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CARLESON MEASURE INEQUALITIES AND KERNEL FUNCTIONS IN $H^2(\mu)$

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Let μ and γ denote finite positive Borel measures with compact support in the complex plane, **C**. Define $H^2(\mu)$ to be the closure in $L^2(\mu)$ of the polynomials in z. A general problem of interest is to characterize (or give sufficient conditions) on measures γ, so that the densely defined operator given by $p \to p$, where p is a polynomial, extends to a bounded operator from $H^2(\mu)$ into $H^2(\gamma)$ for a fixed measure μ. That is, those measures γ should be determined for which

$$(1) \qquad \int |p|^2 \,\mathrm{d}\gamma \leqslant C \int |p|^2 \,\mathrm{d}\mu$$

for all polynomials p and a fixed constant C with $0 < C < \infty$.

This question and its variations have arisen in several areas of function theory. For γ a point mass this inequality has been investigated in [2, 3, 4, 5] when μ is absolutely continuous with respect to area measure, and in [1] when μ is a Jensen measure. In the latter context it is attempted to determine when $H^2(\mu) = L^2(\mu)$ in a manner which gives a strong generalization of a theorem of Szegö [26] or else to investigate smoothness properties of functions in $H^2(\mu)$, analogous to those for functions of classical Hardy space. Inequality (1) arose in [6, 7, 15, 20] as part of the necessary and sufficient conditions for a sequence to be interpolating for H^∞. For this case, μ is Lebesgue measure on the boundary of the unit disc and γ is a sum of atoms located at a countable subset of points of the unit disc and weighted by their distance to the boundary of the unit disc. In [25] inequality (1) was used in determining the multipliers on Dirichlet spaces, where the underlying measures are symmetric of "geometric growth". For these same measures, interpolation questions have been investigated in [18,22]. Other places where the inequality has been exploited occur in [8, 11, 13, 14, 17, 28, 29]. Additional references may be found in [18]. Of course, in many of these works (1) is considered for $H^p(\gamma)$

and $H^q(\mu)$ with $1 \leqslant p, q \leqslant \infty$, where μ is a "symmetric type" measure of "geometric" growth carried by a "disc-like" subset of $\mathbf{C}^n$. We will be concerned with the Hilbert space setting and planar measures.

Let $d\sigma$ denote normalized Lebesgue measure on the boundary of the unit disc, ∂D. We say that μ is a (radially) *symmetric* measure if $d\mu = d\omega \times d\sigma$ where ω is a measure carried by the closed interval [0,1] with support containing $\{1\}$. When ω is a point mass at 1, Carleson [7] solved (1) as follows:

CARLESON'S THEOREM [7]. *For $\varphi \in [0,2\pi)$ and $0 < h \leqslant 1$, let $S_h(e^{i\varphi})$ denote the set of z's in $\bar{D}$ with $1 - h \leqslant |z| \leqslant 1$ and $\varphi - \frac{h}{2} \leqslant \arg z \leqslant \varphi + \frac{h}{2}$ for some argument of z. Let γ be carried by D. Then there exists a C with $0 < C < \infty$ so that*

$$\int_D |p|^2 \, d\gamma \leqslant C \int_{\partial D} |p|^2 \, d\sigma$$

for all polynomials p if and only if there exists a C′ with $0 < C' < \infty$ satisfying

$$\mu(S_h(e^{i\varphi})) \leqslant C'\sigma(S_h(e^{i\varphi})) \tag{2}$$

for all "(Carleson) windows", $S_h(e^{i\varphi})$.

Motivated by Carleson's theorem, Hastings [14] proved a similar theorem with σ replaced by area measure on D. Other variations of (1) have been proven where μ is a symmetric measure of "geometric growth", that is $d\mu = (1-r)^\alpha dr \times d\sigma$ where $\alpha > -1$, [17,25]. For all these cases the appropriate "window" conditions (2) characterize when inequality (1) holds.

We begin with two examples. The first example shows that the usual "window" conditions fail to suffice for (1) whenever μ is a symmetric measure of exponential decay. In the second example, μ is carried by ∂D and not necessarily symmetric. It is shown that the window condition is not in general sufficient for (1). For measures carried by ∂D necessary and sufficient conditions for (1) are obtained. This second example combined with the approach of Shapiro-Shields [23] and a method of Vinogradov (see [21]) when $\mu = \sigma$ provides the motivation for using an appropriate kernel function hypothesis (see Theorem 2) as a possible replacement for the "window" conditions. We have been unable to show that the kernel function hypothesis characterizes (1) for general symmetric measures. However, this method provides a relatively easy and uniform approach for μ equal to σ and for μ a measure of geometric growth, whereas previous approaches handle the two cases separately and directly involve window conditions. Moreover the method actually shows that a "kernel function extension" from $L^2(\mu)$ embeds continuously in $L^2(\gamma)$.

1. EXAMPLES

EXAMPLE 1. Let $d\mu = d\mu_1 \times d\sigma$ be a symmetric measure of exponential decay. By this we mean that $\frac{1}{(1-r)^n} \in L^2(d\mu_1)$ for n=1,2, ... (For example, take $d\mu_1(r) = e^{-\frac{1}{1-r}}\,dr$.) Define the measure γ supported on [0,1] by $d\gamma(r) = (1-r)d\mu_1(r)$. Let S_h denote a window in D. Then

$$\gamma(S_h) \leqslant \int_{1-h}^{1} (1-x)\,d\mu_1(x) \leqslant h \int_{1-h}^{1} d\mu_1(x).$$

Also $\mu(S_h) = h \int_{1-h}^{1} d\mu_1(x)$. Thus γ satisfies the Carleson condition (2) with respect to μ. Now

$$\left\| \frac{1}{(1-z)^n} \right\|_{\gamma}^{2} = \int_0^1 \left(\frac{1}{1-x}\right)^{2n} (1-x)\,d\mu_1 = \int_0^1 \left(\frac{1}{1-x}\right)^{2n-1} d\mu_1(x).$$

But

$$\left\| \frac{1}{(1-z)^n} \right\|_{\mu}^{2} = \int_0^1 \int_{-\pi}^{\pi} \left| \frac{1}{(1-z)^2} \right|^n d\theta\,d\mu_1(r) \leqslant$$

$$\leqslant \left(\int_0^1 \left(\frac{1}{1-r}\right)^{2n-1} d\mu_1(r)\right)\left(C\int_0^\infty \frac{1}{(1+t^2)^n}\,dt\right)$$

by estimates in [11, p. 66], where C is independent of n. Since $\int_0^\infty \frac{1}{(1+t^2)^n}\,dt \to 0$ as $n \to \infty$, it is clear that (1) cannot hold.

It would be of interest to know a characterization of (1) for μ as above. Note that the above example does not rule out the possibility that the appropriate windows, S_h, to be considered should have the ratio of length and width vary as a function of h.

EXAMPLE 2. We begin with a lemma which is required only for the case when both exponents equal 2. In this case it follows from a Putnam-Fuglede type theorem of [24] or from [10]. Since a corollary of the general lemma might be useful in determining when $H^2(\mu) \neq L^2(\mu)$ we include a proof.

Let $0 < m < \infty$. When E is a Borel set we say that $L^m(\mu|E) \subset H^m(\mu)$, if every function in $L^m(\mu|E)$ is the restriction μ-a. e. on E of a function in $H^m(\mu)$ which is 0 μ-a. e. on E^c.

LEMMA 1. *Suppose that* $L^m(\mu|E) \subset H^m(\mu)$ *for some Borel set E and*

$$\int_C |p|^m \, d\gamma \leqslant C \int_C |p|^m \, d\mu \quad \text{for } 0 < C < \infty \tag{3}$$

and all polynomials p. Then

(i) $\int_{C \setminus E} |p|^m \, d\gamma \leqslant C \int_{C \setminus E} |p|^m \, d\mu,$

(ii) $\int_E |p|^m \, d\gamma \leqslant C \int_E |p|^m \, d\mu,$

(iii) $L^m(\gamma|E) = H^m(\gamma|E)$ *and* $L^m(\gamma|E) \subset H^m(\gamma)$,

and

(iv) $\gamma|E \ll \mu|E$ *and* $\frac{d\gamma}{d\mu}|E \in L^\infty(\mu)$.

We assume $m \geqslant 1$. A modification of the proof takes care of the case $0 < m < 1$. Let α denote the measure $\gamma + \mu$.

Proof. Let F be a Borel subset of E and choose polynomials p_n so that $\|p_n - \chi_F\|_{\mu, m} < \frac{1}{n}$ in $L^m(\mu)$. Then by (3) $p_n \to f$ in $L^m(d\alpha)$ for some $f \in L^m(d\alpha)$. Now $\|p_n p_m - \chi_F\|_{\mu, m} \leqslant \|p_n\|_\infty \|p_m - \chi_F\|_{\mu, m} + \frac{1}{n}$. Choose m so large that $\|p_m - \chi_F\|_{\mu, m} < \frac{1}{n\|p_n\|_\infty}$. If we denote this m by m_n we have that $p_n p_{m_n} \to \chi_F$ in $L^m(\mu)$, so by (3)

$$p_n p_{m_n} \to g \quad \text{for some } g \text{ in } L^m(\alpha).$$

But

$$\operatorname{Lim}_{n \to \infty} \|p_n - p_n p_{m_n}\|_{\mu, m} = 0$$

so

$$\operatorname{Lim}_{n \to \infty} \|p_n - p_n p_{m_n}\|_{\alpha, m} = 0$$

and thus $g = f$. By choosing subsequences converging α-a. e. we see that $p_n p_{m_n}$ converges to f^2 pointwise α-a. e.. Thus $f^2 = f$.

Let T denote the linear operator from $H^2(\mu)$ into $H^2(\alpha)$ fixing polinomials, which by equality (3) is bounded. For G a Borel subset of E, the previous computation shows that $T(\chi_G) = \chi_{G'}$ in $H^m(\alpha)$. Denote G' by TG.

Choose Borel subsets E_{jn} of E so that (as $n \to \infty$)

$$\sum_i a_{in}\chi_{E_{jn}} \to \chi_E z \text{ in } H^m(\mu).$$

Then

$$\sum_i a_{in}\chi_{TE_{jn}} \to T(\chi_E z) = z\chi_{TE}$$

and $\sum_i \bar{a}_{in}\chi_{TE_{jn}} \to \bar{z}\chi_{TE}$. Now $H^\infty(\alpha) = H^m(\alpha) \cap L^\infty(\gamma)$ is an algebra and contains $z\chi_{TE}$ and $\bar{z}\chi_{TE}$. Thus by the Stone-Weierstrass theorem $L^m(\alpha|TE) \subset H^m(\alpha)$.

From (3)

$$\int |p|^m d\mu \leqslant \int |p|^m d\alpha \leqslant (C+1)\int |p|^m d\mu$$

for all polynomials p. Approximate F, a Borel subset of E, in $L^m(\alpha)$ by polynomials. Thus from the previous inequality

$$\mu F \leqslant \alpha(TF) \leqslant (C+1)\mu F$$

and $TF = F$ μ-a.e. . Then $\gamma \circ T \ll \mu$, so $\gamma \ll \mu \circ T^{-1} = \mu$ and (iv) holds. Now (i), (ii), and (iii) follow easily. ▨

This theorem leads to

COROLLARY 1. *If there exists a measure γ so that γ is not absolutely continuous with respect to μ but*

$$\int |p|^2 \, d\gamma \leqslant C\int |p|^2 \, d\mu \tag{4}$$

for some $0 < C < \infty$ and all polynomials p, then $H^2(\mu) \neq L^2(\mu)$.

If γ has an atom at some point, (4) is just the requirement for the existence of a bounded point evaluation for $H^2(\gamma)$.

The following theorem, characterizing (1) for μ carried by ∂D, will be established by appealing to Lemma 1, Carleson's Theorem, and Hardy space considerations [16]. See Clary [9] for related results.

THEOREM 1. *Let μ be carried by ∂D and let γ be carried by $\bar{D}$. Then there exists a constant C with $0 < C < \infty$ and*

$$\int_{\bar{D}} |p|^2 \, d\gamma \leqslant C \int_{\partial\bar{D}} |p|^2 \, d\mu \tag{5}$$

for all polynomials, if and only if either

$$\log\frac{d\mu}{d\sigma} \notin L^1(d\sigma), \quad \gamma \ll \mu, \quad \textit{and } \frac{d\gamma}{d\mu} \in L^\infty(\mu) \tag{6}$$

or else

$$(7)\qquad \log\frac{d\mu}{d\sigma}\in L^1(d\sigma),\quad \gamma|\partial D\ll\mu,\quad \frac{d\gamma}{d\mu}\Big|\partial D\in L^\infty(\mu)$$

and for $F(z)=\exp\left(\frac{1}{2}\int_{\partial D}\frac{e^{it}+z}{e^{it}-z}\log\frac{d\mu}{d\sigma}\,d\sigma\right)$ *for* $z\in D$, *then* $\frac{1}{|F|^2}\,d\gamma$ *satisfies a "window" condition* (2) *with respect to* σ.

Proof. First assume $\log\frac{d\mu}{d\sigma}\notin L_1(d\sigma)$. By a theorem of Szegö [26] $H^2(\mu)=L^2(\mu)$. Thus if (5) holds, (6) follows from Lemma 1. Conversely, (6) obviously implies (5).

Next, assume that $\log\frac{d\mu}{d\sigma}\in L^1(d\sigma)$. Let $d\mu=\frac{d\mu}{d\sigma}d\sigma+d\mu_s$ be the Lebesgue decomposition of μ with respect to $d\sigma$ on ∂D. Using [9, 10, 16, or 27] it can be shown that $L^2(d\mu_s)\subset H^2(d\mu)$. Similarly write $d\gamma=d\gamma|D+\frac{d\gamma}{d\sigma}d\sigma+d\gamma_s$.

If (5) holds, then by Lemma 1 (iv) $\gamma_s\ll\mu_s$. Applying (5) to z^np and letting $n\to\infty$ gives

$$\int_{\partial D}|p|^2\,d\gamma\leqslant C\int_{\partial D}|p|^2\,d\mu.$$

Setting $p(e^{it})=\frac{1-|\alpha|^2}{(1-\bar{\alpha}e^{it})}$ and using Fatou's theorem (see [16, p. 34]) we get $\frac{d\gamma}{d\sigma}\leqslant\frac{d\mu}{d\sigma}$. Thus $\gamma|\partial D\ll\mu$ and $\frac{d\gamma|\partial D}{d\mu}\in L^\infty(d\mu)$. From Lemma 1(i) and (5) we have

$$(8)\qquad \int_D|p|^2\,d\gamma\leqslant C\int_{\partial D}|p|^2\frac{d\mu}{d\sigma}\,d\sigma=C\int_{\partial D}|pF|^2\,d\sigma.$$

Since F is outer, $\{pF:p \text{ a polynomial}\}$ is dense in $H^2(d\sigma)$. Hence for any polynomial q, there is a sequence of polynomials p_n with $p_nF\to q$ in $H^2(d\sigma)$. By Szegö's theorem $p_nF\to q$ uniformly on compact subsets of D. Hence $p_n\to\frac{q}{F}$ uniformly on compacta in D so by (8),

$$(9)\qquad \int_D\frac{|q|^2}{|F|^2}\,d\gamma\leqslant C\int_{\partial D}|q|^2\,d\sigma.$$

Applying Carleson's theorem to (9), we have (7). (5) follows from (7) by reversing some of the previous arguments. ▨

The main point to notice is that $\frac{1}{|F|^2}\,d\gamma$ satisfying a "window" condition with respect to σ is not in general equivalent to γ satisfying a "window" condition with respect to $\mu = |F|^2\,d\sigma = \omega d\sigma$. So the usual window condition does not characterize (5).

For a specific example with an outer function F let

$$\omega(e^{i\theta}) = \begin{cases} 1 & \theta \in [0, \pi) \\ -\theta & \theta \in [-\pi, 0). \end{cases}$$

Let $d\gamma$ be arc length measure for a "small" line segment, ℓ, of negative slope containing 1 and contained in $D \cup \{1\}$. Define

$$F(z) = \exp\left(\frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log\omega(e^{it})\,d\sigma(t)\right) \quad \text{for } z \in D.$$

By Fatou's theorem $|F(e^{i\theta})|^2 = \omega(e^{i\theta})$ σ-a. e.. It is easy to see that for S_h a Carleson window $\gamma(S_h) \leqslant Ch$ and, if $1 \in S_h$, and S_h is symmetric with respect to the real axis, that $\gamma(S_h) \geqslant dh$, where C and d are constants independent of h. If $1 \in S_h$,

$$\int_{S_h \cap \partial D} |F|^2\,d\sigma \geqslant \int_{S_h \cap [0,\pi]} 1\,d\sigma.$$

The last term is the length of the radial projection of S_h onto ∂D in the upper half-plane. But this is of the same order as the length of $S_h \cap \ell$. Thus for some, constant N independent of h,

$$\gamma(S_h) \leqslant N \int_{S_h \cap \partial D} |F|^2\,d\sigma.$$

On the other hand, by routine estimates on the Poisson kernel (see [11, p. 66]) using the fact that for $z = re^{i\theta}$ on ℓ, $\frac{\theta}{1-r}$ is bounded, we get

$$\frac{1}{|F(re^{i\theta})|^2} \geqslant \exp\left(\frac{L}{1-r}\int_0^{1-r} -\log t\,d\sigma(t) \geqslant M\exp(-L\log(1-r)\right) \geqslant M\left(\frac{1}{1-r}\right)^L$$

for $re^{i\theta}$ on ℓ, r close to 1 and L and M positive constants independent of r close to 1.

Thus if $1 \in S_h$ and S_h is symmetric with respect to the real axis, then the above calculation combined with $\gamma(S_h) \geqslant dh$ shows that $\frac{1}{|F|^2}\,d\gamma$ is not a Carleson measure.

For this second example, we had to modify the measure and then apply a window condition. In the next section we give a method which explains this example and measures of geometric growth.

2. REPRODUCING KERNEL

To establish the necessity of the "window" condition (2) for inequality (1) in Carleson's theorem, only the fact that (1) holds for functions of the form $\frac{1}{1-\bar{\alpha}z}$ *for $\alpha \in D$ is needed. But a simple geometric calculation (see [13, p. 239]) shows that* (1) holding for $\frac{1}{1-\bar{\alpha}z}$ with $\alpha \in D$ implies the window condition. Thus Carleson's Theorem has the following formulation: For all $\alpha \in D$

$$\int_D \frac{1}{|1-\bar{\alpha}z|^2}\, d\mu \leqslant C \int_{\partial D} \frac{d\sigma}{|1-\bar{\alpha}z|^2}$$

if and only if for all polynomials

$$\int_D |p|^2\, d\mu \leqslant C' \int_{\partial D} |p|^2 d\sigma.$$

Here C and C' denote fixed constants. This formulation of Carleson's theorem has been independently observed by several people, but seems to be due to Vinogradov (see [21, p. 83]).

From now on μ will denote a symmetric measure on the disc. We use the symbols $\|\cdot\|_\mu$ and $\|\cdot\|_{\mu,1}$ for the norm in $L^2(\mu)$ and $L^1(\mu)$, respectively. From the symmetry of μ we see that $\left\{\frac{z^n}{(\mu_n)^{1/2}}\right\}_{n=0}^{\infty}$, where $\mu_n = \|z^n\|_\mu^2$, is a basis of orthogonal polynomials for $H^2(\mu)$. Let $k_\omega^\mu(z) = \sum_{n=0}^{\infty} \frac{z^n\bar{\omega}^n}{\mu_n}$ for $\omega \in D$ and z in the support of μ. Then is easy to check that k_ω^μ and all derivatives and powers of k are bounded analytic functions in D. Because of the reproducing property, $p(\omega) = \langle p, k_\omega^\mu\rangle_\mu$ for all polynomials p, k_ω^μ is called a *reproducing kernel* (function) or r.k.. By calculating we see that if $\omega \in D$ and $z \in \bar{D}$, then

$$(10) \qquad \frac{1}{(1-z\bar{\omega})^{s+1}} \text{ is a r.k. for } \begin{cases} 2sr(1-r^2)^{s-1}dr\, d\sigma & \text{if } s > 0 \\ d\sigma & \text{if } s = 0. \end{cases}$$

In the case of Example 2, the r.k. for $\omega\, d\sigma$ is $[(1-z\bar{\omega})F(z)\overline{F(\omega)}]^{-1}$. Note that $\{k_\alpha^\mu : \alpha \in D\}$ is a total set in $H^2(\mu)$, i.e. its closed span is dense. But $\{k_\alpha^\mu : \alpha \in D\}$

itself is never dense in $H^2(\mu)$, since the function z is never in its closure, else $1 = \langle 1, k^\mu_{\alpha_n}\rangle_\mu \to \langle 1, z\rangle_\mu = 0$.

For a measure μ of geometric growth (which is equivalent to the measures in (10)), a "window" condition is equivalent to inequality (1) holding for the r. k. in $H^2(\mu)$ by an argument similar to one used in Garnett [13, p. 239] and we have the following:

THEOREM 2. *Let μ denote Lebesgue measure, a symmetric measure of geometric growth, or a nonsymmetric measure as in Example 2. Then*

$$\|k^\mu_\omega\|_\gamma \leqslant M\|k^\mu_\omega\|_\mu \tag{11}$$

for some $0 < M < \infty$ and all $\omega \in D$ if and only if

$$\|p\|_\gamma \leqslant C' \|p\|_\mu \tag{12}$$

for some $0 < C' < \infty$ and all polynomials p.

For the measure of Example 2, (11) holding for the measures γ and $|F|^2 d\sigma$ is equivalent to (11) holding for the measures $\frac{1}{|F|^2}d\gamma$ and $d\sigma$. So we may assume that μ is symmetric.

We must show that (11) $\Rightarrow$ (12). If k^μ_ω is a r. k. then, since

$$\langle p, |k^\mu_\omega|^2/\|k^\mu_\omega\|^2_\mu\rangle_\mu = \langle pk^\mu_\omega, k^\mu_\omega/\|k^\mu_\omega\|^2_\mu\rangle_\mu = p(\omega),$$

$(|k^\mu_\omega|^2/\|k^\mu_\omega\|^2_\mu)d\mu$ is a representing measure for polynomials. Denote this kernel by P_ω. (When $\mu = \sigma$ this is just the Poisson kernel.) Notice that (11) is just the hypothesis that $\sup\{\|P_\omega\|_{\gamma,1} : \omega \in D\} < \infty$. Let T denote the operator defined (initially) on a dense subset of $L^2(\mu)$ by $Tf(\omega) = \langle f, P_\omega\rangle_\mu$, $\omega \in D$. Formally the densely defined adjoint of T from $L^2(\gamma)$ into $L^2(\mu)$ is given by $T^*g(z) = \langle g(x), P_x(z)\rangle_\gamma$, $z \in$ support of μ. We need only show that T^* is bounded. The following lemma is duet to Vinogradov (see [21, p. 83]), when $\mu = \sigma$.

LEMMA 2. *Let μ denote Lebesgue measure or a symmetric measure of geometric growth. There exists a $C < \infty$ so that for all $z \in \overline{D}$ and $x, t \in D$ we have*

$$P_x(z)P_t(z) \leqslant C[P_t(x) + P_x(t)]\,[P_x(z) + P_t(z)]. \tag{13}$$

Proof. An elementary calculation for $x, t \in D$ and $z \in \overline{D}$ gives $|1 - \bar{x}t| \leqslant |1 - \bar{x}z| + |1 - \bar{t}z|$ so

$$\frac{1}{|1-\bar{x}z|}\,\frac{1}{|1-\bar{t}z|} \leqslant \frac{1}{|1-\bar{x}t|}\left[\frac{1}{|1-\bar{x}z|} + \frac{1}{|1-\bar{t}z|}\right]. \tag{14}$$

Now for $s \geqslant 0$ we have $(|a| + |b|)^{2s+2} \leqslant 2^{2s+1}(|a|^{2s+2} + |b|^{2s+2})$. Applying this to (14) and using the fact that for some $s \geqslant 0$

$$P_x(z) = \frac{(1 - |x|^2)^{s+1}}{|1 - \bar{x}z|^{2s+2}} \quad \text{(for the cases under consideration),}$$

we get

$$P_x(z)P_t(z) \leqslant 2^{2s+1}P_t(x)\ [P_x(z) + P_t(z)] \quad \text{for } |t| \leqslant |x| \text{ in } D.$$

(13) follows from this. ▨

Integrating (13) with respect to μ or γ and applying (11) we get.

LEMMA 3. *Let μ be as in Lemma 2. Suppose* (11) *holds with constant M and* (13) *holds with constant C.*

$$\int_{\bar{D}} P_x(z)P_t(z)\mathrm{d}\mu(z) \leqslant 2C[P_t(x) + P_x(t)] \tag{15}$$

and

$$\int_{D} P_x(z)P_t(z)\mathrm{d}\gamma(z) \leqslant 2CM^2[P_t(x) + P_x(t)]. \tag{16}$$

Proof of Theorem 2.

$$\int_{\bar{D}} |T^*g(z)|^2\,\mathrm{d}\mu(z) = \int_D\int_D g(s)\overline{g(t)} \int_{\bar{D}} P_s(z)P_t(z)\,\mathrm{d}\mu(z)\,\mathrm{d}\gamma(s)\,\mathrm{d}\gamma(t) \leqslant$$

$$\leqslant 4C \int_D\int_D |g(s)g(t)|P_s(t)\,\mathrm{d}\gamma(s)\,\mathrm{d}\gamma(t) \leqslant \qquad \text{by (15)}$$

$$\leqslant 4C\|g\|_\gamma \left[\int_D\int_D |g(s)g(u)| \int_D P_s(t)P_u(t)\,\mathrm{d}\gamma(t)\,\mathrm{d}\gamma(s)\,\mathrm{d}\gamma(u)\right]^{1/2} \leqslant$$

$$\leqslant C_1\|g\|_\gamma \left[\int_D\int_D |g(s)|\,|g(u)|P_s(u)\,\mathrm{d}\gamma(s)\,\mathrm{d}\gamma(u)\right]^{1/2} \qquad \text{by (16)}$$

where C_1 is an absolute constant. Let R denote the quantity in brackets. Then we have shown that $\|T^*g\|_\mu^2 \leqslant 4CR \leqslant C_1\|g\|_\gamma R^{1/2}$. Now for g bounded, $R \leqslant \leqslant \|g\|_\infty^2 M\gamma(D) < \infty$. Thus we conclude that $R^{1/2} \leqslant C_2\|g\|_\gamma$ for C_2 independent of g and so T^* is bounded. ▨

Let S denote the restriction of T to $H^2(\mu)$. It might be of interest to determine when S is a compact mapping. We use the notation from the proof of Theorem 2.

THEOREM 3. *Let μ denote Lebesgue measure, a symmetric measure of geometric growth, or a nonsymmetric measure as in Example 2. Then S is compact if and only if* $\operatorname{Lim}_{|\alpha|\to 1} \|P_\alpha^\mu\|_{\gamma,1} = 0$.

Proof. We may assume that μ is symmetric. Any weakly convergent subsequence of the unit vectors $\left\{\frac{k_\alpha^\mu}{\|k_\alpha^\mu\|_\mu}\right\}$ converges weakly to 0 in $H^2(\mu)$ as $|\alpha| \to 1$. If $\overline{\operatorname{Lim}}_{n\to\infty} \|P_{\alpha_n}^\mu\|_{\gamma,1} > 0$ for a subsequence $|\alpha_n| \to 1$, then $\overline{\operatorname{Lim}}_{n\to\infty} \frac{\|S(k_{\alpha_n}^\mu)\|_\gamma}{\|k_{\alpha_n}^\mu\|_\mu} > 0$. Thus S is not compact.

Conversely, let $\operatorname{Lim}_{|\alpha|\to 1} \|P_\alpha^\mu\|_{\gamma,1} = 0$. For $E \subset D$, let S_E denote the linear operator from $H^2(\mu)$ into $L^2(\gamma)$, defined on polynomials by $p \mapsto pX_E$. For $\{p_n\}_0^\infty$ an orthonormal basis in $H^2(\mu)$, we have

$$\|S_E\|^2 \leqslant \|S_E\|_2^2 = \sum_{n=0}^{\infty} \|S_E p_n\|_\gamma^2 = \sum_{n=0}^{\infty} \int_E |p_n(z)|^2 \, d\gamma = \int_E \|k_z^\mu\|_\mu^2 \, d\gamma.$$

Thus for $E = D_{|\alpha|}$, S_E is Hilbert-Schmidt and hence compact. But the mapping S_{E^c} is the restriction of T_{E^c} to $H^2(\mu)$, where $T_{E^c} : L^2(\mu) \to L^2(\gamma|E^c)$ sends $f \to \langle f, k_\omega^\mu\rangle_\mu$. A close inspection of the proof of Theorem 2 shows that

$$\|S_{E^c}\| \leqslant \|T_{E^c}\| \leqslant 4C \sup\left\{\frac{\|k_\omega^\mu\|_\gamma}{\|k_\omega^\mu\|_\mu} : \omega \in E^c\right\} = 4C \sup\{\|P_\omega\|_{\mu,1} : \omega \in E^c\}$$

where C denotes an absolute geometric constant. Thus by hypothesis as $|\alpha| \to 1$, $\|S_{E^c}\| \to 0$, so $S = S_E + S_{E^c}$ can be uniformly approximated by compact operators. Hence S is compact. ▨

Combining Theorem 3 with the estimates in Garnett [13, p. 239] gives the following window version, due to McDonald and Sundberg [19] for the case when μ is area Lebesgue measure on D.

COROLLARY. *For μ a symmetric measure of geometric growth or Lebesgue measure, the mapping from $H^2(\mu)$ into $H^2(\gamma)$ sending $p \mapsto p$, for polynomials p, is compact if and only if* $\operatorname{Lim}_{h\to 0} \frac{\gamma(S_h(e^{i\varphi}))}{\mu(S_h(e^{i\varphi}))} = 0$, *independently of φ.*

REMARKS. (a) We point out that for Lemma 2 to hold for μ a symmetric measure, it is necessary that $k_r^\mu(r) \leqslant C k_{r^2}^\mu(r^2)$ for C independent of r. For the measure $d\mu = (1-r)^{-3/2} e^{-\frac{1}{1-r}} dr d\sigma$ we have $k_r(r) \geqslant C e^{\frac{D}{(1-r)}}$, so Lemma 2 fails in this case. The computations are based on estimates from [12].

Lemma 3 may also fail for measures as in Example 2, although Theorem 2 is valid in this case.

(b) For Lebesgue measure and symmetric measures of geometric growth the window condition is actually equivalent to (1) holding not only for $\{k_\alpha^\mu : \alpha \in D\}$, but for all derivatives with respect to α or z of the r.k. as well. Again the closure of $\{\partial_z^n \partial_\alpha^m k_\alpha^\mu : \alpha \in D\}$ in $H^2(\mu)$ does not contain z. Thus for general symmetric measures, (11) might need to be replaced by $\|\partial_z^n \partial_\alpha^m k_\alpha^\mu\|_\gamma \leqslant C \|\partial_z^n \partial_\alpha^m k_\alpha^\mu\|_\mu$ for all $\alpha \in D$ and $n = 0,1, \ldots$.

(c) It is easy to verify that if γ and μ are both symmetric measures, then

$$\|\partial_\alpha^m k_\alpha^\mu\|_\gamma \leqslant C \|\partial_\alpha^m k_\alpha^\mu\|_\mu \quad \text{for } m = 0,1, \ldots, \alpha \in D$$

if and only if

$$\|p\|_\gamma \leqslant C' \|p\|_\mu \quad \text{for all polynomials } p,$$

where C and C' are constants.

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