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COMPACT OPERATORS IN THE ALGEBRA OF A PARTIALLY ORDERED MEASURE SPACE

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In [1, p. 498] Arveson conjectured that $\operatorname{alg} L(2^\infty, \leqslant, m_p)$ contains no nonzero compact operators. By establishing a general result about the lattice of invariant subspaces of a compact operator, we show that both $\operatorname{alg} L(2^\infty, \leqslant, m_p)$, $(0 < p < 1)$, and $\operatorname{alg} L(2^\infty, \leqslant)$ contain no nonzero compact operators.

Throughout this paper (X, Σ, μ) will denote a probability measure space. If $B \in \Sigma$ we shall denote by P_B the projection on $L^2(X, \Sigma, \mu)$ induced by B.

LEMMA 1. *Let $\{B_n\}$ be a sequence of measurable sets such that for each subsequence $\{B_{n_i}\}$, $\mu\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) = 1$. If $f \in L^1(X, \Sigma, \mu)$ satisfies $\liminf \int_{B_n} |f| \mathrm{d}\mu = 0$ then $f = 0$ a.e. $[\mu]$.*

Proof. Suppose $\liminf \int_{B_n} |f| \mathrm{d}\mu = 0$, and pick $\varepsilon > 0$. Then there is . sequence $\{n_i\}$ such that $\int_{B_{n_i}} |f| \mathrm{d}\mu \leqslant \varepsilon/2^i$. Since $\mu\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) = 1$ we have that

$$\int_X |f| \mathrm{d}\mu = \int_{\bigcup_{i=1}^{\infty} B_{n_i}} |f| \mathrm{d}\mu \leqslant \sum_{i=1}^{\infty} \int_{B_{n_i}} |f| \mathrm{d}\mu \leqslant \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Therefore $\int_X |f| \mathrm{d}\mu = 0$ so $f = 0$ a.e. $[\mu]$.

LEMMA 2. *Let $\{B_n\}$ be a sequence of measurable sets such that for each subsequence $\{B_{n_i}\}$, $\mu\left(\bigcap_{i=1}^{\infty} B_{n_i}\right) = 0$. If $\{f_i\}$ is a sequence in $L^2(X, \Sigma, \mu)$ such that* support$(f_i) \subseteq B_{n_i}$ *for some subsequence $\{n_i\}$ and $f_i \xrightarrow{L^2} f$ then $f = 0$ a.e. $[\mu]$.*

Proof. Since $f_i \to f$ we can extract a subsequence, also called $\{f_i\}$, which we may assume converges pointwise to f. Set $E_K = \bigcap_{i \geq K} B_{n_i}$ and $E = \bigcup_{K=1}^{\infty} E_K$. From $\mu(E_K) = 0$ we obtain $\mu(E) = 0$. Suppose $x \notin E$. Then there is a subsequence $\{B_{n_{i_j}}\}$ such that $x \notin B_{n_{i_j}}$ for each n_{i_j}. This gives $f_{i_j}(x) = 0$. Since $f_{i_j}(x) \to f(x)$ we conclude that $f(x) = 0$. So $f = 0$ a.e. $[\mu]$.

THEOREM. *Suppose $\{B_n\}$ is a sequence of measurable sets such that for each subsequence $\{B_{n_i}\}$, $\mu\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) = 1$ and $\mu\left(\bigcap_{i=1}^{\infty} B_{n_i}\right) = 0$. If $K: L^2(X, \Sigma, \mu) \to L^2(X, \Sigma, \mu)$ is a compact operator and $P_{B_n} \in \mathrm{Lat}(K)$ for each n, then $K = 0$.*

Proof. Let $f \in L^2(X, \Sigma, \mu)$ and consider the sequence $\{P_{B_n} K^* f\}$. We have $\|P_{B_n} K^* f\| \leqslant \|K^*\| \, \|f\|$ and since K is compact we can find a subsequence $\{n_i\}$ such that $KP_{B_{ni}} K^* f$ converges. Since $P_{B_{ni}} \in \mathrm{Lat}(K)$,

$$\text{support}(KP_{B_{n_i}} K^* f) \subseteq B_{n_i}.$$

Hence by Lemma 2, $KP_{B_{ni}} K^* f \to 0$. Therefore $(KP_{B_{ni}} K^* f, f) \to 0$ so $(P_{B_{n_i}} K^* f, K^* f) \to 0$. Whence $\int_{B_{n_i}} |K^* f|^2 d\mu \to 0$. Lemma 1 now gives $K^* f = 0$. Therefore $K^* = 0$, and so $K = 0$.

As an immediate consequence we have

COROLLARY. *Let $(X, \leqslant, \mu)$ be a standard partially ordered probability measure space and $\{B_n\}$ a sequence of decreasing Borel sets such that for any subsequence $\{B_{n_i}\}$, $\mu\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) = 1$ and $\mu\left(\bigcap_{i=1}^{\infty} B_{n_i}\right) = 0$. Then $\mathrm{alg}\, L(X, \leqslant, \mu)$ contains no nonzero compact operators.*

For the partially ordered measure spaces $(2^\infty, \leqslant, m_p)$, $0 < p < 1$, take $B_n = \{x : x_n = 0\}$ for $n = 1, 2, \ldots$. It is an easy measure-theoretic exercise to verify that if $\{B_{n_i}\}$ is a subsequence then $m_p\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) = 1$ and $m_p\left(\bigcap_{i=1}^{\infty} B_{n_i}\right) = 0$. Therefore $\mathrm{alg}\, L(2^\infty, \leqslant, m_p)$ contains no nonzero compact operators. Similarly, by taking $B_n = \{x : x_{2n} = 0\}$, we can show that $\mathrm{alg}\, L(2^\infty, \leqslant)$ contains no nonzero compact operators (here $(2^\infty, \leqslant)$ denotes the even partial order on 2^∞). This settles Arveson's conjecture [1].

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