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SINGLY GENERATED HYPONORMAL C^*-ALGEBRAS

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1.

Let H be a separable, infinite dimensional Hilbert space and T a completely hyponormal operator on H. Thus, if T has the Cartesian representation $T = A + \mathrm{i}B$, then

(1.1) $$T^*T - TT^* = 2\mathrm{i}(AB - BA) = D \geqslant 0,$$

and T has no nontrivial reducing subspace on which it is normal. It is known that

(1.2) $$\pi\|D\| \leqslant \mathrm{meas}_2\,\sigma(T);$$

see [15], and, in the special case where D is compact, [2]. Let $A = \mathrm{Re}(T)$ have the spectral resolution

(1.3) $$A = \int t\,\mathrm{d}E_t,$$

and let Δ be any open interval of the real axis for which $E(\Delta) \neq 0$. Then $T_\Delta = E(\Delta)TE(\Delta)$, regarded as an operator on $E(\Delta)H$ with spectrum $\sigma(T_\Delta)$, is also completely hyponormal (on $E(\Delta)H$) and

(1.4) $$\sigma(T_\Delta) \subset \sigma(T).$$

See [15] and, in case T is compact, [2]. Moreover,

(1.5) $$\sigma(T_\Delta) \cap \{z : \mathrm{Re}(z) \in \Delta\} = \sigma(T) \cap \{z : \mathrm{Re}(z) \in \Delta\};$$

see [16]. (For a recent survey of hyponormal operators including, in particular, a discussion of the "cut down" operators T_Δ, see [5].)

It follows from (1.4) that if β is any Borel subset of the real line and if $F(t)$ denotes the Lebesgue linear measure of the vertical cross section $\sigma(T) \cap \{z : \mathrm{Re}(z) = t\}$

then

$$\pi\|E(\beta)DE(\beta)\| \leqslant \int_\beta F(t)\mathrm{d}t. \tag{1.6}$$

This paper will be concerned mainly with completely hyponormal operators for which equality holds in (1.2), that is,

$$\pi\|D\| = \mathrm{meas}_2\,\sigma(T). \tag{1.7}$$

In this case, equality also holds in (1.6). For, otherwise, let α be a Borel set of the real line satisfying $\pi\|E(\alpha)DE(\alpha)\| \equiv a < \int_\alpha F(t)\,\mathrm{d}t$, so that $\pi^{1/2}\|D^{1/2}E(\alpha)\| = a^{1/2} < \left(\int_\alpha F(t)\mathrm{d}t\right)^{1/2}$. If $\gamma = (-\infty, \infty)\setminus\alpha$, then for any x in H, and in view of (1.6) with $\beta = \gamma$,

$$\pi^{1/2}\|D^{1/2}x\| \leqslant \pi^{1/2}(\|D^{1/2}E(\alpha)x\| + \|D^{1/2}E(\gamma)x\|) \leqslant$$

$$\leqslant a^{1/2}\|E(\alpha)x\| + \left(\int_\gamma F(t)\mathrm{d}t\right)^{1/2}\|E(\gamma)x\|.$$

Since $\|x\|^2 = \|E(\alpha)x\|^2 + \|E(\gamma)x\|^2$, and x is arbitrary, an application of the Schwarz inequality yields

$$\pi\|D\| \leqslant a + \int_\gamma F(t)\,\mathrm{d}t < \int_\alpha F(t)\,\mathrm{d}t + \int_\gamma F(t)\,\mathrm{d}t = \mathrm{meas}_2\,\sigma(T),$$

in contradiction with (1.7).

If T is completely hyponormal then $A = \mathrm{Re}(T)$ and $B = \mathrm{Im}(T)$ are absolutely continuous and $\sigma(A)$ and $\sigma(B)$ are the (real) projections on the real and imaginary axes of $\sigma(T)$; see [14], pp. 42, 46. If A has the spectral resolution (1.3) then $\|E_t x\|^2$ is an absolutely continuous function of t for every x in H. Also, if $g(t)$ is a real-valued function in $L^\infty(\sigma(A))$ then $g(A)$ is defined and is a bounded selfadjoint operator. In view of (1.6), one has

$$\pi\|g(A)Dg(A)\| \leqslant \int g^2(t)F(t)\,\mathrm{d}t, \quad g \text{ real and in } L^\infty(\sigma(A)). \tag{1.8}$$

In fact, for any unit vector x, it follows from (1.6) that $\pi\|E(\beta)D^{1/2}x\|^2 \leqslant$

$\leqslant \int_\beta F(t)\mathrm{d}t$ for any Borel set β and hence $\pi\|\mathrm{d}E_t D^{1/2}x\|^2 \leqslant F(t)\mathrm{d}t$ as a Stieltjes differential inequality and so (a.e.)

$$g^2(t)\|\mathrm{d}E_t D^{1/2}x\|^2 = \|\mathrm{d}E_t g(A) D^{1/2}x\|^2 \leqslant g^2(t)F(t)\mathrm{d}t.$$

An integration leads to (1.8).

It is clear that if the equality (1.7) is assumed, so that equality holds in (1.6), then equality also holds in (1.8). In the sequel, it will be supposed that $g(t)$ is even continuous.

Let $T = A + \mathrm{i}B$ be the completely hyponormal operator of (1.1) and let $f(t)$ and $g(t)$ be continuous, real-valued functions on $(-\infty, \infty)$ where $g(t) \neq 0$ a.e. on $\sigma(A)$. Then

$$T_{fg}^* T_{fg} - T_{fg} T_{fg}^* = g(A) D g(A) \geqslant 0, \tag{1.9}$$

where

$$T_{fg} = A + \mathrm{i}(f(A) + g(A) B g(A)),$$

so that T_{fg} is hyponormal. Since $g(t) \neq 0$ a.e. on $\sigma(A)$ then T_{fg} is also completely hyponormal. (Cf. [6], p. 452, when D has rank 1.)

In view of (1.5), one has

$$\sigma(T_{fg}) \cap \{z : \mathrm{Re}(z) = t\} = f(t) + g^2(t)\,[\sigma(T) \cap \{z : \mathrm{Re}(z) = t\}]. \tag{1.10}$$

Thus, if $z = t + \mathrm{i}y$ belongs to $\sigma(T)$, then $z = t + \mathrm{i}(f(t) + g^2(t)y)$ belongs to $\sigma(T_{fg})$, and, if $g(t) \neq 0$, also conversely. If (1.7) holds, then, as was shown above, equality holds in (1.8), so that the relation (1.7) for T implies the corresponding equality for T_{fg}.

Henceforth, it will be supposed that T is completely hyponormal, that T satisfies (1.7), and that

$$D = T^*T - TT^* \text{ has rank } 1. \tag{1.11}$$

In addition, let $a < b$ and suppose that

$$\sigma(T) = \{t + \mathrm{i}s : a \leqslant t \leqslant b,\ \ y_1(t) \leqslant s \leqslant y_2(t)\}, \tag{1.12}$$

where $y_1(t)$ and $y_2(t)$ are continuous real-valued functions on $[a, b]$ satisfying

$$y_1(t) < y_2(t) \quad \text{a.e. on } [a, b]. \tag{1.13}$$

Then, in view of results of Pincus [11, 12, 13] (see [18]), T is unitarily equivalent to the (completely hyponormal) singular integral operator

(1.14)
$$T_0 = A_0 + iB_0 \quad \text{on } L^2(a, b), \quad \text{where } (A_0x)(t) = tx(t) \text{ and}$$
$$(B_0x)(t) = \alpha(t)x(t) - (i\pi)^{-1}\beta(t)\int_a^b x(s)\,\beta(s)(s-t)^{-1}\,ds,$$

where $x \in L^2(a, b)$ and

(1.15)
$$\alpha(t) = \frac{1}{2}(y_1(t) + y_2(t)) \quad \text{and} \quad \beta(t) = \left[\frac{1}{2}(y_2(t) - y_1(t))\right]^{1/2}.$$

In the special case that $[a, b] = [-1, 1]$, $\alpha(t) \equiv 0$ and $\beta(t) = (1 - t^2)^{1/4}$, then the above $T_0 = A_0 + iB_0$ is the simple unilateral shift V_0; see [3], [4], p. 528, also [18], p. 136.

It is to be noted that hyponormality, and even complete hyponormality, is preserved under each of the operations

(1.16)
$$T \to cT + d \quad (c, d = \text{complex constants}, \ c \neq 0) \quad \text{and} \quad T \to T_{fg},$$

where T_{fg} is defined in (1.9) (and $f(t)$, $g(t)$ are real-valued continuous functions on $(-\infty, \infty)$ with $g(t) \neq 0$ a.e. on $\sigma(A)$). What is important for this paper is that if T satisfies the additional restrictions (1.7), (1.11) and (1.12) then T_{fg} (as well as $cT + d$) satisfies similar conditions. Thus, $\pi\|g(A)Dg(A)\| = \text{meas}_2\,\sigma(T_{fg})$, $g(A)Dg(A)$ has rank 1, and (cf. [6])

(1.17)
$$\sigma(T_{fg}) = \{t + is : a \leq t \leq b, \ y_1(t) + f(t) - g^2(t) \leq s \leq y_2(t) + f(t) + g^2(t)\}.$$

For any bounded opeator S on H, let $A(S)$ and $A(I,S)$ denote the C^*-algebras of bounded operators on H generated, respectively, by S and by S and the identity. Thus, if $B(H)$ is the algebra of all bounded operators on H, then $A(I,S)$, for instance, is the norm closure in $B(H)$ of the space of all (noncommutative) polynomials in I, S and S^* (or, equivalently, in I, $\text{Re}(S)$ and $\text{Im}(S)$).

It is seen that if T is completely hyponormal then, in view of the Weierstrass approximation theorem, the new operators obtained from T under the transformations of (1.16) all belong to $A(I, T)$, so that $A(I, T_{fg}) \subset A(I, T)$.

For certain results on C^*-algebras generated by a hyponormal operator, and which will be alluded to later, see Bunce [1] and Howe [10]. For results relating to the structure of C^*-algebras generated by isometries, see Coburn [7,8].

2.

THEOREM. *Let T be a completely hyponormal operator on H satisfying* (1.7), (1.11) *and* (1.12), *where $y_1(t)$ and $y_2(t)$ are continuous real-valued functions satisfying*

(1.13). *Then, for some simple unilateral shift, V, on H,*

$$A(I, T) \subset A(V). \tag{2.1}$$

Furthermore, there exists some representative, W, on H of the simple unilateral shift satisfying

$$A(I, T) = A(W) \tag{2.2}$$

if and only if

$$y_1(t) < y_2(t) \quad \textit{on } (a, b). \tag{2.3}$$

That (2.2) implies (2.3) is a consequence of a result of Howe [10], p. 636 (Proposition 4). See also the proof below.

Note that, strictly speaking, even for a fixed Hilbert space H, the simple unilateral shift is any one of a certain set of unitarily equivalent operators on this space. Thus, the operators V and W in the Theorem are particular representatives of this set. Note also that, since $V^*V = I$, the C^*-algebra $A(I, V)$ coincides with the C^*-algebra, $A(V)$, generated by V only; similarly, $A(I, W) = A(W)$.

It is convenient to prove first two lemmas.

LEMMA 1. *Consider on $L^2(-1, 1)$*

$$\begin{gathered} S_0 = H_0 + iK_0, \quad \textit{where } (H_0x)(t) = tx(t) \textit{ and} \\ (K_0x)(t) = -(i\pi)^{-1}\int_{-1}^{1} x(s)(s-t)^{-1}ds, \end{gathered} \tag{2.4}$$

and the simple unilateral shift

$$\begin{gathered} V_0 = H_0 + iJ_0, \quad \textit{where } (H_0x)(t) = tx(t) \textit{ and} \\ (J_0x)(t) = -(i\pi)^{-1}(1-t^2)^{1/4}\int_{-1}^{1} x(s)(1-s^2)^{1/4}(s-t)^{-1}ds. \end{gathered} \tag{2.5}$$

Then

$$A(V_0) \subset A(I, S_0). \tag{2.6}$$

Also, there exists a unitary transformation U on $L^2(-1,1)$ for which

$$A(I, S_0) \subset A(I, UV_0U^*). \tag{2.7}$$

Proof. Note that $\sigma(S_0) = [-1, 1]\times[-1, 1]$. Further, if $h(t) = (1-t^2)^{1/4}$, then $V_0 = H_0 + ih(H_0)K_0h(H_0) \in A(I, S_0)$, and so (2.6) is proved. The argument is slightly harder to prove (2.7).

First, it is clear that one can now choose $k(t)$ continuous on $[-1, 1]$ so that $g(t) = k(t)(1-t^2)^{1/4} = 1$ on $[-1/2, 1/2]$ and $g(t)$ is (strictly) monotone increasing on $[-1, -1/2]$ and (strictly) monotone decreasing on $[1/2, 1]$. Let $S_1 = H_0 + ik(H_0)J_0k(H_0) \in A(V_0)$, where J_0 is defined in (2.5). Next, let $S_2 = iS_1$. It is clear that $S_2 \in A(V_0)$ and that S_2 is unitarily equivalent to $T_1 = A_1 + iB_1$ on $L^2(-1, 1)$, where $(A_1x)(t) = tx(t)$ and

$$(B_1x)(t) = -(i\pi)^{-1} g_1(t)\int_{-1}^{1} x(s)g_1(s)(s-t)^{-1}ds,$$

and where $g_1(t)$ is continuous and positive on $[-1, 1]$. Since $g_1^{-1}(t)$ is also continuous, then $g_1^{-1}(A_1)B_1g_1^{-1}(A_1) = K_0$, the (finite Hilbert transform) operator of (2.4), so that $S_0 = A_1 + ig_1^{-1}(A_1)B_1g_1^{-1}(A_1)$ belongs to $A(I, T_1)$.

Thus, $S_0 \in A(I, T_1)$, T_1 is unitarily equivalent to S_2 and $S_2 \in A(V_0)$. Relation (2.7) now follows and the proof of Lemma 1 is complete.

If m and n are real constants, $m \neq 0$, one can replace the variable t on $[a, b]$ by $t = mt' + n$, so that, on dropping the prime, the operator T_0 of (1.14) is replaced by

(2.8) $$T_0 = A_0 + iB_0 \quad \text{on } L^2(-1, 1),$$

where

$$(A_0x)(t) = (mt + n)\, x(t) \quad (n \neq 0)$$

and

$$(B_0x)(t) = \alpha(t)x(t) - (i\pi)^{-1}\beta(t)\int_{-1}^{1} x(s)\beta(s)(s-t)^{-1}ds,$$

where $x \in L^2(-1, 1)$ and $\alpha(t)$ and $\beta(t)$ are defined by (1.15), but now for $-1 \leqslant t \leqslant 1$. Corresponding to (1.12) and (1.13) one now has

(2.9) $$\sigma(T_0) = \{mt + n + is : -1 \leqslant t \leqslant 1,\ y_1(t) \leqslant s \leqslant y_2(t)\}$$

and

(2.10) $$y_1(t) < y_2(t) \quad \text{a.e. on } [-1, 1].$$

LEMMA 2. *Let T_0 be defined on $L^2(-1, 1)$ by (2.8), where $x \in L^2(-1, 1)$, $\alpha(t)$ and $\beta(t)$ are defined on $[-1, 1]$ by (1.15), and $y_1(t)$ and $y_2(t)$ are continuous, real-valued functions on $[-1, 1]$ satisfying (2.9) and (2.10). If V_0 is defined on $L^2(-1, 1)$ by (2.5) then there exists a simple unilateral shift V_1 on $L^2(-1, 1)$ for which*

(2.11) $$A(I, T_0) \subset A(V_1).$$

Further, if

(2.12) $$y_1(t) < y_2(t) \quad on \ (-1, 1),$$

then there exists a simple unilateral shift V_2 *on* $L^2(-1, 1)$ *for which*

(2.13) $$A(I, T_0) = A(V_2).$$

Proof. By using the mapping $T_0 \to T_1$ of (1.16) defined by $T_1 = A_1 + iB_1 = m^{-1}(T_0 - n)$, where $m \neq 0$ and n are real constants as in (2.8), and following with the mapping $T_2 = A_2 + iB_2 = A_1 + i(-\alpha(A_1) + B_1)$, one sees that $A(I, T_0) = A(I, T_1) = A(I, T_2)$ and that

(2.14) $$(T_2x)(t) = tx(t) + i(-(i\pi)^{-1}\beta(t)\int_{-1}^{1} x(s)\beta(s)(s-t)^{-1}ds),$$

where, by (2.9),

(2.15) $$\beta(t) = \left[\frac{1}{2}(y_2(t) - y_1(t))\right]^{1/2}.$$

Thus, $T_2 = H_0 + i\beta(H_0)K_0\beta(H_0) \in A(I, S_0)$, where S_0 is defined by (2.4). Relation (2.11) now follows from (2.7) of Lemma 1.

Next, (2.13) will be proved under the hypothesis (2.12).

First, we consider the special case where

(2.16) $$y_1(t) < y_2(t) \quad \text{on } (-1, 1) \text{ and } y_1(1) = y_2(1), y_1(-1) = y_2(-1).$$

In view of the mappings $T_0 \to T_1 \to T_2$ used above, it is sufficient to show that

(2.17) $$A(I, T_2) = A(V_0),$$

where T_2 is defined in (2.14) and $\beta(t)$ is defined in (2.15) and (2.12).

For each $\varepsilon > 0$ define a continuous function $h_\varepsilon(t)$ on $(-1, 1)$ so that $0 < h_\varepsilon(t) \leqslant 1$ on $(-1, 1)$, $h_\varepsilon(t) = 1$ on $[-1 + \varepsilon, 1 - \varepsilon]$ and $g_\varepsilon(t) = h_\varepsilon(t)(1 - t^2)^{1/4}\beta^{-1}(t) \to 0$ as $t \to -1 + 0$ and as $t \to 1 - 0$. If one puts $g_\varepsilon(-1) = g_\varepsilon(1) = 0$ then $g_\varepsilon(t)$ is continuous on $[-1, 1]$. Let T_2 of (2.14) be expressed as $T_2 = H_2 + iJ_2$, and define $T_{2\varepsilon} = H_2 + ig_\varepsilon(H_2)J_2g_\varepsilon(H_2) \in A(I, T_2)$. Explicitly,

(2.18) $$(T_{2\varepsilon}x)(t) = tx(t) + i(-(i\pi)^{-1}h_\varepsilon(t)(1 - t^2)^{1/4}\int_{-1}^{1} x(s)h_\varepsilon(s)(1-s^2)^{1/4}(s-t)^{-1}ds).$$

Clearly, $|h_\varepsilon(t)(1 - t^2)^{1/4} - (1 - t^2)^{1/4}| \leqslant (2\varepsilon)^{1/4}$ on $(-1, 1)$, and hence $\|T_{2\varepsilon} - V_0\| \to 0$ as $\varepsilon \to 0$, where V_0 is defined by (2.5). Thus $V_0 \in A(I, T_2)$.

In view of (2.16), an argument similar to that of the preceding paragraph, but with the roles of V_0 and T_2 reversed, shows that $T_2 \in A(V_0)$ and hence (2.17). Thus, (2.13) is proved under the assumption (2.16).

To accommodate the case where

$$(2.19) \qquad y_1(-1) < y_2(-1) \quad \text{or} \quad y_1(1) < y_2(1)$$

a modification of the above argument is needed. First, suppose that T_0 and T_0' each is an operator on $L^2(-1, 1)$ of the type (2.8), as in Lemma 2, and suppose that a relation of the type (2.12) holds for the spectrum (cf. (2.9)) of each operator T_0 and T_0'. The above argument shows that

$$(2.20) \qquad A(I, T_0) = A(I, T_0')$$

if both operators have corresponding functions $y_1(t)$ and $y_2(t)$ satisfying (2.16).

It will next be shown that (2.20) holds if both operators have corresponding functions $y_1(t)$ and $y_2(t)$ satisfying

$$(2.21) \qquad y_1(t) < y_2(t) \quad \text{on } [-1, 1].$$

For let $\beta'(t)$ denote the function for T_0' corresponding to $\beta(t) = [(1/2)(y_2(t) - y_1(t))]^{1/2}$ for T_0 (see (2.15)). Then, as before, one has $A(I, T_0) = A(I, T_2)$, with $T_2 = H_2 + iJ_2$ of the form (2.14). If $g(t) = \beta'(t)\beta^{-1}(t)$ then the operator T_2' for T_0' corresponding to T_2 for T_0 is given by $T_2' = H_2 + ig(H_2)J_2g(H_2) \in A(I, T_2)$. Since $g^{-1}(t)$ is also continuous on $[-1, 1]$, one similarly obtains $T_2 \in A(I, T_2')$ and (2.20) follows.

Next, it will be shown that (2.20) holds if both T_0 and T_0' have corresponding functions $y_1(t)$ and $y_2(t)$ satisfying

$$(2.22) \qquad y_1(t) < y_2(t) \quad \text{on } (-1, 1] \quad \text{and} \quad y_1(-1) = y_2(-1),$$

or the corresponding relation

$$(2.23) \qquad y_1(t) < y_2(t) \quad \text{on } [-1, 1) \quad \text{and} \quad y_1(1) = y_2(1).$$

It is suffcient to treat only the case (2.22) for the other can be treated similarly. The argument is similar to that given earlier. Thus, one can define $h_\varepsilon(t)$ on $(-1, 1)$ so that $0 < h_\varepsilon(t) \leq 1$ on $(-1, 1)$, $h_\varepsilon(t) = 1$ on $[-1 + \varepsilon, 1)$ and so that $g_\varepsilon(t) = h_\varepsilon(t)\beta'(t) \cdot \cdot \beta^{-1}(t) \to 0$ as $t \to -1 + 0$. If one puts $g_\varepsilon(-1) = 0$ then $g_\varepsilon(t)$ is continuous on $[-1, 1]$. Again, let T_2 of (2.14), corresponding to T_0, be expressed as $T_2 = H_2 + iJ_2$ and note that $T_{2\varepsilon} = H_2 + ig_\varepsilon(H_2)J_2g_\varepsilon(H_2) \in A(I, T_2)$. Corresponding to (2.18) one has explicitly

$$(T_{2\varepsilon}x)(t) = tx(t) + \mathrm{i}(-(\mathrm{i}\pi)^{-1}h_\varepsilon(t)\,\beta'(t)\int_{-1}^{1} x(s)h_\varepsilon(s)\beta'(s)(s-t)^{-1}\mathrm{d}s).$$

Clearly, $|h_\varepsilon(t)\,\beta'(t) - \beta'(t)| \to 0$ as $\varepsilon \to 0+$ holds uniformly on $(-1, 1)$, so that $\|T_{2\varepsilon} - T_0'\| \to 0$ as $\varepsilon \to 0$, and hence $T_0' \in A(I, T_2) = A(I, T_0)$. A similar argument shows that $T_0 \in A(I, T_0')$, so that (2.20) is established under the condition (2.22).

Note that (2.20) has now been proved when both pairs $\{y_1(t), y_2(t)\}$, corresponding to each operator T_0 and T_0', satisfy any one of the relations (2.16), (2.21) or (2.22).

Now we can prove (2.13), in general, under the hypothesis (2.12). There remains to establish (2.13) in case either (2.21) or (2.22) is assumed. (The case (2.16) was treated first, and the case (2.23) is essentially the same as (2.22).)

First, if (2.21) holds and if $T_2 = H_2 + \mathrm{i}J_2$ is defined by (2.14), then let $g(t) = \beta^{-1}(t)$, so that $T_3 = H_2 + \mathrm{i}g(H_2)J_2g(H_2) = S_0$ of (2.4). Since both $g(t)$ and $g^{-1}(t)$ are continuous it is clear that $A(I, T_2) = A(I, S_0)$. Let $S_1 = 2^{-1/2}e^{\mathrm{i}\pi/4}S_0$, so that $\sigma(S_1)$ is a square with diagonals $[-1, 1]$ and $\mathrm{i}[-1, 1]$. Clearly, $A(I, T_0) = A(I, S_1)$. Since S_1, is unitarily equivalent to an operator of the type T_0 of (2.8) satisfying (2.16), then (2.20) holds and it follows that $A(I, S_1) = A(V_2)$ (and hence (2.13)) for some simple unilateral shift V_2 on $L^2(-1, 1)$. (Note that the corresponding $y_1(t)$ and $y_2(t)$ associated with $\sigma(S_1)$ and with $\sigma(V_2)$ satisfy conditions of type (2.16).)

Finally, suppose that (2.22) holds. Then, let T_3 be an operator of the form (2.14) where $\sigma(T_3)$ is the (solid) triangle with vertices $(-1, 0)$, $(1, -1)$ and $(1, 1)$. By (2.20), $A(I, T_2) = A(I, T_3)$. If $T_4 = \mathrm{i}T_3$ then $A(I, T_2) = A(I, T_4)$. In view of the discussion in Section 1, T_4 is unitarily equivalent to an operator T_{04} of the type (2.8) (even (2.14)) satisfying a condition of type (2.16). Since $A(I, T_0) = A(I, T_2)$, it follows then from (2.20) that $A(I, T_0) = A(I, T_4) = A(V_2)$ for some simple unilateral shift V_2 on $L^2(-1, 1)$ and (2.13) is proved. (Incidentally, it is clear that the case (2.23) can be reduced to the case (2.22) by replacing the pair $\{T_0, T_0'\}$ by $\{-T_0, -T_0'\}$.) This completes the proof of Lemma 2.

3. PROOF OF THE THEOREM

In view of the discussion of Section 1, as well as that preceding the statement of Lemma 2, one sees that there exists a unitary transformation $U: H \to L^2(-1, 1)$ for which $UTU^* = T_0$, where T_0 is defined in (2.8). Note that if $T = A + \mathrm{i}B$ and $T_0 = A_0 + \mathrm{i}B_0$ then $UAU^* = A_0$ and $UBU^* = B_0$. Further, if T_{fg} is defined by (1.9) then, since $f(A)$ and $g(A)$ each is a limit in the norm topology of polynomials in A, it is clear that $UT_{fg}U^* = T_{0fg}$, where $T_{0fg} = A_0 + \mathrm{i}(f(A_0) + g(A_0)B_0g(A_0))$.

Relation (2.1) of the Theorem now follows from (2.11) of Lemma 2. In addition, it also follows from Lemma 2 that (2.3) implies (2.2).

There remains to be shown that, conversely, (2.2) implies (2.3). The idea of the proof will be to show that (2.2) implies that P_T, the approximate point spectrum of T, is homeomorphic to the approximate point spectrum of W, that is, to the unit circle $\mathbf{C} = \{z : |z| = 1\}$. In view of (1.13) and the fact that $\partial(\sigma(T)) \subset P_T$, this clearly implies (2.3).

That, in fact, P_T is homeomorphic to $\mathbf{C}$ as a consequence of (2.2) follows from a result of Howe [10], p. 636 (Proposition 4). The argument depends upon some results of Bunce [1]. (See also Conway [9], p. 144.) For completeness, a related and unsophisticated argument for the case at hand will be given below.

Thus, we assume (2.2) and we shall define a mapping of P_T into $\mathbf{C}$. Let $z \in P_T$. Since $A(I, T) = A(I, T-z)$ then, by (2.2), $W \in A(I, T-z)$. Hence there exists a sequence of constants $\{c_n\}$ and a sequence of polynomials in $T-z$ and $(T-z)^*$ without constant terms, $\{p_n(T-z, (T-z)^*)\}$, satisfying

$$\|W - [c_n + p_n(T-z, (T-z)^*)]\| \to 0, \quad n \to \infty. \tag{3.1}$$

Since $z \in P_T$ and $T-z$ is hyponormal there exists a sequence, $\{x_k(z)\}$, of unit vectors satisfying

$$(T-z)x_k(z) \to 0 \quad \text{and} \quad (T-z)^* x_k(z) \to 0, \quad k \to \infty. \tag{3.2}$$

In view of (3.1) and (3.2), for any $\varepsilon > 0$ and positive integer n, there exist positive numbers N_ε and $K_{\varepsilon n}$ for which

$$\|(W - c_n)x_k(z)\| z < \varepsilon, \quad \text{where } n \text{ (fixed)} > N_\varepsilon, k > K_{\varepsilon n}. \tag{3.3}$$

If $n, m > N_\varepsilon$ (n, m fixed) then $|c_n - c_m| < 2\varepsilon$ if $k > \max(K_{\varepsilon n}, K_{\varepsilon m})$. Since $|c_n - c_m|$ is independent of k and since W is an isometry it follows that $c = \lim_{n\to\infty} c_n$ exists and that $|c| = 1$, so that $c \in \mathbf{C}$.

It is clear from (3.3) that

$$(W - c)x_k(z) \to 0 \quad \text{and} \quad (W - c)^* x_k(z) \to 0, \quad k \to \infty. \tag{3.4}$$

Further, the mapping

$$z \to c(z) \tag{3.5}$$

of P_T into $\mathbf{C}$ is independent of the sequences of constants $\{c_n\}$ and of polynomials $\{p_n(T-z, (T-z)^*)\}$ (without constant terms) satisfying (3.1) and, in addition, the mapping is independent of the choice of the sequence $\{x_k(z)\}$ of (3.2).

Next, note that an interchange of the roles of T and W above leads to a similar mapping from $\mathbf{C}$ into P_T. Thus, if $u \in \mathbf{C}$, there exists a sequence, $\{y_k(u)\}$, of unit vectors satisfying $(W-u)y_k(u) \to 0$ and $(W-u)^* y_k(u) \to 0$ as $k \to \infty$, and, in addition, there exists some point $z \in P_T$ for which $(T-z)y_k(u) \to 0$ and $(T-z)^* y_k(u) \to$ $\to 0$, as $k \to \infty$. Clearly, (3.5) maps z into u and so (3.5) maps P_T onto $\mathbf{C}$.

Further, distinct points of P_T are mapped into distinct points of $\mathbf{C}$. For, suppose that $z \neq z'$ and $u = c(z) = c(z')$. Then there exist sequences of unit vectors, say $\{x_k\}$ and $\{x'_k\}$, such that

3.6) $(T-z)x_k \to 0, \quad (W-u)x_k \to 0, \quad (T-z')\,x'_k \to 0 \quad \text{and} \quad (W-u)x'_k \to 0.$

If the sequence $\{y_k\}$ is defined by $y_{2k-1} = x_k$ and $y_{2k} = x'_k$ $(k = 1,2, \ldots)$ then $(W-u)y_k \to 0$. Hence, there exists some z'' for which $(T-z'')y_k \to 0$, and hence, by (3.6), $z = z'' = z'$, a contradiction. Thus, the mapping (3.5) is one to one from P_T onto $\mathbf{C}$.

Next, we show that the mapping (3.5) is continuous. Suppose that $z_n \to z$ $(z_n, z \in P_T)$ as $n \to \infty$. It is to be shown that $c(z_n) \to c(z)$. Choose sequences $\{x_k(z_n)\}$ of unit vectors such that, for each fixed $n = 1,2, \ldots, (T-z_n)x_k(z_n) \to 0$ as $k \to \infty$, and hence $(W - c(z_n))x_k(z_n) \to 0$ as $k \to \infty$. Since $z_n \to z$ as $n \to \infty$, one may choose $k = k(n) \to \infty$ so that both

$$(W - c(z_n))x_{k(n)}(z_n) \to 0, \quad n \to \infty, \tag{3.7}$$

and $(T-z)x_{k(n)}(z_n) \to 0$ as $n \to \infty$. However, this last relation implies that

$$(W - c(z))\, x_{k(n)}(z_n) \to 0, \quad n \to \infty, \tag{3.8}$$

so that, by (3.7) and (3.8), $c(z_n) \to c(z)$, as was to be shown.

Thus, the mapping (3.5) is continuous. A similar argument shows that the inverse mapping is also continuous, and so P_T is homeomorphic to $\mathbf{C}$. As noted earlier this implies (2.3). This completes the proof of the theorem.

4.

REMARKS. We note that the proof of the theorem shows that the conditions on the completely hyponormal T occurring there may be relaxed. In particular, one can allow any completely hyponormal T satisfying (1.7) and (1.11) and for which $A(I, T) = A(I, T')$, where T' satisfies, in addition to relations corresponding to (1.7) and (1.11), also one corresponding to (1.12). However, by applying a succession of transformations of the type (1.16), one can often distort the spectrum of T into the spectrum of an operator T'. In the light of this observation we offer the following

CONJECTURE. *If T is completely hyponormal and satisfies* (1.7) *and* (1.11) *and if $\sigma(T)$ is the closure of the interior of a simple closed curve then* (2.2) *holds for some simple unilateral shift W.*

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