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ABSOLUTE CONTINUITY AND DISJOINTNESS OF STATES IN C^*-DYNAMICAL SYSTEMS

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INTRODUCTION

The notions of absolute continuity and disjointness of measures are fundamental in the measure theory. These notions and the Radon-Nikodym theorem are extended in several ways to the noncommutative case on von Neumann algebras or C^*-algebras. In this paper, the absolute continuity and disjointness of states of C^*-algebras are discussed. In Section 1, we consider some types of absolute continuity of states. For states φ and ψ of a C^*-algebra A, we say that ψ is absolutely continuous (resp. quasi-absolutely continuous) with respect to φ if the support (resp. the central support) of ψ is contained in that of φ, where the supports (the central supports) are taken for the normal extensions of φ and ψ to the von Neumann algebra induced by a certain representation of A. Let a C^*-dynamical system (A, G, τ) be given. For invariant states φ and ψ, it is shown that ψ is absolutely continuous with respect to φ if and only if ψ is in the norm closure of the invariant states dominated by φ. If A is G-abelian, then the absolute continuity of invariant states is equivalent to that of their maximal representing measures. We show that A is G-central if and only if the absolute continuity and quasi-absolute continuity are identical for each invariant states. We also present the relation between the absolute continuity and the relative entropy of states.

The Kubo-Martin-Schwinger (KMS) condition was introduced to describe thermodynamical equilibrium states of a quantum system. Combined with the-Tomita-Takesaki theory [22], the KMS condition has played a vital role in the ana lysis of operator algebras. The lowest energy states of a quantum system are formulated by ground states which are regarded as limits of KMS states when the inverse temperature tends to $+\infty$. After the concept of passivity given by Pusz and Woronowicz [19] from the thermodynamical viewpoint, de Cannière [11] introduced the similar concept of spectral passivity by using the Arveson spectral subspace given a one-parameter C^*-dynamical system. The spectrally passive states contain the

KMS states and ground states. In Section 2, we show that absolutely continuous states with respect to a KMS (resp. ground, spectrally passive) state are also KMS (resp. ground, spectrally passive) under some assumptions.

In Section 3, we discuss the notions of disjointness of states and obtain the results analogous to those in Section 1. In particular, the G-centrality of a C^*-dynamical system is characterized by conditions on the disjointness of invariant states. We finally mention the Lebesgue type decompositions of states and some properties of absolute continuity and disjointness of limit states in norm.

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1. ABSOLUTE CONTINUITY OF STATES

Let A be a unital C^*-algebra and $\mathrm{S}(A)$ the set of all states of A. We denote by $(\mathscr{H}_\varphi, \pi_\varphi, \xi_\varphi)$ the cyclic representation of A associated with $\varphi \in \mathrm{S}(A)$. Each $\varphi \in \mathrm{S}(A)$ is uniquely extended to a normal state of the enveloping von Neumann algebra A^{**}. Throughout this paper, the normal extension of φ to A^{**} is denoted by $\overline{\varphi}$ and the support and the central support of $\overline{\varphi}$ are denoted by $\mathrm{s}(\varphi)$ and $\mathrm{c}(\varphi)$, respectively. For $\varphi, \psi \in \mathrm{S}(A)$, we say that ψ is *absolutely continuous* with respect to φ (and write $\psi \ll \varphi$) if $\mathrm{s}(\psi) \leqslant \mathrm{s}(\varphi)$, and ψ is *quasi-absolutely continuous* with respect to φ (and write $\psi \prec \varphi$) if $\mathrm{c}(\psi) \leqslant \mathrm{c}(\varphi)$. Instead of the universal representation, take any representation π of A such that φ and ψ have the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi(A)''$, that is, $\tilde{\varphi}$ and $\tilde{\psi}$ are normal states of $\pi(A)''$ with $\varphi = \tilde{\varphi} \circ \pi$ and $\psi = \tilde{\psi} \circ \pi$. For example, let $\pi = \pi_\varphi \oplus \pi_\psi$ or $\pi = \pi_\rho$ with $\rho = (\varphi + \psi)/2$. Let $\bar{\pi}$ be the normal extension of π to A^{**} (see [23, p. 121]) and $\mathrm{s}(\bar{\pi})$ the support of $\bar{\pi}$. As to the support $\mathrm{s}(\tilde{\varphi})$ and the central support $\mathrm{c}(\tilde{\varphi})$ of $\tilde{\varphi}$, we have $\mathrm{s}(\tilde{\varphi}) = \bar{\pi}(\mathrm{s}(\varphi))$ with $\mathrm{s}(\varphi) \leqslant \mathrm{s}(\bar{\pi})$, so that $\mathrm{c}(\tilde{\varphi}) = \bar{\pi}(\mathrm{c}(\varphi))$ with $\mathrm{c}(\varphi) \leqslant \mathrm{s}(\bar{\pi})$. Hence $\psi \ll \varphi$ (resp. $\psi \prec \varphi$) if and only if $\mathrm{s}(\tilde{\psi}) \leqslant \mathrm{s}(\tilde{\varphi})$ (resp. $\mathrm{c}(\tilde{\psi}) \leqslant \mathrm{c}(\tilde{\varphi})$). If φ and ψ are normal states of a von Neumann algebra, then $\psi \ll \varphi$ means that $\varphi(x^*x) = 0$ implies $\psi(x^*x) = 0$. There is another notion of absolute continuity given in [14]. A state ψ is said to be *strongly absolutely continuous* with respect to φ (we write $\psi \lll \varphi$) if $\lim_n \psi(a_n^* a_n) = 0$ for any sequence $\{a_n\}$ in A satisfying

$$\lim_n \varphi(a_n^* a_n) = \lim_{n,m} \psi((a_n - a_m)^*(a_n - a_m)) = 0.$$

We here arrange some rather known characterizations of the above notions of absolute continuity. For $\varphi, \psi \in \mathrm{S}(A)$, consider the following conditions:

(i) $\psi \lll \varphi$;

(ii) There exists a positive self-adjoint operator h' on $\mathscr{H}_\varphi$ affiliated with $\pi_\varphi(A)'$ such that

$$\psi(a) = \langle \pi_\varphi(a) h' \xi_\varphi, h' \xi_\varphi \rangle, \quad a \in A;$$

(iii) $\psi \ll \varphi$;

(iv) There exists a positive self-adjoint operator h on $\mathscr{H}_\varphi$ affiliated with $\pi_\varphi(A)''$ such that

$$\psi(a) = \langle \pi_\varphi(a)h\xi_\varphi, h\xi_\varphi \rangle, \quad a \in A,$$

and $s(h) \leqslant s(\tilde{\varphi})$ were $s(h)$ is the support of h and $\tilde{\varphi}$ is the normal extension of φ to $\pi_\varphi(A)''$;

(v) $\psi \prec \varphi$;

(vi) π_ψ is quasi-equivalent to a subrepresentation of π_φ;

(vii) ψ has a normal extension to $\pi_\varphi(A)''$.

Then the following implications hold:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow \text{(v)} \Leftrightarrow \text{(vi)} \Leftrightarrow \text{(vii)}.$$

See [17, Proposition 4.3] for (v) $\Leftrightarrow$ (vi), and (v) $\Leftrightarrow$ (vii) is easily verified. Obviously (iii) $\Rightarrow$ (v). Since $\psi \ll \varphi$ if and only if ψ has the normal extension $\tilde{\psi}$ to $\pi_\varphi(A)''$ and $s(\tilde{\psi}) \leqslant s(\tilde{\varphi})$, (iii) $\Leftrightarrow$ (iv) follows from the Radon-Nikodym theorem [22, Theorem 15.1] on the von Neumann algebra $s(\tilde{\varphi})\pi_\varphi(A)''s(\tilde{\varphi})$. (i) $\Leftrightarrow$ (ii) is shown in [14, Corollary 2] for states of a Banach $*$-algebra. (ii) $\Rightarrow$ (iii) is immediate. The statements (ii) and (iv) are the Radon-Nikodym theorems for states of a C^*-algebra.

When a C^*-dynamical system (A, G, τ) is given where G is a group and τ is an action of G as $*$-automorphisms of A, we denote by $S_G(A)$ the set of all τ-invariant states of A. If $\varphi \in S_G(A)$, there is a unique unitary representation u_φ of G on $\mathscr{H}_\varphi$ such that $u_\varphi(g)\xi_\varphi = \xi_\varphi$ and

$$\pi_\varphi(\tau_g(a)) = u_\varphi(g)\pi_\varphi(a)u_\varphi(g)^*, \quad a \in A, \ g \in G.$$

For $\varphi \in S_G(A)$, we denote by $F(\varphi)$ (resp. $F_G(\varphi)$) the face of $S(A)$ (resp. $S_G(A)$) generated by φ which is the set of all $\psi \in S(A)$ (resp. $\psi \in S_G(A)$) dominated by φ, i.e. $\psi \leqslant \lambda\varphi$ for some $\lambda > 0$.

THEOREM 1.1. *Let (A, G, τ) be a C^*-dynamical system and $\varphi, \psi \in S_G(A)$. Then the following conditions are equivalent:*

(i) $\psi \ll \varphi$;

(ii) $\psi \in \overline{F(\varphi)}^{n}$, *the norm closure of* $F(\varphi)$;

(iii) $\psi \in \overline{F_G(\varphi)}^{n}$, *the norm closure of* $F_G(\varphi)$.

Proof. (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). If there is a sequence $\{\psi_n\}$ in $F(\varphi)$ satisfying $\|\psi_n - \psi\| \to 0$, then $\tilde{\varphi}(x^*x) = 0$ implies $\tilde{\psi}(x^*x) = \lim_n \tilde{\psi}_n(x^*x) = 0$. Hence we have $\psi \ll \varphi$.

(i) $\Rightarrow$ (iii). Assume $\psi \ll \varphi$, then φ and ψ have the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi_\varphi(A)''$ such that $s(\tilde{\psi}) \leqslant s(\tilde{\varphi})$. Let $e = s(\tilde{\varphi})$ which is the projection onto $[\pi_\varphi(A)'\xi_\varphi]$ and is in $u_\varphi(G)'$. Define $M = e\pi_\varphi(A)''e$ on $\mathscr{H} = e\mathscr{H}_\varphi$, $\hat{\varphi} = \tilde{\varphi} \restriction M$, $\hat{\psi} = \tilde{\psi} \restriction M$, $\hat{u}(g) = u_\varphi(g)e$ and $\hat{\tau}_g(x) = \hat{u}(g)x\hat{u}(g)^*$ for $x \in M, g \in G$. Since ξ_φ is a $\hat{u}$-invariant, cyclic and separating vector for M, by [1, Theorems 6 and 11] there exists a $\hat{u}$-invariant vector η in the natural positive cone associated with (M, ξ_φ) such that $\hat{\psi}(x)=\langle x\eta, \eta\rangle$ for all $x \in M$. Let p be the projection onto the $\hat{u}$-invariant vectors in $\mathscr{H}$. Choosing a sequence $\{h'_n\}$ in M' such that $\|h'_n\xi_\varphi - \eta\| \to 0$ and hence $\|ph'_n\xi_\varphi - \eta\| \to 0$, we define $\hat{\tau}$-invariant normal states $\hat{\psi}_n$ of M by

$$\hat{\psi}_n(x) = \|ph'_n\xi_\varphi\|^{-2}\langle xph'_n\xi_\varphi, ph'_n\xi_\varphi\rangle, \quad x \in M.$$

We then have $\|\hat{\psi}_n - \hat{\psi}\| \to 0$. By Kovács-Szücs theorem [18], there exists aconditional expectation ε of M onto $M \cap \hat{u}(G)'$ satisfying $\varepsilon(x)p = pxp = p\varepsilon(x)$ for all $x \in M$. For every $x \in M_+$, we have

$$h'^*_n pxph'_n = \varepsilon(x)^{1/2}h'^*_n ph'_n\varepsilon(x)^{1/2} \leqslant \|h'^*_n ph'_n\|\varepsilon(x),$$

so that

$$\hat{\psi}_n(x) \leqslant \lambda_n \langle \varepsilon(x)\xi_\varphi, \xi_\varphi\rangle = \lambda_n\hat{\varphi}(x),$$

where $\lambda_n = \|ph'_n\xi_\varphi\|^{-2}\|h'^*_n ph'_n\|$. Now define states ψ_n of A by

$$\psi_n(a) = \hat{\psi}_n(e\pi_\varphi(a)e), \quad a \in A.$$

Then

$$\psi_n(a) \leqslant \lambda_n\hat{\varphi}(e\pi_\varphi(a)e) = \lambda_n\varphi(a), \quad a \in A_+,$$

and

$$\psi_n(\tau_g(a)) = \hat{\psi}_n(eu_\varphi(g)\pi_\varphi(a)u_\varphi(g)^*e) =$$

$$= \hat{\psi}_n(\hat{\tau}_g(e\pi_\varphi(a)e)) = \psi_n(a), \quad a \in A.$$

Moreover we have $\|\psi_n - \psi\| \leqslant \|\hat{\psi}_n - \hat{\psi}\| \to 0$ since $\psi(a) = \hat{\psi}(e\pi_\varphi(a)e)$. Thus (iii) holds. Q.E.D.

THEOREM 1.2. *For each $\varphi \in \mathrm{S}(A)$, the following conditions are equivalent:*
(i) $\psi \ll \varphi$ and $\psi \prec \varphi$ are identical for every $\psi \in \mathrm{S}(A)$;
(ii) ξ_φ is separating for $\pi_\varphi(A)''$.

Proof. Let $\tilde{\varphi}$ be the normal extension of φ to $\pi_\varphi(A)''$. The condition (i) means that $\mathrm{s}(\tilde{\psi}) \leqslant \mathrm{s}(\tilde{\varphi})$ holds for every normal state $\tilde{\psi}$ of $\pi_\varphi(A)''$. This is the case when $\mathrm{s}(\tilde{\varphi}) = 1$, that is, ξ_φ is separating for $\pi_\varphi(A)''$. Q.E.D.

In the abelian case $A = C(X)$ with a compact Hausdorff space X, all the conditions $\psi \lll \varphi$, $\psi \ll \varphi$ and $\psi \prec \varphi$ are identical and mean the absolute continuity of measures on X representing ψ and φ.

Let (A, G, τ) be a C^*-dynamical system. It is said that A is *G-abelian* if $\pi_\varphi(A)' \cap \cap u_\varphi(G)'$ is abelian for every $\varphi \in \mathrm{S}_G(A)$. It is known (cf. [10, Theorem 1]) that A is G-abelian if and only if $\mathrm{S}_G(A)$ is a Choquet simplex. In this case, each $\varphi \in \mathrm{S}_G(A)$ has a unique maximal representing measure μ_φ on $\mathrm{S}_G(A)$ which is the $\pi_\varphi(A)' \cap \cap u_\varphi(G)'$-orthogonal measure of φ (cf. [20, Theorem 3.6] or [8, Proposition 4.3.3]), and the mapping $\varphi \mapsto \mu_\varphi$ is affine (cf. [8, Corollary 4.1.17]) and isometric, i.e. $\|\varphi - - \psi\| = \|\mu_\varphi - \mu_\psi\|$ (cf. [16, IV.4]). A state $\varphi \in \mathrm{S}_G(A)$ is called to be *G-subcentral* if $\pi_\varphi(A)' \cap u_\varphi(G)' \subset \pi_\varphi(A)''$. It is said that A is *G-central* (or (A, G, τ) is *quasi-large*) if every $\varphi \in \mathrm{S}_G(A)$ is G-subcentral. This condition is weaker than some other conditions of asymptotic abelianness (see [10, 13]).

THEOREM 1.3. *Let (A, G, τ) be a C^*-dynamical system such that A is G-abelian. Then for each $\varphi, \psi \in \mathrm{S}_G(A)$, the following conditions are equivalent:*

(i) $\psi \lll \varphi$;

(ii) $\psi \ll \varphi$;

(iii) $\mu_\psi \ll \mu_\varphi$.

Proof. (i) $\Rightarrow$ (ii) is valid generally.

(ii) $\Rightarrow$ (iii). If $\psi \ll \varphi$, by Theorem 1.1 there is a sequence $\{\psi_n\}$ in $\mathrm{F}_G(\varphi)$ satisfying $\|\psi_n - \psi\| \to 0$. From the facts mentioned above, each μ_{ψ_n} is dominated by μ_φ and $\|\mu_{\psi_n} - \mu_\psi\| = \|\psi_n - \psi\| \to 0$. Hence we have $\mu_\psi \ll \mu_\varphi$.

(iii) $\Rightarrow$ (i). Assume $\mu_\psi \ll \mu_\varphi$ and let $g = \mathrm{d}\mu_\psi/\mathrm{d}\mu_\varphi$. Because μ_φ is the $\pi_\varphi(A)' \cap \cap u_\varphi(G)'$-orthogonal measure, there is a $*$-isomorphism θ of $L^\infty(\mu_\varphi)$ onto $\pi_\varphi(A)' \cap \cap u_\varphi(G)'$ such that

$$\langle \theta(f)\pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle = \int f(\omega)\omega(a)\, \mathrm{d}\mu_\varphi(\omega), \qquad a \in A, \;\; f \in L^\infty(\mu_\varphi).$$

Letting $g_k = \min(g^{1/2}, k)$ for $k \geq 1$, we have

$$\psi(a^*a) = \int g(\omega)\omega(a^*a)\, \mathrm{d}\mu_\varphi(\omega) =$$

$$= \sup_k \int g_k(\omega)^2 \omega(a^*a)\, \mathrm{d}\mu_\varphi(\omega) = \sup_k \|\theta(g_k)\pi_\varphi(a)\xi_\varphi\|^2, \qquad a \in A.$$

Hence a positive quadratic form q on $\mathscr{H}_\varphi$ with the domain $\mathrm{D}(q) = \pi_\varphi(A)\xi_\varphi$ defined by

$$q(\pi_\varphi(a)\xi_\varphi) = \psi(a^*a), \qquad a \in A,$$

is lower semi-continuous on $\mathrm{D}(q)$. Therefore q is closable and we have $q(\pi_\varphi(a_n)\xi_\varphi) \to 0$ for any sequence $\{a_n\}$ in A with $\|\pi_\varphi(a_n)\xi_\varphi\| \to 0$ and $q(\pi_\varphi(a_n - a_m)\xi_\varphi) \to 0$ (see [21, p. 467]). This shows $\psi \lll \varphi$. Q.E.D.

It seems that the conditions $\psi \lll \varphi$ and $\psi \ll \varphi$ are different in general. It is interesting to provide an explicit example showing that $\psi \ll \varphi$ does not imply $\psi \lll \varphi$.

THEOREM 1.4. *Let (A, G, τ) be a C^*-dynamical system. Then the following conditions are equivalent:*

(i) *$\psi \ll \varphi$ and $\psi \prec \varphi$ are identical for every $\varphi, \psi \in S_G(A)$;*

(ii) *A is G-central.*

Proof. Let $\bar{\tau}$: $G \to \mathrm{Aut}(A^{**})$ be the extension of τ and e the finite part of $(A^{**}, G, \bar{\tau})$, i.e. $e = \bigvee\{s(\varphi): \varphi \in S_G(A)\}$ (see [9]). Let Z be the center of A^{**} and $\mathscr{I}$ (resp. Z^G) the set of $\bar{\tau}$-invariant elements in A^{**} (resp. Z). According to [10, Theorem 2],A is G-central if and only if $\mathscr{I}_e = (Z^G)_e$ holds. We show that $\mathscr{I}_e = (Z^G)_e$ is equivalent to the following condition:

(iii) $s(\varphi) = c(\varphi)e$ for every $\varphi \in S_G(A)$.

If $\mathscr{I}_e = (Z^G)_e$ holds, then the condition (iii) is easily verified since $c(\varphi) \in Z^G$ for $\varphi \in S_G(A)$. Conversely if (iii) is satisfied, then for any projection p in $\mathscr{I}_e$ we have

$$p = \bigvee\{s(\varphi): \varphi \in S_G(A) \text{ with } s(\varphi) \leqslant p\} =$$

$$= (\bigvee\{c(\varphi): \varphi \in S_G(A) \text{ with } s(\varphi) \leqslant p\})e,$$

and hence $p \in (Z^G)_e$. Thus $\mathscr{I}_e = (Z^G)_e$ is obtained.

It now suffices to show that the conditions (i) and (iii) are equivalent. (iii) $\Rightarrow$ (i) is obvious. Suppose that $s(\varphi) \neq c(\varphi)e$ for some $\varphi \in S_G(A)$ and let $p = c(\varphi)e - s(\varphi)$. Then we can choose a $\rho \in S_G(A)$ satisfying $\bar{\rho}(p) > 0$. Defining a $\psi \in S_G(A)$ by $\psi(a) = \bar{\rho}(pap)/\bar{\rho}(p)$ for $a \in A$, we have $s(\psi) \leqslant p \leqslant c(\varphi)$, so that $c(\psi) \leqslant c(\varphi)$ but $s(\psi) \perp s(\varphi)$, contradicting (i). Q.E.D

EXAMPLE 1.5. Let A be the C^*-algebra of $n \times n$ complex matrices. Since $A = A^{**}$ is a factor, it is clear that $\psi \prec \varphi$ holds for every $\varphi, \psi \in S(A)$. On the other hand, $\psi \ll \varphi$ means that ψ is dominated by φ. A C^*-dynamical system $(A, \mathbf{R}, \alpha)$ is given by $\alpha_t(a) = e^{ith}ae^{-ith}$ where h is a self-adjoint element in A. Assume that h has the spectral decomposition $h = \sum_{i=1}^{n} \lambda_i p_i$ with $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Then each $\varphi \in S_{\mathbf{R}}(A)$ is defined by a density matrix commuting with h. Hence $S_{\mathbf{R}}(A)$ is a Choquet simplex and A is $\mathbf{R}$-abelian. However A is not $\mathbf{R}$-central for $n \geqslant 2$ as we have φ, $\psi \in S_{\mathbf{R}}(A)$ such that $\psi \ll \varphi$ does not hold.

We conclude this section with some notes on the relation between the absolute continuity and the relative entropy of states. Since Umegaki [25], the relative entropy in noncommutative systems has been studied by several authors. Araki [3, 4] extended the notion of relative entropy to the case of normal positive linear functionals of a von Neumann algebra. Furthermore Uhlmann [24] defined the relative entropy of positive linear functionals of an arbitrary $*$-algebra. For each normal positive linear functionals of a von Neumann algebra, the Uhlmann's relative entropy is equal to the Araki's relative entropy (see [15]).

Let $\varphi, \psi \in S(A)$ be given. The relative entropy of ψ with respect to φ is denoted by $S(\psi \mid \varphi)$. Let γ be a positive linear map of A into the abelian C^*-algebra $C(S(A))$ of continuous functions on $S(A)$ defined by $(\gamma a)(\omega) = \omega(a)$ for $a \in A$ and $\omega \in S(A)$. For any probability measures μ and ν on $S(A)$ with the barycenters φ and ψ, applying the monotonicity of relative entropy (cf. [24, Proposition 18]) we have

$$S(\psi \mid \varphi) = S(\nu \circ \gamma \mid \mu \circ \gamma) \leqslant S(\nu \mid \mu),$$

where $S(\nu \mid \mu)$ is given by the classical relative entropy:

$$S(\nu \mid \mu) = \begin{cases} \int \log \dfrac{d\nu}{d\mu} d\nu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Suppose that $\psi \leqslant \lambda\varphi$ for some $\lambda > 0$. Taking probability measures μ, ν on $S(A)$ which have the barycenters φ, ψ and satisfies $\nu \leqslant \lambda\mu$, we have $S(\psi \mid \varphi) \leqslant S(\nu \mid \mu) \leqslant \leqslant \log \lambda$. Hence it is seen that if ψ is dominated by φ, then $S(\psi \mid \varphi) < +\infty$. Since $S(\psi \mid \varphi) = S(\overline{\psi} \mid \overline{\varphi})$ (cf. [15, Lemma 3.1]), the condition $S(\psi \mid \varphi) < +\infty$ is sufficient for $\psi \ll \varphi$. This fact is a stronger version of [2, Lemma 2]. We finally note (cf. [15, Theorem 3.2]) that if A is G-central, then $S(\psi \mid \varphi) = S(\mu_\psi \mid \mu_\varphi)$ for each φ $\psi \in S_G(A)$ where μ_φ and μ_ψ are as in Theorem 1.3.

2. ABSOLUTE CONTINUITY AND KMS STATES

In this section, let $(A, \mathbf{R}, \alpha)$ be a fixed strongly continuous one-parameter C^*-dynamical system. For $0<\beta<+\infty$, a state φ of A is said to be *β-KMS* with respect to α if for every pair $a, b \in A$ there exists a bounded continuous function F on the strip $0 \leqslant \operatorname{Im} z \leqslant \beta$ which is holomorphic in the interior and has boundary values:

$$F(t) = \varphi(a\alpha_t(b)), \quad F(t + i\beta) = \varphi(\alpha_t(b)a), \quad t \in \mathbf{R}.$$

Also φ is said to be a *ground state* with respect to α if

$$- i\varphi(a^*\delta(a)) \geqslant 0$$

for all $a \in D(\delta)$ where δ is the generator of α. If φ is a β-KMS or ground state, then φ is α-invariant. A ground state may be called to be $(+\infty)$-KMS (see [8, Propositions 5.3.19, 5.3.23]). On the other hand, a tracial state is regarded as a 0-KMS state. We denote by K_β (resp. K_∞) the set of all β-KMS (resp. ground) states of A, and by K_0 the set of α-invariant tracial states.

Since K_∞ is a weak* closed face of $S(A)$ (cf. [8, Theorem 5.3.37]), it follows that if $\varphi \in K_\infty$ and $\psi \leqslant \varphi$, then $\psi \in K_\infty$. As for the quasi-absolute continuity, we give:

EXAMPLE 2.1. Let $\varphi \in K_\infty$. Then the following conditions are equivalent:
(i) any $\psi \in S(A)$ with $\psi \prec \varphi$ is in K_∞;
(ii) $u_\varphi(t) = 1$ for all $t \in \mathbf{R}$.

In fact, if u_φ is trivial, then $\pi_\varphi(\delta(a)) = 0$ for all $a \in D(\delta)$ and hence any $\psi \in S(A)$ having the normal extension to $\pi_\varphi(A)''$ is in K_∞. Conversely if (i) holds, then any vector state $\psi(a) = \langle \pi_\varphi(a)\eta, \eta\rangle$ with $\eta \in \mathscr{H}_\varphi$ is α-invariant, so that u_φ is trivial by [5, Lemma 4.1]. Now let $\varphi \in K_\infty$ be such that u_φ is nontrivial. Then there exists a $\psi \in S(A)$ such that $\psi \prec \varphi$ and $\psi \notin K_\infty$. In connection with Theorem 1.1, this shows that the condition $\psi \prec \varphi$ does not necessarily imply that ψ is in the weak* closure of $F(\varphi)$.

THEOREM 2.2. *Let (A, G, τ) be a C^*-dynamical system. Assume that $\varphi \in K_\beta$ with $0 < \beta < +\infty$ and φ is G-subcentral. Then any $\psi \in S_G(A)$ satisfying $\psi \prec \varphi$ is in K_β.*

Proof. Since $\varphi \in K_\beta$, ξ_φ is separating for $\pi_\varphi(A)''$. Theorems 1.1 and 1.2 assert that $\psi \in \overline{F_G(\varphi)}^n$. Hence it is sufficient to show that $\psi \in F_G(\varphi)$ implies $\psi \in K_\beta$. If $\psi \in F_G(\varphi)$, there is a positive operator h in $\pi_\varphi(A)' \cap u_\varphi(G)'$ such that $\psi(a) = \langle \pi_\varphi(a)h\xi_\varphi, \xi_\varphi\rangle$ for all $a \in A$. Since φ is G-subcentral, we get $h \in \pi_\varphi(A)' \cap \pi_\varphi(A)''$. Therefore $\psi \in K_\beta$. Q.E.D.

A state φ of A is said to be *spectrally passive* with respect to α if φ is α-invariant and

$$\varphi(a^*a) \leqslant \varphi(aa^*)$$

for all a in the spectral subspace $R(-\infty, 0)$ which is the closed subspace spanned by all elements of the form $\int f(t)\alpha_t(b)\,dt$ with $b \in A$ and $f \in L^1(\mathbf{R})$ whose inverse Fourier transform $\hat{f}$ has compact support in $(-\infty, 0)$. The notion of spectral passivity was introduced by de Cannière [11]. He showed that φ is spectrally passive if and only if

$$-\mathrm{i}\,\varphi(a\delta(a)) \geqslant 0$$

for all $a = a^* \in D(\delta)$, which is satisfied if φ is passive in the sense of Pusz and Woronowicz [19]. The set of (spectrally) passive states is a weak* closed convex set containing $\bigcup_{0 \leqslant \beta \leqslant \infty} K_\beta$. Let (A, G, τ) be a C^*-dynamical system such that τ commutes with α. If $\varphi \in S_G(A)$ is G-subcentral and spectrally passive, then φ is in the weak* closed convex hull of $\bigcup_{0 \leqslant \beta \leqslant \infty} K_\beta$ (cf. [7, Theorem 5]).

THEOREM 2.3. *Let A be separable and τ be a strongly continuous action of a locally compact separable group G commuting with α. Assume that A is G-central and has at most one τ-invariant tracial state. If $\varphi \in S_G(A)$ is spectrally passive, then any $\psi \in S_G(A)$ with $\psi \prec \varphi$ is spectrally passive.*

Proof. Let u_φ and v_φ be the unitary representations of G and $\mathbf{R}$, respectively, on $\mathscr{H}_\varphi$ associated with φ. From Theorems 1.1 and 1.4, it may be assumed that $\psi \in F_G(\varphi)$. We first suppose that ψ is further α-invariant. There then exists a positive operator h in $\pi_\varphi(A)' \cap u_\varphi(G)' \cap v_\varphi(\mathbf{R})'$ such that $\psi(a) = \langle \pi_\varphi(a)h\xi_\varphi, \xi_\varphi\rangle$ for all $a \in A$. Since $h \in \pi_\varphi(A)' \cap \pi_\varphi(A)''$, it follows as in the proof of [7, Theorem 5] that

$$\langle \pi_\varphi(a^*a)h\xi_\varphi, \xi_\varphi\rangle \leqslant \langle \pi_\varphi(aa^*)h\xi_\varphi, \xi_\varphi\rangle$$

for all $a \in R(-\infty, 0)$. Thus ψ is spectrally passive. In this case, we have used only the condition of φ being G-subcentral.

We now assume all the conditions in the theorem and prove that each $\psi \in F_G(\varphi)$ is α-invariant. To do this, it suffices to show that

$$(*) \qquad \pi_\varphi(A)' \cap \pi_\varphi(A)'' \cap u_\varphi(G)' \subset v_\varphi(\mathbf{R})'.$$

Since A is G-central, A is also $G \times \mathbf{R}$-central. Let μ be the $\pi_\varphi(A)' \cap u_\varphi(G)' \cap v_\varphi(\mathbf{R})'$--orthogonal measure of φ, which is a unique maximal measure on $S_{G\times\mathbf{R}}(A)$ representing φ. By the separability assumptions, we obtain the spatial decomposition of $(\mathscr{H}_\varphi, \pi_\varphi, \xi_\varphi)$ as follows:

$$\mathscr{H}_\varphi = \int^{\oplus} \mathscr{H}_\omega \, d\mu(\omega),$$

$$\pi_\varphi = \int^{\oplus} \pi_\omega \, d\mu(\omega),$$

$$\xi_\varphi = \int^{\oplus} \xi_\omega \, d\mu(\omega).$$

Moreover u_φ and v_φ are decomposable by

$$u_\varphi(g) = \int^{\oplus} u_\omega(g) \, d\mu(\omega), \quad g \in G,$$

$$v_\varphi(t) = \int^{\oplus} v_\omega(t) \, d\mu(\omega), \quad t \in \mathbf{R}.$$

It is shown in the proof of [7, Theorem 5] that μ is supported by $\bigcup_{0\leqslant\beta\leqslant\infty} K_\beta$. If $\omega \in K_\beta$ with $0 < \beta \leqslant \infty$, then

$$\pi_\omega(A)' \cap \pi_\omega(A)'' \subset v_\omega(\mathbf{R})'$$

(cf. [8, Propositions 5.3.28, 5.3.19]). If $\omega \in K_0 \cap S_{G\times\mathbf{R}}(A)$, then ω is the unique τ-invariant tracial state. Hence it is seen from the G-centrality of A that ω is G-ergodic, so that

$$\pi_\omega(A)' \cap u_\omega(G)' = \mathbf{C}\,1.$$

Therefore we have

$$\pi_\omega(A)' \cap \pi_\omega(A)'' \cap u_\omega(G)' \subset v_\omega(\mathbf{R})'$$

for μ-almost all ω. This implies the desired inclusion (*). Q.E.D.

EXAMPLE 2.4. The C^*-algebra A of a d-dimensional quantum lattice system (see [8, Section 6.2.1]) is asymptotic abelian under the action τ of $\mathbf{Z}^d$ and hence $\mathbf{Z}^d$-central. Let α be the one-parameter automorphism group defined by a translationally invariant interaction Φ. Then Theorems 2.2 and 2.3 can be applied since A has a unique tracial state. Thus if φ is a τ-invariant β-KMS (resp. ground, spectrally passive) state, then any τ-invariant state ψ with $\psi \prec \varphi$ is β-KMS (resp. ground, spectrally passive). For KMS states and ground states of quantum lattice systems, see [8, Theorems 6.2.42, 6.2.58].

3. DISJOINTNESS OF STATES

For $\varphi, \psi \in S(A)$, it is said that φ and ψ are *disjoint* (denoted by $\varphi \between \psi$) if π_φ and π_ψ are disjoint or equivalently if $c(\varphi) \perp c(\psi)$. We call φ and ψ to be *orthogonal* (denoted by $\varphi \perp \psi$) if $s(\varphi) \perp s(\psi)$, and to be *singular* ($\varphi \wedge \psi = 0$) if $s(\varphi) \wedge s(\psi) = 0$. Obviously $\varphi \between \psi \Rightarrow \varphi \perp \psi \Rightarrow \varphi \wedge \psi = 0$. It is known (cf. [12, Proposition 12.3.1]) that $\varphi \perp \psi$ if and only if $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$ $(= 2)$. Take any representation π of A such that φ and ψ have the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi(A)''$. Then we see as in Section 1 that $\varphi \between \psi$ (resp. $\varphi \perp \psi$, $\varphi \wedge \psi = 0$) if and only if $c(\tilde{\varphi}) \perp c(\tilde{\psi})$ (resp. $s(\tilde{\varphi}) \perp s(\tilde{\psi})$, $s(\tilde{\varphi}) \wedge s(\tilde{\psi}) = 0$).

THEOREM 3.1. *Let (A, G, τ) be a C^*-dynamical system. For each $\varphi, \psi \in S_G(A)$, the following conditions are equivalent:*

(i) $\varphi \wedge \psi = 0$;
(ii) $\overline{F(\varphi)}^{\,n} \cap \overline{F(\psi)}^{\,n} = \emptyset$;
(iii) $\overline{F_G(\varphi)}^{\,n} \cap \overline{F_G(\psi)}^{\,n} = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 1.1, and (ii) $\Rightarrow$ (iii) is trivial. Suppose (iii) and let $p = s(\varphi) \wedge s(\psi)$ which is $\bar{\tau}$-invariant where $\bar{\tau}$ is the extension of τ to A^{**}. If $\bar{\varphi}(p) > 0$, then we can define a $\rho \in S_G(A)$ by $\rho(a) = \bar{\varphi}(pap)/\bar{\varphi}(p)$

for $a \in A$. Since $s(\rho) = p$, Theorem 1.1 implies that $\rho \in \overline{F_G(\varphi)}^n \cap \overline{F_G(\psi)}^n$, contradicting (iii). Hence $\overline{\varphi}(p) = 0$, so that $p = 0$. Thus (iii) $\Rightarrow$ (i). Q.E.D.

In [8, Definition 4.1.20], φ and ψ are said to be orthogonal if $F(\varphi) \cap F(\psi) = \emptyset$ in our terminology. This definition of orthogonality is somewhat different from ours. In fact, Theorem 3.1 shows that if $\varphi \wedge \psi = 0$ then φ and ψ are orthogonal in the sense of [8].

THEOREM 3.2. *Let (A, G, τ) be a C^*-dynamical system such that A is G-abelian. Then for each $\varphi, \psi \in S_G(A)$, the following conditions are equivalent:*

(i) $\varphi \perp \psi$;

(ii) $\varphi \wedge \psi = 0$;

(iii) $F_G(\varphi) \cap F_G(\psi) = \emptyset$;

(iv) μ_φ *and* μ_ψ *are singular.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold generally.

(iii) $\Rightarrow$ (i). Let $\rho = (\varphi + \psi)/2 \in S_G(A)$. There then exists an operator h in $\pi_\rho(A)' \cap u_\rho(G)'$ with $0 \leqslant h \leqslant 1$ such that

$$\varphi(a) = 2\langle \pi_\rho(a) h \xi_\rho, \xi_\rho \rangle,$$
$$\psi(a) = 2\langle \pi_\rho(a)(1 - h)\xi_\rho, \xi_\rho \rangle,$$

for all $a \in A$. If $F_G(\varphi) \cap F_G(\psi) = \emptyset$, then we get $h(1 - h) = 0$ and hence h is a projection in $\pi_\rho(A)' \cap u_\rho(G)'$. The supports $s(\tilde{\varphi})$ and $s(\tilde{\psi})$ of the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi_\rho(A)''$ are the projections onto $[\pi_\rho(A)' h \xi_\rho]$ and $[\pi_\rho(A)'(1 - h)\xi_\rho]$, respectively. By Kovács-Szücs theorem, we can take a conditional expectation ε of $\pi_\rho(A)'$ onto $\pi_\rho(A)' \cap u_\rho(G)'$ satisfying

$$\langle \varepsilon(x)\xi_\rho, \xi_\rho \rangle = \langle x\xi_\rho, \xi_\rho \rangle, \quad x \in \pi_\rho(A)'.$$

Since $\pi_\rho(A)' \cap u_\rho(G)'$ is abelian, we have

$$\langle xh\xi_\rho, y(1 - h)\xi_\rho \rangle = \langle (1 - h)\varepsilon(y^* x) h \xi_\rho, \xi_\rho \rangle = 0$$

for every $x, y \in \pi_\rho(A)'$. This implies $s(\tilde{\varphi}) \perp s(\tilde{\psi})$ and thus $\varphi \perp \psi$.

(iii) $\Rightarrow$ (iv). Suppose that μ_φ and μ_ψ are not singular. Then there is a probability measure ν on $S_G(A)$ dominated by both μ_φ and μ_ψ. If ρ is the barycenter of ν, then it follows that $\rho \in F_G(\varphi) \cap F_G(\psi)$.

(iv) $\Rightarrow$ (ii). If (ii) is not true, then we obtain a $\rho \in S_G(A)$ such that $\rho \ll \varphi$ and $\rho \ll \psi$, so that $\mu_\rho \ll \mu_\varphi$ and $\mu_\rho \ll \mu_\psi$ by Theorem 1.3. This contradicts (iv). Q.E.D.

THEOREM 3.3. *Let (A, G, τ) be a C^*-dynamical system. Then the following conditions are equivalent:*

(i) $\varphi \perp \psi$ *implies* $\varphi \,\delta\, \psi$ *for every* $\varphi, \psi \in S_G(A)$;

(ii) $F_G(\varphi) \cap F_G(\psi) = \emptyset$ *implies* $\varphi \,\delta\, \psi$ *for every* $\varphi, \psi \in S_G(A)$;

(iii) *A is G-central.*

Proof. (ii) $\Rightarrow$ (i) is clear. Let e be as in the proof of Theorem 1.4. The G-centrality of A is equivalent to that $s(\varphi) = c(\varphi)e$ holds for every $\varphi \in S_G(A)$. If $s(\varphi) \neq \neq c(\varphi)e$ for some $\varphi \in S_G(A)$, then we obtain a $\psi \in S_G(A)$ such that $c(\psi) \leqslant c(\varphi)$ but $s(\psi) \perp s(\varphi)$ (see the proof of Theorem 1.4). Thus (i) $\Rightarrow$ (iii). Suppose that A is G-central. Let $\rho = (\varphi + \psi)/2$ and h be as in the proof of (iii) $\Rightarrow$ (i) in Theorem 3.2. If $F_G(\varphi) \cap F_G(\psi) = \emptyset$, then it follows in this case that h is a central projection in $\pi_\rho(A)''$. Therefore $c(\tilde{\varphi}) \leqslant h$ and $c(\tilde{\psi}) \leqslant 1 - h$ for the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi_\rho(A)''$, so that $\varphi \mathbin{\rotatebox[origin=c]{180}{$\perp$}} \psi$. Thus (iii) $\Rightarrow$ (ii). Q.E.D.

REMARK 3.4. Assume that A is G-central. If φ is centrally ergodic (i.e. $\pi_\varphi(A)' \cap \cap \pi_\varphi(A)'' \cap u_\varphi(G)' = \mathbf{C}\,1$) then φ is G-ergodic and so $F_G(\varphi) = \{\varphi\}$. Hence Theorem 3.3 has the corollary that distinct centrally ergodic states of A are disjoint (see [8, Theorem 4.3.19], [6]). The conditions given in Theorems 1.4 and 3.3 seem to be new characterizations of G-centrality.

Note that all definitions of absolute continuity and disjointness in this paper remain valid for any positive linear functionals of A. We can obtain the Lebesgue type decompositions of states. For each φ, $\psi \in S(A)$, it is immediately seen that ψ has a unique decomposition $\psi = \psi_1 + \psi_2$ where ψ_1 and ψ_2 are positive linear functionals of A with $\psi_1 \prec \varphi$ and $\psi_2 \mathbin{\rotatebox[origin=c]{180}{$\perp$}} \varphi$. Now let (A, G, τ) be a C^*-dynamical system such that A is G-abelian. Then for each φ, $\psi \in S_G(A)$, by using Theorems 1.3, 3.2 and the Lebesgue decomposition of μ_ψ with respect to μ_φ we see that there exist unique τ-invariant positive linear functionals ψ_1 and ψ_2 of A such that $\psi = \psi_1 + \psi_2$, $\psi_1 \ll \varphi$ and $\psi_2 \perp \varphi$.

Lastly we note some properties of absolute continuity and disjointness of limit states. According to [17, Theorem 3.2], if $\{\varphi_n\}$ is a sequence in $S(A)$ with $\|\varphi_n - \varphi\| \to 0$, then $\{s(\varphi_n)s(\varphi)\}$ and $\{c(\varphi_n)\,c(\varphi)\}$ converge strongly to $s(\varphi)$ and $c(\varphi)$, respectively. From these facts, we can easily show the following:

(1)Let $\{\psi_n\}$ be a sequence in $S(A)$ with $\|\psi_n - \psi\| \to 0$. If $\psi_n \ll \varphi$ (resp. $\psi_n \prec \varphi$) for all n, then $\psi \ll \varphi$ (resp. $\psi \prec \varphi$).

(2) Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences in $S(A)$ with $\|\varphi_n - \varphi\| \to 0$ and $\|\psi_n - \psi\| \to 0$. If $\varphi_n \perp \psi_n$ (resp. $\varphi_n \mathbin{\rotatebox[origin=c]{180}{$\perp$}} \psi_n$) for all n, then $\varphi \perp \psi$ (resp. $\varphi \mathbin{\rotatebox[origin=c]{180}{$\perp$}} \psi$).

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