_0, A_1, \ldots, A_n)$ *is positive semidefinite, then so is the operator Toeplitz matrix* $\mathbf{T}(B_0, B_1, \ldots, B_n)$, *where* $\{B_k\}$ *is the Cesáro average of the sequence* $\{A_k\}$.

Proof. It may be assumed that $A_0 = I$. Then by an operator version of a trigonometric moment problem (see [2]) and the Naimark dilation theorem (see [6]), there is a unitary operator U on a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ such that

$$A_k = PU^k|_{\mathscr{H}} \quad (k = 0, 1, \ldots, n)$$

where P is the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$. The last dilation representation of the sequence $\{A_k\}$ yields that

$$B_k = P\int_0^1 B(t)^k \mathrm{d}t|_{\mathscr{H}} \quad (k = 0, 1, \ldots, n),$$

where

$$B(t) = tU + (1 - t)I.$$

Since $B(t)$ is a contraction, i.e. $\|B(t)\| \leqslant 1$, it follows that

$$\mathbf{T}(B(t)^0, B(t)^1, \ldots, B(t)^n) \geqslant 0,$$

by using a theorem of Sz.-Nagy (see [6], p. 28). Now the positive semidefiniteness of $\mathbf{T}(B_0, B_1, \ldots, B_n)$ follows. ▨

5. SOME REMARKS

1) The stability theorem (Theorem 6) yields the positive definiteness of the following Toeplitz matrices;

$$\mathbf{T}(1, (1 + \cos t)/2, \ldots, (1 + \cos t + \ldots + \cos nt)/(n + 1))$$

for $0 \leqslant t \leqslant 2\pi$, and

$$\mathbf{T}(\sin t, (\sin 2t)/2, \ldots, (\sin nt)/n)$$

for $0 \leqslant t \leqslant \pi$.

2) The system of inequalities

$$\mathbf{T}(A_0, A_1, \ldots, A_n) \geqslant 0$$

and

$$\mathrm{Re}[\mathbf{T}(A_1, A_2, \dots, A_{n+1})] \geqslant 0$$

is not stable under Cesáro average.

3) A continuous analogue to Theorem 6 holds, too. A continuous (operator valued) function on the real axis is *nonnegative definite* if for any set of real values $x_1, x_2, \dots, x_n$ and any vectors $z_1, z_2, \dots, z_n$ in $\mathscr{H}$, the inequality

$$\sum_{r,s=1}^{n} \langle f(x_r - x_s) z_r \mid z_s \rangle \geqslant 0$$

is satisfied.

THEOREM 7. *If f is a continuous nonnegative definite function, then so is the function F defined by*

$$F(x) = \begin{cases} \left(\int_0^x f(y)\,\mathrm{d}y\right) \Big/ x & \text{if } x \neq 0, \\ f(0) & \text{if } x = 0. \end{cases}$$

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