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SEMINORMAL COMPOSITION OPERATORS

DAVID J. HARRINGTON and ROBERT WHITLEY

1. INTRODUCTION

A composition operator C_T on $L^2(S, \Sigma, \mu)$, (S, Σ, μ) sigma-finite throughout this paper, is a bounded linear operator given by composition with a map $T: S \to S$ via $C_T f(s) = f(T(s))$. The classical necessary and sufficient conditions for this are [4, p. 39 and 2, pp. 663—5] T measurable, $\mu \circ T^{-1}$ absolutely continuous with respect to μ, and $h = \mathrm{d}\mu \circ T^{-1}/\mathrm{d}\mu$ in $L^\infty(S, \Sigma, \mu)$.

Various properties of these operators can be characterized in measure theoretic terms. For example, answering a question raised by Nordgren in his interesting lectures [4], (see [6] and [7]):

0. THEOREM. a) *C_T is normal iff*
   i) *$T^{-1}(\Sigma)$ is essentially all of Σ, and*
   ii) *$h = h \circ T$ a.e., where $h = \mathrm{d}\mu \circ T^{-1}/\mathrm{d}\mu$.*

b) *If $\mu(S)$ is finite, then C_T is normal iff*
   i) *$T^{-1}(\Sigma)$ is essentially all of Σ, and*
   ii) *$h = 1$ a.e., i.e., T is measure preserving.*

In this case C_T is unitary.

An operator A is said to be *hyponormal* in case $A^*A - AA^* \geqslant 0$; A is *seminormal* if either A or A^* is hyponormal. A is called *quasinormal* when A commutes with A^*A. It is our purpose here to obtain characterizations analogous to Theorem 0 for C_T^* hyponormal, C_T^* quasinormal, and C_T hyponormal (C_T quasinormal being characterized in [7]).

We will often consider a sigma-algebra $\mathscr{A}$ of subsets of S and the subspace of $L^2(S, \Sigma, \mu)$ consisting of all $\mathscr{A}$-measurable functions. Strictly speaking, this subspace consists of all functions in L^2 which are equal a.e. to an $\mathscr{A}$-measurable function, or, alternately, consists of all equivalence classes each of which contains an $\mathscr{A}$-measurable function. To avoid circumlocutions like these, we will take $\mathscr{A}$ to be *relatively complete* with respect to Σ, that is we will pass to the sigma-algebra generated by $\mathscr{A}$ and $\{F \text{ in } \Sigma : \mu(F) = 0\}$, i.e., $\{A \Delta F: A \text{ in } \mathscr{A},$

F in Σ, $\mu(F)=0\}$, once again calling it $\mathscr{A}$. In keeping with this practice $T^{-1}(\Sigma)$ will denote the relative completion of the sigma-algebra $\{T^{-1}(A): A \text{ in } \Sigma\}$. With this convention, condition i) in Theorem 0 above becomes $T^{-1}(\Sigma)=\Sigma$.

1. LEMMA. *Let P denote the projection of $L^2(S, \Sigma, \mu)$ onto $\overline{R(C_T)}$.*

a)

$$C_T^* C_T f = hf \quad \text{and} \quad C_T C_T^* f = (h \circ T)Pf, \quad \forall f \in L^2. \tag{1}$$

b)

$$\overline{R(C_T)} = \{f \text{ in } L^2 : f \text{ is } T^{-1}(\Sigma) \text{ measurable}\}. \tag{2}$$

c) *If f is $T^{-1}(\Sigma)$ measurable and g and fg belong to L^2, then*

$$P(fg) = fPg, \tag{3}$$

(f need not be in L^2.)

Proof. The first equation in a) is well-known [5]:

$$(C_T^* C_T f, g) = (C_T f, C_T g) = \int hfg \, d\mu = (hf, g).$$

Given f, there are functions $C_T g_n$ in the range of C_T converging to Pf. First, $C_T C_T^* Pf = \lim C_T C_T^* C_T g_n = \lim C_T(hg_n) = h \circ TPf$. Second, $f - Pf$ belongs to $R(C_T)^{\perp} = \ker(C_T^*)$, and so $C_T C_T^* f = C_T C_T^* Pf$.

Given g in $\overline{R(C_T)}$, there are functions g_n from $R(C_T)$ converging to g in L^2, and by passing to a subsequence, converging a.e.. As these g_n are $T^{-1}(\Sigma)$ measurable, so is g since $T^{-1}(\Sigma)$ is relatively complete. Suppose f is the characteristic function of a set A in $T^{-1}(\Sigma)$. By adjusting f on a set of measure zero we may suppose that $A = T^{-1}(B)$ for some B in Σ. Write B as the union $\cup B_n$ of an increasing sequence of sets of finite measure in Σ. Then, $\|C_T \chi_{B_n} - f\|^2 = \mu(T^{-1}(B) - T^{-1}(B_n))$ which converges to zero. It follows that the closure of the range of C_T contains all $T^{-1}(\Sigma)$ measurable functions.

Let f be $T^{-1}(\Sigma)$ measurable and essentially bounded. Then for any l in L^2,

$$(P(fg), l) = (fg, Pl) = (g, \bar{f}Pl) = (Pg, \bar{f}Pl) = (fPg, Pl) = (fPg, l)$$

and (3) holds. Given an arbitrary $T^{-1}(\Sigma)$ measurable f there are f_n simple and $T^{-1}(\Sigma)$ measurable, $|f_n| \leqslant |f|$, and f_n converging to f a.e.. As $|f_n g - fg|^2 \leqslant (2|fg|)^2$, the Lebesgue Dominated Convergence Theorem shows that $f_n g \to fg$ in L^2 and so $P(f_n g)$ converges to $P(fg)$. From above, $P(f_n g) = f_n Pg$ which converges a.e. to fPg.

Q.E.D.

Let $\mathscr{A}$ be a (relatively complete) sigma-subalgebra of Σ and P be the projection of $L^2(S, \Sigma, \mu)$ onto the subspace $L^2(\mathscr{A})$ of $\mathscr{A}$-measurable L^2 functions.

Since $Pf-f$ is orthogonal to $L^2(\mathscr{A})$,

$$\int_A Pf\,d\mu = \int_A f\,d\mu, \tag{4}$$

for all A in $\mathscr{A}$ of finite measure. The equation (4) and the $\mathscr{A}$-measurability of Pf determine Pf completely: for if g is $\mathscr{A}$-measurable and $\int_A g\,d\mu = \int_A f\,d\mu$ for all A in $\mathscr{A}$, $\mu(A)$ finite, then, from (4), $(g-Pf, \chi_A) = 0$ and the $\mathscr{A}$-measurable function $g-Pf$ is orthogonal to $L^2(\mathscr{A})$ and therefore is zero a.e.. Thus, in the case $\mu(S) = 1$, the projection Pf is the conditional expectation $E(f\,|\,\mathscr{A})$ of f given $\mathscr{A}$. Lemma 1 c) is, in the case $\mu(S) = 1$, an important property of conditional expectations [1, p. 300]. Although Lemma 1 c) is stated for $T^{-1}(\Sigma)$, the proof is valid for an arbitrary $\mathscr{A}$.

The projection P is order-preserving, for if $f \geqslant g$, then $\int_A Pf - Pg = \int_A f - g \geqslant 0$ for all A in $\mathscr{A}$, and so $Pf \geqslant Pg$. Consequently, if g is essentially bounded then so is Pg.

Let $\Sigma_{\sigma(h)}$ be the relative completion of the sigma-algebra generated by $\{A \cap$ support of $h : A$ in $\Sigma\}$ where, as always, h is the Radon-Nikodym derivative $d\mu \circ T^{-1}/d\mu$.

2. LEMMA. *The following are equivalent:*

a) $\ker(C_T^*) \subseteq \ker(C_T)$

b) $\Sigma_{\sigma(h)} \subseteq T^{-1}(\Sigma)$.

Proof. Suppose that b) holds but that, contrary to a), there is a function f which belongs to $\ker(C_T^*) = R(C_T)^{\perp}$ but not to $\mathrm{Ker}(C_T)$. Since $0 < \|C_T f\|^2 = \int |f \circ T|^2 d\mu = \int h|f|^2 d\mu$, there is a set A in $\Sigma_{\sigma(h)}$, $0 < \mu(A) < \infty$, with, say, $\mathrm{Re}(f) > 0$ on A. By Lemma 1 b) χ_A belongs to $\overline{R(C_T)}$ and so $0 = (\mathrm{Re}(f), \chi_A) = \int_A \mathrm{Re}(f) d\mu > 0$, a contradiction.

Suppose that a) holds but that, contrary to b), there is a set A of finite measure belonging to $\Sigma_{\sigma(h)}$ but not to $T^{-1}(\Sigma)$. We may assume $A \subset \mathrm{support}(h)$. Because, by Lemma 1, χ_A does not belong to $\overline{R(C_T)}$, there is a function f in $R(C_T)^{\perp}$ with $(f, \chi_A) \neq 0$ which, together with $0 = C_T^* C_T f = hf$, shows that $\mu(A\text{-support}\,h) > 0$, a contradiction. Q.E.D.

2. HYPONORMAL AND QUASINORMAL C_T^*

3. THEOREM. *The adjoint C_T^* is hyponormal iff*

a) $\Sigma_{\sigma(h)} \subseteq T^{-1}(\Sigma)$

and

b) $h \circ T \geqslant h$ a.e. .

Proof. Suppose C_T^* is hyponormal. Then $\ker(C_T^*) \subseteq \ker(C_T)$, and a) follows from Lemma 2. One consequence of a) is that h is $T^{-1}(\Sigma)$ measurable. Therefore the set $A = \{s: h(T(s)) < h(s)\}$ belongs to $T^{-1}(\Sigma)$ and so can be written as a disjoint union of sets A_n of finite measure which also belong to $T^{-1}(\Sigma)$. Since C_T^* is hyponormal,

$$0 \leqslant \|C_T^*\chi_{A_n}\|^2 - \|C_T\chi_{A_n}\|^2 =$$

$$= (h \circ TP\chi_{A_n}, \chi_{A_n}) - (h\chi_{A_n}, \chi_{A_n}) = \int_{A_n} (h \circ T - h)\, d\mu \leqslant 0,$$

which implies that $\mu(A_n) = 0$ for each n and b) holds.

Conversely, suppose that a) and b) hold. Write any f as $f = f_1 + f_2$ with f_1 in $\overline{R(C_T)}$ and f_2 in $R(C_T)^{\perp}$. We have

$$\|C_T^* f\|^2 - \|C_T f\|^2 = (h \circ TPf - hf, f) =$$

$$= (h \circ Tf_1, f_1) + (h \circ Tf_1, f_2) - (hf_1, f_1) - (hf_1, f_2) - (hf_2, f_1) - (hf_2, f_2).$$

Since $h \circ Tf_1$ is $T^{-1}(\Sigma)$ measurable it belongs to $\overline{R(C_T)}$ and $(h \circ Tf_1, f_2) = 0$. Since f_2 belongs to $\ker(C_T)$ by Lemma 2, $hf_2 = C_T^* C_T f_2 = 0$, and $(hf_1, f_2) = (hf_2, f_1) = (hf_2, f_2) = 0$. Thus

$$\|C_T^* f\|^2 - \|C_T f\|^2 = \int (h \circ T - h) \,|f_1|^2\, d\mu \geqslant 0. \qquad \text{Q.E.D.}$$

4. THEOREM. *The adjoint C_T^* is quasinormal iff*

a) $\Sigma_{\sigma(h)} \subseteq T^{-1}(\Sigma)$

and

b) $h = h \circ T$ a.e., *on the support of h.*

Proof. By definition, C_T^* is quasinormal iff $C_T^* C_T C_T^* = C_T C_T^* C_T^*$, i.e., $hC_T^* f = h \circ TP(C_T^* f)$ for all f. Since $\mathrm{Ker}(P) = R(C_T)^{\perp} = \ker(C_T^*)$ this becomes

$$hC_T^* Pf = h \circ TP(C_T^* Pf). \tag{5}$$

Suppose that a) and b) hold. By a) and Lemma 2, for g in $R(C_T)^{\perp}$, $hg = C_T^* C_T g = 0$, and (5) reduces to

$$hP(C_T^* Pf) = h \circ TP(C_T^* Pf) \tag{6}$$

which clearly holds on the support of h by b). It will follow that C_T^* is quasinormal if we can show that $\operatorname{supp}(PC_T^*Pf) \subseteq \operatorname{supp}(h)$. To see this, given f, choose f_n so that $C_T f_n \to Pf$; then $hf_n = C_T^* C_T f_n \to C_T^* Pf$. Now $hPf_n = P(hf_n)$ by a) and Lemma 1 c), and $P(hf_n) = P(C_T^* C_T f_n) \to P(C_T^* Pf)$.

Suppose that C_T^* is quasinormal. By Theorem 3, a) holds and, consequently, h is $T^{-1}(\Sigma)$ measurable; write $\operatorname{supp}(h) = \cup A_n$ with A_n in $T^{-1}(\Sigma)$ and each A_n of finite measure. Set $f = C_T \chi_{A_n}$ in equation (5) to obtain $h^2 \chi_{A_n} = h \circ T h \chi_{A_n}$. Thus $h = h \circ T$ on $\operatorname{supp}(h)$. Q.E.D.

5. Corollary. *The operator C_T is normal iff C_T^* is quasinormal and $h > 0$* a.e. .

Proof. If C_T^* is quasinormal and $h > 0$, then $h = h \circ T$ a.e. and $\Sigma_{\sigma(h)} = \Sigma \subseteq \subseteq T^{-1}(\Sigma)$. Normality follows from Lemma 2 of [7]. Set $Z = \{s : h(s) = 0\}$; then $\mu\{s : h(Ts) = 0\} = \mu T^{-1}(Z) = \int_Z h d\mu = 0$, and we see that $h \circ T > 0$ a.e. . If C_T is normal, then by Lemma 2 in [7] $h = h \circ T$ a.e. and so $h > 0$ a.e. . Q.E.D.

6. Examples. Let S be (0,1), Σ the Borel sets, and μ Lebesgue measure.

a) C_T^* quasinormal, $T^{-1}(\Sigma) \neq \Sigma$, and $h \circ T = h$ only on $\operatorname{supp}(h)$.
Let

$$Tx = (3/4 - x)\chi_{(0,1/4]} + (1/4 + x)\chi_{(1/4,1/2]} + (1/2 + x/2)\chi_{(1/2,1)}.$$

A computation shows that $h = 2\chi_{(1/2,1)}$ and $h \circ T = 2$ on (0,1). Every open subinterval of (1/2,1) has the form $T^{-1}(a, b)$ with $3/4 < a < b < 1$, and therefore belongs to $T^{-1}(\Sigma)$; hence $\Sigma_{\sigma(h)} \subseteq T^{-1}(\Sigma)$. The induced operator C_T^* is thus quasinormal, but, for example, (0,1/4) does not belong to $T^{-1}(\Sigma)$.

b) C_T^* with $h = h \circ T$, $hT^{-1}(\Sigma)$ measurable, but C_T^* not quasinormal.

Let $Tx = 2x\chi_{(0,1/2)} + (2 - 2x)\chi_{[1/2,1)}$. The induced C_T is quasinormal, and therefore hyponormal, but not normal, and the range of C_T is not dense as $T^{-1}(\Sigma) \neq \neq \Sigma$ [7]. Being equal to 1 a.e., h is $T^{-1}(\Sigma)$ measurable and $h \circ T = h$ also holds, yet $\Sigma_{\sigma(h)} = \Sigma$ is not contained in $T^{-1}(\Sigma)$.

In [7] it was shown that, when $\mu(S)$ is finite, C_T normal implies that T is measure preserving. Lemma 7 below shows that this is also true if C_T is hyponormal. However, as shown by Example 8, C_T^* may be hyponormal without T being measure preserving.

7. Lemma. *Let S be of finite measure.*

a) *C_T is hyponormal iff $h = 1$* a.e. .

b) *C_T^* quasinormal implies that $h \circ T \geqslant 1$* a.e. *or, equivalently, $h \geqslant 1$* a.e. *on* $\operatorname{supp}(h)$.

Proof. Suppose C_T is hyponormal. Then χ_S belongs to $\ker(C_T - I) \subseteq$ $\subseteq \ker(C_T^* - I)$ and $h\chi_S = C_T^* C_T \chi_S = \chi_S$. Conversely, if $h = 1$ a.e., then $C_T^* C_T -$ $- C_T C_T^* = I - P \geqslant 0$.

Suppose C_T^* is quasinormal. For $0 < \varepsilon < 1$, set $A = \{s : 0 < h(s) \leqslant 1 - \varepsilon\}$. By Theorem 4 b), $A \subseteq T^{-1}(A)$ and thus $\mu(A) \leqslant \mu(T^{-1}(A)) = \int_A h \mathrm{d}\mu \leqslant (1 - \varepsilon)\mu(A)$.

So $\mu(A) = 0$ and, as ε is arbitrary, $h \geqslant 1$ a.e. on the support of h. The equivalence of this with $h \circ T \geqslant 1$ a.e. follows from

$$\mu\{s : 0 \leqslant h(T(s)) < 1\} = \mu T^{-1}\{s : 0 \leqslant h(s) < 1\} = \int_{\{0 \leqslant h < 1\}} h \mathrm{d}\mu. \qquad \text{Q.E.D.}$$

8. Example. S of finite measure, C_T^* hyponormal, not normal, and h not constant.

Let $S = (0,1)$, Σ the Borel sets, and μ Lebesgue measure. Define $Tx = \log(1 + x)/\log(2)$. Since T is increasing $T^{-1}(\Sigma) = \Sigma$. Thus C_T has dense range and, further, is invertible as $h(x) = 2^x \log(2)$ is bounded below. Since $Tx > x$ and h increases, $h(Tx) > h(x)$. By Theorem 3 C_T^* is hyponormal.

3. HYPONORMAL C_T

9. Theorem. a) *C_T is hyponormal iff*

$$\|h^{1/2} f\| \geqslant \|(h \circ T)^{1/2} Pf\|, \quad \textit{for all } f. \tag{7}$$

Consequently, if $h \geqslant h \circ T$ then C_T is hyponormal.

b) *If C_T is hyponormal, then*

$$\|h^{1/2} Pf\| \geqslant \|(h \circ T)^{1/2} Pf\| \quad \textit{for all } f. \tag{8}$$

c) *Inequality* (8) *holds iff* $\mathrm{d}\mu \circ T^{-2}/\mathrm{d}\mu \geqslant h^2$ a.e..

d) *If C_T is hyponormal, then* $\mathrm{d}\mu \circ T^{-2} \mathrm{d}\mu \circ T^{-1} \geqslant h > 0$ a. e..

Proof. First, $((C_T^* C_T - C_T C_T^*)f, f) = (hf, f) - (h \circ TPf, f)$. Factor to get $(h \circ TPf, f) = ((h \circ T)^{1/2} Pf, (h \circ T)^{1/2} f)$, which by Lemma 1 b) equals $((h \circ T)^{1/2} Pf,$ $P((h \circ T)^{1/2} f))$, which by Lemma 1 c) equals $\|(h \circ T)^{1/2} Pf\|^2$, and a) follows. If $h \geqslant h \circ T$, then

$$\|h^{1/2} f\|^2 = \int h |f|^2 \mathrm{d}\mu \geqslant \int h \circ T |f|^2 \mathrm{d}\mu \geqslant$$

$$\geqslant \|P((h \circ T)^{1/2} f)\|^2 = \|(h \circ T)^{1/2} Pf\|^2.$$

Condition b) is a direct consequence of a).

Suppose that (8) holds. For E a set of finite measure in Σ, let $A = T^{-1}(E)$. Since A is $T^{-1}(\Sigma)$ measurable, $P\chi_A = \chi_A$ and

$$0 \leqslant \|h^{1/2}P\chi_A\|^2 - \|(h \circ T)^{1/2}P\chi_A\|^2 = \int_A (h - h \circ T)\mathrm{d}\mu = \mu T^{-1}(A) -$$

$$-\int (h \circ T)C_T\chi_E \mathrm{d}\mu = \mu T^{-2}(E) - \int_E h^2 \mathrm{d}\mu = \int_E \left(\frac{\mathrm{d}\mu \circ T^{-2}}{\mathrm{d}\mu} - h^2\right)\mathrm{d}\mu$$

and the inequality of c) follows. Conversely suppose that $\mathrm{d}\mu \circ T^{-2}/\mathrm{d}\mu \geqslant h^2$ a.e. . Then for any E in Σ of finite measure the argument above shows that (8) holds for $f = \chi_{T^{-1}(E)}$. Suppose that f is $T^{-1}(\Sigma)$ measurable and simple, $f = \sum a_j \chi_{A_j}$, A_j disjoint sets in $T^{-1}(\Sigma)$. Then

$$\|h^{1/2}Pf\|^2 = \sum \|a_j h^{1/2}\chi_{A_j}\|^2 \geqslant \sum \|a_j (h \circ T)^{1/2}\chi_{A_j}\|^2 = \|(h \circ T)Pf\|^2.$$

Since the $T^{-1}(\Sigma)$ measurable simple functions are dense in $\overline{R(C_T)}$, (8) holds for all f.

Suppose that C_T is hyponormal. We will first show that $h > 0$ a.e.. Let A in $T^{-1}(\Sigma)$ be of finite measure and set $E = A \cap (\mathrm{supp}(h))^c$. Then

$$0 \geqslant ((h \circ T)P\chi_E, \chi_E) - (h\chi_E, \chi_E) =$$

$$= (h \circ TP\chi_E, \chi_E) = \int (h \circ T)\,(P\chi_E)^2 \mathrm{d}\mu,$$

using the fact that $h \circ TP\chi_E$ is $T^{-1}(\Sigma)$ measurable and that $P\chi_E \geqslant 0$ a.e. . So $P\chi_E = 0$ a.e. and $\mu(E) = \mu(E \cap A) = \int_A \chi_E \mathrm{d}\mu = \int_A P\chi_E \mathrm{d}\mu = 0$. Thus $A \subseteq \mathrm{supp}(h)$ a.e. ; hence $T^{-1}(\Sigma) \subseteq \Sigma_{\sigma(h)}$ and $h > 0$ a.e. . By c), $h\mathrm{d}\mu \circ T^{-2}/\mathrm{d}\mu \circ T^{-1} = \mathrm{d}\mu \circ T^{-2}/\mathrm{d}\mu \geqslant h^2$, and we can now divide by h. Q.E.D.

10. Example. The necessary condition of c) and d) above is not sufficient for hyponormality.

Let S be $(0, \infty)$, Σ the Borel sets, and μ Lebesgue measure. Define T by

$$Tx = x\chi_{(0,1)} + (2x - 2)\chi_{[1,3/2)} + (4x - 5)\chi_{[3/2,13/8)} +$$

$$+ (x - 13/8)\chi_{[13/8,21/8)} + (x/5 + 39/40)\chi_{[21/8,\infty)}.$$

Calculations, which are best made from a graph of T, show that

$$h = 5/2\chi_{(0,1)} + 1/4\chi_{[1,3/2)} + 5\chi_{[3/2,\infty)},$$

$$h \circ T = 5/2\chi_{(0,3/2)} + 1/4\chi_{[3/2,13/8)} + 5/2\chi_{[13/8,21/8)} + 5\chi_{[21/8,\infty)},$$

and $d\mu \circ T^{-2}/d\mu \circ T^{-1} = 61/20\chi_{(0,1)} + 5\chi_{[1,\infty)}$. Thus $d\mu \circ T^{-2}/d\mu \circ T^{-1} \geqslant h > 0$ a.e. . A simple but lengthy calculation shows that $\|(h \circ T)^{1/2} P\chi_{[1,3/2)}\|^2 = 1/4$ and $\|h^{1/2}\chi_{[1,3/2)}\|^2 = 1/8$. Therefore, C_T is not hyponormal.

11. Corollary. *If h is $T^{-1}(\Sigma)$ measurable, then C_T is hyponormal iff $h \geqslant h \circ T$ a.e.* .

Proof. Let $N = \{s : h(s) < h(Ts)\}$. We need only show that if C_T is hyponormal and h is $T^{-1}(\Sigma)$ measurable, then $\mu(N) = 0$. Choose A_n of finite measure in $T^{-1}(\Sigma)$ such that $N = \cup A_n$, which is possible as $N \in T^{-1}(\Sigma)$. Then $P\chi_{A_n} = \chi_{A_n}$ and

$$0 \leqslant ((C_T^*C_T - C_TC_T^*)\chi_{A_n}, \chi_{A_n}) = ((h - h \circ T)\chi_{A_n}, \chi_{A_n}) = \int_{A_n} (h - h \circ T)d\mu \leqslant 0.$$

So $\mu(A_n) = 0$ for all n and thus $\mu(N) = 0$. Q.E.D.

12. Example. The inequality $h \geqslant h \circ T$ holds a.e., thus C_T is hyponormal, yet h is not $T^{-1}(\Sigma)$ measurable.

Let $S = (0, \infty)$, Σ the Borel sets, and μ Lebesgue measure. Define

$$Tx = x\chi_{(0,1)} + (2x - 5/4)\chi_{[1,9/8)} + (x/3 + 5/8)\chi_{[9/8,\infty)}.$$

Then

$$h = \chi_{(0,3/4)} + 3/2\chi_{[3/4,1)} + 3\chi_{[1,\infty)},$$

and

$$h \circ T = \chi_{(0,3/4)} + 3/2\chi_{[3/4,9/8)} + 3\chi_{[9/8,\infty)}.$$

Hence $h \geqslant h \circ T$ and C_T is hyponormal. However h is not $T^{-1}(\Sigma)$ measurable since, for example, $h^{-1}\{3/2\} = [3/4,1)$ which does not belong to $T^{-1}(\Sigma)$.

With this example in mind we show that the conditions $h \geqslant h \circ T$ and $hT^{-1}(\Sigma)$ measurable, which are strictly stronger than C_T hyponormal, correspond to a norm condition similar to (7) and (8).

13. Theorem. *The following are equivalent:*

a) $\|P(h^{1/2}f)\| \geqslant \|P((h \circ T)^{1/2}f)\|$ *for all* f.

b) $h \geqslant h \circ T$ a.e. *and h is $T^{-1}(\Sigma)$ measurable.*

Proof. If b) holds, then C_T is hyponormal by Theorem 9 a) and $\|h^{1/2}f\| \geqslant \|(h \circ T)^{1/2}Pf\|$ for all f. Since h is $T^{-1}(\Sigma)$ measurable, $\|P(h^{1/2}f)\| = \|h^{1/2}Pf\| \geqslant \|(h \circ T)^{1/2}Pf\|$ and a) follows.

Suppose that a) holds. By Theorem 9 a) C_T is hyponormal and, by Corollary 11, the proof will be complete once we have shown that h is $T^{-1}(\Sigma)$ measurable. Towards this goal, note that S can be written as an increasing sequence of sets A_n of finite measure belonging to $T^{-1}(\Sigma)$. As $h\chi_{A_n}$ converges to h a.e., it will suffice to show that $h\chi_A$ is $T^{-1}(\Sigma)$ measurable for A in $T^{-1}(\Sigma)$ with $\mu(A)$ finite. Set $k=h^{1/2}\chi_A$, and put $k_1 = Pk$, $k_2 = k - Pk$. With this notation, we want to show that $k_2 = 0$ a.e. . For any f in L^2, set $f_1=Pf$ and $f_2 = f - Pf$. Now $\|P(kf)\| = \|k_1 f_1 + P(k_2 f_2)\|$, using the $T^{-1}(\Sigma)$ measurability of k_1 and Lemma 1 c). Also, since $(h\circ T)^{1/2}\chi_A$ is $T^{-1}(\Sigma)$ measurable, $\|P((h \circ T)^{1/2}\chi_A f)\| = \|(h \circ T)^{1/2}\chi_A f_1\|$. Therefore a) implies that

$$\|k_1 f_1 + P(k_2 f_2)\| \geqslant \|(h \circ T)^{1/2}\chi_A f_1\| \tag{9}$$

for all f. Given g_2 in $\mathrm{R}(C_T)^\perp$, select f by setting $f_1=-P(k_2g_2)$, $f_2=k_1g_2$, and $f=f_1+f_2$. Apply (9) to this f to obtain $\|(h \circ T)^{1/2}\chi_A f_1\| \leqslant 0$. So $0 = (h \circ T)^{1/2}\chi_A f_1 = (h \circ T)^{1/2}\chi_A P(k_2 g_2)$ a.e. for all g_2 in $\mathrm{R}(C_T)^\perp$. Equivalently, as $h \circ T>0$ a.e., $\chi_A P(k_2g_2)=0$ a.e. . But as $k_2 = P^\perp(h^{1/2}\chi_A) = \chi_A P^\perp(h^{1/2}\chi_A) = \chi_A k_2$, so $P(k_2g_2) = P(\chi_A k_2 g_2) = \chi_A P(k_2 g_2)$. Hence $P(k_2g_2) = 0$ a.e., i.e., k_2g_2 is in $\mathrm{R}(C_T)^\perp$ for all g_2 in $\mathrm{R}(C_T)^\perp$. Equivalently, $k_2 g_1$ is in $\overline{\mathrm{R}(C_T)}$ for each g_1 in $\mathrm{R}(C_T)$. Thus $k_2 g_1 = P(k_2 g_1) = g_1 P(k_2) = 0$ for all g_1 in $\mathrm{R}(C_T)$ and consequently $k_2 = 0$ a.e. . Q.E.D.

14. EXAMPLE. The conditions of Theorem 13, while stronger than hyponormality, do not imply subnormality.

Let S be the integers, $\{p_n\}$ a sequence of strictly positive real numbers, and $\mu(n) = p_n$ for all $n = 0, \pm 1, \dots$. Define $T(n) = n - 1$. Then $h(n) = p_{n+1}/p_n$ and C_T is bounded iff the sequence $\{p_{n+1}/p_n\}$ is bounded. The operator C_T is unitarily equivalent to the weighted shift on ℓ^2 with weights $\{(p_{n+1}/p_n)^{1/2}\}$ which is hyponormal iff h is increasing iff $h \geqslant h \circ T$. Note that $T^{-1}(\Sigma) = \Sigma$ so h is $T^{-1}(\Sigma)$ measurable. For the choice $p_1 = 4$, $p_n = 1, n = 2, 3, \dots$, and $p_n = 8^{2(1-n)}$ for $n = 0, -1, \dots$ the induced operator is not subnormal ([3], pp. 241—2 and pp. 311—12), yet h is $T^{-1}(\Sigma)$ measurable and $h \geqslant h \circ T$.

15. LEMMA. *The operator C_T is hyponormal iff for any f, setting $f_1 = Pf$ and $f_2 = f - Pf$,*

$$|(hf_1, f_2)|^2 \leqslant (hf_2, f_2)((h - h \circ T)f_1, f_1). \tag{10}$$

If the essential infimum of h is positive, then

$$\int_A (h \circ T - (h \circ T)^2/h)\,d\mu \geqslant 0 \quad \textit{for all } A \textit{ in } T^{-1}(\Sigma) \tag{11}$$

implies that C_T is hyponormal.

Proof. That C_T is hyponormal is equivalent to $0 \leqslant ((h - h \circ TP)(f_1 - \lambda f_2), (f_1 - \lambda f_2))$ for all f in L^2 and all real λ. Expanding the inner product yields a real quadratic in λ from which we see that this inequality holds iff $(\mathrm{Re}(hf_1, f_2))^2 \leqslant$ $\leqslant (hf_2, f_2)((h - h \circ T)f_1, f_1)$. Given f, choose α a complex number of modulus one so that $(hf_1\alpha, f_2)$ is real and replace f_1 by $f_1\alpha$ to obtain (10).

Suppose that h has a positive essential infimum. By the Cauchy-Schwarz inequality,

$$|(hf_1, f_2)|^2 = \left|\left(\frac{h - h \circ T}{h^{1/2}} f_1, h^{1/2} f_2\right)\right|^2 \leqslant$$

$$\leqslant \left[\int ((h - h \circ T)^2/h)\, |f_1|^2 \mathrm{d}\mu\right] (hf_2, f_2).$$

Using (10), C_T will be hyponormal if

$$\int ((h - h \circ T)^2/h)\, |f_1|^2\, \mathrm{d}\mu \leqslant \int (h - h \circ T)\, |f_1|^2 \mathrm{d}\mu,$$

i.e., if

$$\int (h \circ T - (h \circ T)^2/h)\, |f_1|^2\, \mathrm{d}\mu \geqslant 0$$

for all f in $\mathrm{R}(C_T)$. This is equivalent to (11). Q.E.D.

For the connection of (11) with a conditional expectation, see the discussion following Lemma 1.

16. Example. The condition $h \geqslant h \circ T$ is not necessary for hyponormality. Modify Example 10 by setting

$$Tx = x\chi_{(0,1)} + (2x - 2)\chi_{[1,3/2)} + (x/2 + 1/4)\chi_{[3/2,5/2)} +$$

$$+ (x - 5/2)\chi_{[5/2,7/2)} + (x/5 + 4/5)\chi_{[7/2,\infty)}.$$

Let $f = h \circ T - (h \circ T)^2/h$. For E any Borel set and $A = T^{-1}(E)$,

$$\int_A f = \int_{A \cap [1,3/2)} f + \int_{A \cap [3/2,5/2)} f + \int_{A \cap [5/2,7/2)} f. \tag{12}$$

We have $\mu(A \cap [5/2,7/2)) = 2\mu(A \cap [1,3/2))$, so that the first and last integrals of (12) add to $\mu(A \cap [1,3/2))\,(5/2 - 5/8) \geqslant 0$. Since $f > 0$ on $[3/2, 5/2)$, condition (11) is satisfied and C_T is hyponormal. Yet $h < h \circ T$ on $[1,3/2)$.

An interesting open problem is to find a measure theoretic condition which is both necessary and sufficient for the hyponormality of C_T.

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DAVID J. HARRINGTON
Hughes Aircraft Company,
El Segundo, California,
U.S.A.

ROBERT WHITLEY
Department of Mathematics,
University of California,
Irvine, CA 92717,
U.S.A.

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