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# A SPECTRAL CHARACTERIZATION OF MONOTONICITY PROPERTIES OF NORMAL LINEAR OPERATORS WITH AN APPLICATION TO NONLINEAR TELEGRAPH EQUATION

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## 1.

Let $H$ be a real Hilbert space and $T: H \supset \mathrm{dom}(T) \to H$ be a densely defined, closed linear operator with $\mathrm{ran}(T) = \ker(T)^{\perp}$. The inverse $T^{-1}$ of $T \mid (\mathrm{dom}(T) \cap \mathrm{ran}(T))$ is a bounded linear operator on $\mathrm{ran}(T)$ according to the open mapping theorem, hence

$$\alpha(T) := \sup\{c \mid c \in \mathbf{R}_+ \setminus \{0\}, \forall\, x \in \mathrm{dom}(T) : \langle Tx, x\rangle \geqslant -c^{-1}\|Tx\|^2\}$$

is a positive real number or $\infty$.

The interest in $\alpha(T)$ arises from the important paper of Brezis and Nirenberg [3] on semilinear problems at resonance, where this quantity serves as a measure of monotonicity for the linear part. Indeed we have $\alpha(T) = \infty$, iff $T$ is monotone, and in case of $\alpha(T) < \infty$ it is the largest positive number, for which $T^{-1} + c^{-1}\mathrm{Id}_{\mathrm{ran}(T)}$ is monotone on $\mathrm{ran}(T)$.

If $T$ is selfadjoint and $\alpha(T) < \infty$, one readily sees that $-\alpha(T)$ is the largest spectral value of $T$ less than 0. This describes the relation between [3] and other papers (e.g. [1, 6, 7]) on resonance problems, which start from spectral properties of the linear part. Moreover, such a characterization turns out to be quite useful for determining $\alpha(T)$ in applications, since the spectrum of various differential operators can be found in the literature. Here we treat the case where $T$ is normal. A nonselfadjoint, normal linear operator is induced e.g. by the telegraph operator. Of course this demands to include the "complex part" of the spectrum of $T$ into consideration. To this end we denote the complexification of $H$ by $H_{\mathbf{C}}$, the elements of which are written as $x + \mathrm{i}y$ $(x, y \in H)$. Thus the linear structure is given by $u + v := (x_u + x_v) + \mathrm{i}(y_u + y_v)$ and $\xi u := (\lambda x_u - \mu y_u) + \mathrm{i}(\lambda y_u + \mu x_u)$ for $u = x_u + \mathrm{i}y_u$ $(x_u, y_u \in H)$, $v = x_v + \mathrm{i}y_v$ $(x_v, y_v \in H)$ and $\xi = \lambda + \mathrm{i}\mu$ $(\lambda, \mu \in \mathbf{R})$. The inner product is defined by

$$\langle u, v\rangle = \langle x_u, x_v\rangle + \langle y_u, y_v\rangle + \mathrm{i}(\langle y_u, x_v\rangle - \langle x_u, y_v\rangle).$$

Moreover, we associate a complex linear operator $A_C: H_C \supset \operatorname{dom}(A_C) \to H_C$ with any linear operator $A: H \supset \operatorname{dom}(A) \to H$ by setting $\operatorname{dom}(A_C) := \{x + iy \mid x, y \in \operatorname{dom}(A)\}$ and $A_C(x + iy) = Ax + iAy$ for $x + iy \in \operatorname{dom}(A_C)$. In particular, $A$ is normal, iff $A_C$ has this property, and $\operatorname{ran}(A) = \ker(A)^{\perp}$ implies $\operatorname{ran}(A_C) = \ker(A_C)^{\perp}$.

THEOREM 1. *Let $H$ be an (infinite dimensional) real Hilbert space and $T: H \supset \operatorname{dom}(T) \to H$ be a normal linear operator with closed range, then*

$$\alpha(T) = \inf\{|\xi|^2/(-\operatorname{Re}\xi) \mid \xi \in \sigma(T_C), \operatorname{Re}\xi < 0\}.$$

Here $\sigma(T_C)$ denotes the spectrum of $T_C$ and Re $\xi$ (Im $\xi$) the real (imaginary) part of $\xi$. Moreover, the infimum is understood to be $\infty$, if there is no $\xi \in \sigma(T_C)$ with $\operatorname{Re}\xi < 0$. The geometrical interpretation of Theorem 1 for $\alpha(T) < \infty$ is that $\alpha(T)$ is the diameter of the largest disk in the complex plane with center on the negative real axis, containing 0 on its boundary and no spectral value of $T_C$ in its interior.

In order to deal with double resonance at 0 and $-\alpha(T)$ ($\alpha(T) < \infty$), Berestycki and de Figueiredo [2] need a further monotonicity property, namely that $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\operatorname{ran}(T)}$ is strictly monotone on $\operatorname{ran}(T)$. Recall that a bounded linear operator $A: Y \to Y$ on a closed subspace $Y$ of $H$ is said to be *strictly monotone*, iff there is a constant $c > 0$ with $\langle Ay, y\rangle \geqslant c\|Ay\|^2$ for all $y \in Y$. As indicated in [2], one knows for selfadjoint $T$ that $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\operatorname{ran}(T)}$ is strictly monotone.

THEOREM 2. *Let $H$ be an (infinite dimensional) real Hilbert space and $T: H \supset \operatorname{dom}(T) \to H$ be a normal linear operator with closed range and $\alpha(T) < \infty$. Moreover assume that $-\alpha(T)$ is an isolated point of $\sigma(T_C)$ or a regular value of $T_C$.*

*Then $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\operatorname{ran}(T)}$ is strictly monotone iff the set $\{\xi \mid \xi \in \sigma(T_C)\setminus\mathbf{R}, |\xi|^2 + \alpha(T)\operatorname{Re}\xi = 0\}$ is empty.*

REMARKS. (1). Theorem 2 has the following geometrical interpretation: $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\operatorname{ran}(T)}$ is strictly monotone, iff the closed disk with center $-\alpha(T)/2$ and radius $\alpha(T)/2$ contains no spectral values of $T_C$ apart from eventually 0 and $-\alpha(T)$.

(2). Though the notion "resonance problem" usually refers to situations where the spectrum of the linear part and the nonlinearity interact at an isolated eigenvalue, perhaps one wants to drop this restriction concerning $-\alpha(T)$ in Theorem 2. In fact, this is possible for the "only if" part, as we shall see in the next section. Here we outline a counterexample for the other direction:

Let $H = \ell_2$ and $(e_j)_{j \in \mathbf{N}}$ be the standard basis of $H$. We set $Y_j := \operatorname{span}\{e_{2j-1}, e_{2j}\}$

for $j \in \mathbf{N}$ and define a linear operator $T_j \colon Y_j \to Y_j$ for $j \in \mathbf{N} \setminus \{1\}$ by:

$$T_j e_{2j-1} = \left(-1 + \frac{1}{j}\right) e_{2j-1} + e_{2j}/\sqrt{j}$$

$$T_j e_{2j} = -e_{2j-1}/\sqrt{j} + \left(-1 + \frac{1}{j}\right) e_{2j}.$$

If $P_j \colon H \to H$ denotes the orthogonal projection on $Y_j$ for $j \in \mathbf{N}$, we set $Tx = \sum_{j=2}^{\infty} T_j \circ P_j x$ for $x \in H$ and obtain a normal, bounded linear operator on $H$. We get with $\xi_j := -1 + \frac{1}{j} + i/\sqrt{j}$ for $j \in \mathbf{N} \setminus \{1\}$:

$$\sigma(T_{\mathbf{C}}) = \{0, -1\} \cup \{\xi_j \mid j \in \mathbf{N} \setminus \{1\}\} \cup \{\bar{\xi}_j \mid j \in \mathbf{N} \setminus \{1\}\}.$$

Hence Theorem 1 yields $\alpha(T) = 1$. Now let $L := T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\mathrm{ran}(T)}$ (recall $T^{-1} := [T \mid (\mathrm{dom}(T) \cap \mathrm{ran}(T)]^{-1}$), then $L_{\mathbf{C}}$ is equal to $T_{\mathbf{C}}^{-1} + \mathrm{Id}_{\mathrm{ran}(T_{\mathbf{C}})}$ and $1 + \xi_j^{-1}$ is an eigenvalue of $L_{\mathbf{C}}$ for $j \in \mathbf{N} \setminus \{1\}$. Choose $x_j, y_j \in \mathrm{ran}(T_{\mathbf{C}})$ for $j \in \mathbf{N} \setminus \{1\}$ such that $\|x_j\|^2 + \|y_j\|^2 = 1$ and $L_{\mathbf{C}}(x_j + iy_j) = (1 + \xi_j^{-1})(x_j + iy_j)$ hold. One obtains for $j \in \mathbf{N} \setminus \{1\}$ (cf. the following section):

$$\langle Lx_j, x_j \rangle + \langle Ly_j, y_j \rangle = \mathrm{Re}(\langle L_{\mathbf{C}}(x_j + iy_j), x_j + iy_j \rangle) = \mathrm{Re}(1 + \xi_j^{-1}),$$

$$\|Lx_j\|^2 + \|Ly_j\|^2 = \|L_{\mathbf{C}}(x_j + iy_j)\|^2 = |1 + \xi_j^{-1}|^2.$$

Now suppose $L$ to be strictly monotone, then we find a $c > 0$ with $\mathrm{Re}(1 + \xi_j^{-1}) \geq c|1 + \xi_j^{-1}|^2$ for $j \in \mathbf{N} \setminus \{1\}$, which implies the boundedness of the sequence $(|1 + \xi_j^{-1}|^2/\mathrm{Re}(1 + \xi_j^{-1}))$. This contradicts

$$|1 + \xi_j^{-1}|^2/\mathrm{Re}(1 + \xi_j^{-1}) = (1 + 2\mathrm{Re}\,\xi_j + |\xi_j|^2)/(|\xi_j|^2 + \mathrm{Re}\,\xi_j) =$$

$$= 1 + (1 + \mathrm{Re}\,\xi_j)/(|\xi_j|^2 + \mathrm{Re}\,\xi_j) = 1 + j$$

for $j \in \mathbf{N} \setminus \{1\}$.

## 2.

Here we give the proofs of Theorems 1 and 2. Throughout $H$ denotes a real infinite dimensional Hilbert space and $T \colon H \supset \mathrm{dom}(T) \to H$ a normal linear operator with closed range.

$\xi \in \mathbf{C}$ belongs to $\sigma(T_{\mathbf{C}})$, iff there is a sequence $(u_j)$ in $\mathrm{dom}(T_{\mathbf{C}})$ with $\|u_j\| = 1$ and $\|T_{\mathbf{C}} u_j - \xi u_j\| \to 0$. Later we need the following fact:

LEMMA. *Let* $\xi \in \sigma(T_C)$ *and* $(u_j) = (x_j + iy_j)$ *be a sequence in* $\mathrm{dom}(T_C)$ *with* $\|u_j\| = 1$ *and* $\|T_C u_j - \xi u_j\| \to 0$, *then we have:*

(i) $\langle Tx_j, x_j \rangle + \langle Ty_j, y_j \rangle \to \mathrm{Re}\,\xi$;

(ii) $\|Tx_j\|^2 + \|Ty_j\|^2 \to |\xi|^2$.

*Proof.* Obviously we have $\langle T_C u_j - \xi u_j, u_j \rangle \to 0$, hence $\langle T_C u_j, u_j \rangle \to \xi$. Observing

$$\langle T_C u_j, u_j \rangle = \langle Tx_j, x_j \rangle + \langle Ty_j, y_j \rangle + i(\langle Ty_j, x_j \rangle - \langle Tx_j, y_j \rangle)$$

for $j \in \mathbf{N}$, we get (i). Using the boundedness of $(\|T_C u_j\|)$ and

$$\langle T_C u_j - \xi u_j, T_C u_j \rangle = \|Tx_j\|^2 + \|Ty_j\|^2 - \xi \langle u_j, T_C u_j \rangle$$

for $j \in \mathbf{N}$, we obtain (ii).

We want to appeal to the spectral theorem for normal linear operators and refer to [8] for the approach, we base upon, and concerning terminology. In particular, resolutions of the identity of $H_C$ are specific mappings from a $\sigma$-algebra into the subset of orthogonal projectors of $[H_C]$, the Banach space of bounded linear operators on $H_C$. If $E$ is a resolution of the identity of $H_C$ and $u, v \in H_C$, $D \mapsto \langle E(D)u, v \rangle$ for $D \in \mathrm{dom}(E)$ is a complex measure, which is denoted by $E_{u,v}$. Let $M \subset \mathbf{C}$ be a Borel set, then in the sequel $\mathfrak{B}(M)$ means the subset of all Borel sets of $M$, and we write $\int_M f(\lambda)\, dE_{u,v}(\lambda)$ for $u, v \in H_C$, $E: \mathfrak{B}(M) \to [H_C]$ a resolution of the identity and $f: M \to \mathbf{C}$ Borel measurable, provided this integral exists.

*Proof of Theorem 1.* Let $E: \mathfrak{B}(\sigma(T_C)) \to [H_C]$ be the resolution of the identity, associated with $T_C$ according to the spectral theorem [8; 13.33]. We have for $x \in \mathrm{dom}(T)$:

$$\langle Tx, x \rangle = \frac{1}{2}[\langle T_C x, x \rangle + \langle T_C^* x, x \rangle] =$$

$$= \frac{1}{2}\left[\int_{\sigma(T_C)} \xi\, dE_{x,x}(\xi) + \int_{\sigma(T_C)} \bar{\xi}\, dE_{x,x}(\xi)\right] =$$

$$= \int_{\sigma(T_C)} \mathrm{Re}\,\xi\, dE_{x,x}(\xi)$$

($T_C^*$ denotes the adjoint of $T_C$). Now let $\gamma \in \mathbf{R}_+$ with $\gamma \leqslant \inf\{|\xi|^2/(-\mathrm{Re}\,\xi) \mid \xi \in \sigma(T_C), \mathrm{Re}\,\xi < 0\}$, then $|\xi|^2 + \gamma\,\mathrm{Re}\,\xi \geqslant 0$ for all $\xi \in \sigma(T_C)$ and we have, since $E_{x,x}$ is a positive measure, that

$$\int_{\sigma(T_C)} (|\xi|^2 + \gamma\,\mathrm{Re}\,\xi)\, dE_{x,x}(\xi) \geqslant 0,$$

which yields $\|Tx\|^2 + \gamma\langle Tx, x\rangle \geqslant 0$ for $x \in \mathrm{dom}(T)$. This shows

$$\alpha(T) \geqslant \inf\{|\xi|^2/(-\operatorname{Re}\xi) \mid \xi \in \sigma(T_{\mathbf{C}}),\ \operatorname{Re}\xi < 0\}.$$

On the other hand let $\xi \in \sigma(T_{\mathbf{C}})$ with $\operatorname{Re}\xi < 0$ and choose sequences $(x_j), (y_j)$ in $\mathrm{dom}(T)$ according to the previous Lemma such that $\|x_j\|^2 + \|y_j\|^2 = 1$ for $j \in \mathbf{N}$, $\langle Tx_j, x_j\rangle + \langle Ty_j, y_j\rangle \to \operatorname{Re}\xi$ and $\|Tx_j\|^2 + \|Ty_j\|^2 \to |\xi|^2$. We have for $\gamma \in (0, \infty)$ with $\gamma \leqslant \alpha(T)$ and $j \in \mathbf{N}$:

$$\langle Tx_j, x_j\rangle \geqslant -\gamma^{-1}\|Tx_j\|^2, \quad \langle Ty_j, y_j\rangle \geqslant -\gamma^{-1}\|Ty_j\|^2.$$

Thus $|\xi|^2 + \gamma\operatorname{Re}\xi \geqslant 0$ holds, which shows $\gamma \leqslant |\xi|^2/(-\operatorname{Re}\xi)$, hence $\alpha(T) \leqslant$ $\leqslant |\xi|^2/(-\operatorname{Re}\xi)$ for all $\xi \in \sigma(T_{\mathbf{C}})$ with $\operatorname{Re}\xi < 0$.

The "only if" part of Theorem 2 is a special case of:

THEOREM 2′. *Let $\alpha(T) < \infty$ and $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\mathrm{ran}(T)}$ be strictly monotone, then $\{\xi \mid \xi \in \sigma(T_{\mathbf{C}}), |\xi|^2 + \alpha(T)\operatorname{Re}\xi = 0,\ \operatorname{Im}\xi \neq 0\}$ is empty.*

*Proof.* Suppose there is a $\xi \in \sigma(T_{\mathbf{C}}) \setminus \mathbf{R}$ with $|\xi|^2 + \alpha(T)\operatorname{Re}\xi = 0$. Set $Y =$ $= \mathrm{ran}(T)$ and $S := T^{-1} + \alpha(T)^{-1}\mathrm{Id}_Y$, then $S$ is a normal, bounded linear operator and $\sigma(S_{\mathbf{C}}) = \{\lambda^{-1} + \alpha(T)^{-1} \mid \lambda \in \sigma(T_{\mathbf{C}}) \setminus \{0\}\}$ holds because of $S_{\mathbf{C}} = T_{\mathbf{C}}^{-1} +$ $+ \alpha(T)^{-1}\mathrm{Id}_{\mathrm{ran}(T_{\mathbf{C}})}$. Applying the Lemma to $S$, $Y$ and $\xi^{-1} + \alpha(T)^{-1}$, we get sequences $(x_j), (y_j) \in Y^{\mathbf{N}}$ with $\|x_j\|^2 + \|y_j\|^2 = 1$, $\langle Sx_j, x_j\rangle + \langle Sy_j, y_j\rangle \to \operatorname{Re}(\xi^{-1} + \alpha(T)^{-1})$ and $\|Sx_j\|^2 + \|Sy_j\|^2 \to |\xi^{-1} + \alpha(T)^{-1}|^2$. Since $S$ is strictly monotone, there is a $c > 0$ with $\langle Sz, z\rangle \geqslant c\|Sz\|^2$ for $z \in Y$. This implies $\operatorname{Re}(\xi^{-1} + \alpha(T)^{-1}) \geqslant c\,|\,\xi^{-1} +$ $+ \alpha(T)^{-1}|^2$, which leads to $\alpha(T)|\xi|^2\operatorname{Re}(\alpha(T)\bar{\xi} + |\xi|^2) \geqslant c\,|\,\alpha(T)\bar{\xi} + |\xi|^2|^2$, hence to $0 \geqslant c|\operatorname{Im}\xi|^2$ because of $|\xi|^2 + \alpha(T)\operatorname{Re}\xi = 0$. This contradicts $\operatorname{Im}\xi \neq 0$.

Instead of proving the "if" part of Theorem 2, we derive:

THEOREM 2″. *Let $\alpha(T) < \infty$ and suppose that there are $\varkappa, \rho \in (0, \infty)$ with*

$$(*) \qquad (\operatorname{Im}\xi)^2 \leqslant \varkappa|\xi|^2(\alpha(T)\operatorname{Re}\xi + |\xi|^2)$$

*for all $\xi \in \sigma(T_{\mathbf{C}})$, satisfying $\alpha(T)\operatorname{Re}\xi + |\xi|^2 \leqslant \rho$. Then $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\mathrm{ran}(T)}$ is strictly monotone.*

In order to see that the "if" part of Theorem 2 is a special case of Theorem 2″, one observes that $-\alpha(T)$ and $0$ are no accumulation points of $\sigma(T_{\mathbf{C}})$. This holds for $-\alpha(T)$ by assumption and for 0, since $T$ is a normal linear operator with closed range. Thus $\sigma(T_{\mathbf{C}}) \setminus \{-\alpha(T), 0\}$ is closed and disjoint to the closed disk $\{\xi \mid \xi \in \mathbf{C}, |\xi|^2 + \alpha(T)\operatorname{Re}\xi \leqslant 0\}$, hence there is some $\rho > 0$ with $\sigma(T_{\mathbf{C}}) \cap \{\xi \mid \xi \in \mathbf{C},$ $|\xi|^2 + \alpha(T)\operatorname{Re}\xi \leqslant \rho\} \subseteq \{-\alpha(T), 0\}$, which yields $(*)$ for any $\varkappa \in (0, \infty)$. More general, $(*)$ also holds in case that $\{\xi \mid \xi \in \mathbf{C}, |\xi|^2 + \alpha(T)\operatorname{Re}\xi \leqslant 0\} \cap (\sigma(T_{\mathbf{C}}) \setminus \mathbf{R}) = \emptyset$

and there is some neighborhood $U$ of $-\alpha(T)$ in $\mathbf{C}$ such that $\xi \in \sigma(T_{\mathbf{C}}) \cap U$ implies $\operatorname{Re}\xi \leqslant -\alpha(T)$. Obviously this condition is satisfied if $T$ is selfadjoint.

*Proof of Theorem* 2''. Theorem 1 and (*) yield $\sigma(T_{\mathbf{C}}) \cap \{\xi \mid \xi \in \mathbf{C},\ |\xi|^2 + \alpha(T)\operatorname{Re}\xi \leqslant 0\} \subset \{0, -\alpha(T)\}$, which shows $\operatorname{Re}(\xi^{-1} + \alpha(T)^{-1}) > 0$ for $\xi \in \omega := \sigma(T_{\mathbf{C}}) \setminus \{0, -\alpha(T)\}$. We understand $E$ and $S$, $Y$ as in the proofs of Theorem 1 and 2', respectively, and set $F(D) := E(D) \mid Y_{\mathbf{C}}$ for $D \in \mathfrak{B}(\tau)$, where $\tau$ denotes $\sigma(T_{\mathbf{C}}) \setminus \{0\}$. We obtain in view of

$$\operatorname{ran}(E(\{0\})) = \ker(T_{\mathbf{C}}) = \operatorname{ran}(T_{\mathbf{C}})^{\perp} = Y_{\mathbf{C}}^{\perp}$$

that $F$ is a resolution of the identity of $Y_{\mathbf{C}}$, associated with $T_{\mathbf{C}} \mid (\operatorname{dom}(T_{\mathbf{C}}) \cap Y_{\mathbf{C}})$. Hence we have for $u, v \in Y_{\mathbf{C}}$:

$$\langle S_{\mathbf{C}}u, v\rangle = \int_{\tau} (\xi^{-1} + \alpha(T)^{-1})\, \mathrm{d}F_{u,v}(\xi)$$

$$\langle S_{\mathbf{C}}^{*}u, v\rangle = \int_{\tau} (\bar{\xi}^{-1} + \alpha(T)^{-1})\, \mathrm{d}F_{u,v}(\xi).$$

This yields for $y \in Y$ because of $\langle Sy, y\rangle = \frac{1}{2}(\langle S_{\mathbf{C}}y, y\rangle + \langle S_{\mathbf{C}}^{*}y, y\rangle)$ and in view of $\omega = \tau \setminus \{-\alpha(T)\}$:

$$\langle Sy, y\rangle = \int_{\tau} \operatorname{Re}(\xi^{-1} + \alpha(T)^{-1})\, \mathrm{d}F_{y,y}(\xi) =$$

$$= \int_{\omega} \operatorname{Re}(\xi^{-1} + \alpha(T)^{-1})\, \mathrm{d}F_{y,y}(\xi).$$

On the other hand we have for $y \in Y$:

$$\|Sy\|^2 = \|S_{\mathbf{C}}y\|^2 = \int_{\tau} |\xi^{-1} + \alpha(T)^{-1}|^2\, \mathrm{d}F_{y,y}(\xi) =$$

$$= \int_{\omega} |\xi^{-1} + \alpha(T)^{-1}|^2\, \mathrm{d}F_{y,y}(\xi).$$

Hence it suffices to derive the boundedness of

$$\theta(\xi) := |\xi^{-1} + \alpha(T)^{-1}|^2 / \operatorname{Re}(\xi^{-1} + \alpha(T)^{-1})$$

on $\omega$, in order to finish the proof. An easy calculation yields for $\xi \in \omega$:

$$\theta(\xi) = \alpha(T)^{-1} + |\xi|^{-2}\mathrm{Re}\,\xi + \alpha(T)[|\xi|^2(|\xi|^2 + \alpha(T)\mathrm{Re}\,\xi)]^{-1}(\mathrm{Im}\,\xi)^2.$$

This implies $\theta(\xi) \leqslant \alpha(T)^{-1} + |\xi|^{-1} + \alpha(T)\rho^{-1}$ for $\xi \in \omega$ with $|\xi|^2 + \alpha(T)\mathrm{Re}\,\xi > \rho$ and with regard to (*) $\theta(\xi) \leqslant \alpha(T)^{-1} + |\xi|^{-1} + \alpha(T)\varkappa$ for $\xi \in \omega$ with $|\xi|^2 + \alpha(T)\mathrm{Re}\,\xi \leqslant \rho$. Hence $0 \notin \bar{\omega}$ delivers the boundedness of $\theta$ on $\omega$.

## 3.

In order to indicate that the previous results may be useful for investigating resonance problems, we give here an application of [2; Theorem 2] to a generalized nonlinear telegraph equation in case that double resonance occurs. We refer to [3; Chapter V.2] for the corresponding simple resonance problem. Of course, the rather modest spectral structure of the linear telegraph operator does not ask for the general approach of Section 1, but it was our main goal to understand the hypotheses of [2, 3], involving $\alpha(T)$, as to their spectral meaning. This, combined with the procedure of [7], will allow an "a priori" bound assertion, similar to that of [2; Theorem 2], for semilinear operator equations in an $L_2$-space over a $\sigma$-finite measure space without any compactness assumption. Details will appear elsewhere.

Here we are looking for a generalized, $2\pi$-time periodic solution of the semilinear "telegraph equation"

(TE) $$\partial_2^2 u(x, t) + Lu(x, t) + a\partial_2 u(x, t) + g(x, t, u(x, t)) = f(x, t)$$

on $\Omega \times \mathbf{R}$ under the following hypotheses:

(I) $\Omega$ is a bounded domain in $\mathbf{R}^n, a \in \mathbf{R}\setminus\{0\}, f\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies $f\,|\,\Omega \times (0, 2\pi) \in L_2(\Omega \times (0, 2\pi))$ and $f(x, \cdot)$ $2\pi$-periodic for $x \in \Omega$.

(II) $L\colon L_2(\Omega) \supset \mathrm{dom}(L) \to L_2(\Omega)$ is linear, selfadjoint and has a compact resolvent (e.g. a symmetric, elliptic differential operator on $\Omega$, subjected to appropriate boundary conditions). $0 \in \sigma(L)$.

(III) $g\colon \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (i.e. $(x, t) \mapsto g(x, t, y)$ is measurable on $\Omega \times \mathbf{R}$ for $y \in \mathbf{R}$ and $y \mapsto g(x, t, y)$ is continuous on $\mathbf{R}$ for $(x, t) \in \Omega \times \mathbf{R}$ a.e.), which fulfils $g(x, \cdot, y)$ $2\pi$-periodic for $(x, y) \in \Omega \times \mathbf{R}$ and the growth condition: there are $b \in L_2(\Omega \times \mathbf{R})$ and $c \in (0, \infty)$ with $|g(x, t, y)| \leqslant b(x, t) + c|y|$ for $(x, t) \in \Omega \times \mathbf{R}$ a.e. and $y \in \mathbf{R}$.

Following [3; Chapter V.2] we call the $2\pi$-periodic extension of any $u \in H := L_2(\Omega \times (0, 2\pi))$ a generalized solution of (TE) iff $u$ satisfies the operator equation

(OE) $$Tu + Bu = f\,|\,(\Omega \times (0, 2\pi))$$

where $T$ is the linear "telegraph operator" in $H$, associated with $L$ and $a$, and $B\colon H \to H$ is given by $(Bu)(x, t) = g(x, t, u(t, x))$ for $u \in H$, $x \in \Omega$ and $t \in (0, 2\pi)$.

In order to introduce $T$, we denote by $(\mu_j)_{j\in J}$ the nondecreasing sequence of all eigenvalues of $L$, counted according multiplicity, with either $J = \mathbf{Z}$ or for some $1 \in \mathbf{Z}_+ J = \{j \mid j \in \mathbf{Z},\ j \geq -1\}$ or $J = \{j \mid j \in \mathbf{Z},\ j \leq 1\}$ and with $\mu_j < 0$ iff $j < 0$. Moreover, let $(w_j)_{j\in J}$ be an orthonormal basis of $L_2(\Omega)$ with $w_j \in \mathrm{dom}(L)$ for $j \in J$ and $Lw_j = \mu_j w_j$ for $j \in J$ and set $v_{k,j}(x,t) = \frac{1}{2\pi} w_j(x)\exp(ikt)$ for $x \in \Omega$, $t \in (0, 2\pi)$, $k \in \mathbf{Z}$ and $j \in J$. Then $T\colon H \supset \mathrm{dom}(T) \to H$ is defined by $\mathrm{dom}(T) := \{u \mid u \in H, \sum_{j\in J}\sum_{k\in\mathbf{Z}} |\mu_j - k^2 + iak|^2 |\langle u, v_{k,j}\rangle|^2 < \infty\}$ and $Tu := \sum_{j\in J}\sum_{k\in\mathbf{Z}} (\mu_j - k^2 + iak)\langle u, v_{k,j}\rangle v_{k,j}$ for $u \in \mathrm{dom}(T)$.

It is well-known that $T$ is a normal linear operator with compact resolvent, and that $\sigma(T_{\mathbf{C}}) = \{\mu_j - k^2 + iak \mid j \in J,\ k \in \mathbf{Z}\}$. Hence Theorem 1 or the direct computation of [3] yield

$$\alpha(T) = \inf\left\{k^2 - \mu_j + \frac{a^2k^2}{k^2 - \mu_j} \;\middle|\; j \in J, k \in \mathbf{Z}, \mu_j < k^2\right\}.$$

Moreover, we have according to Theorem 2 that $T^{-1} + \alpha(T)^{-1}\mathrm{Id}_{\mathrm{ran}(T)}$ is strictly monotone, iff

(IV)

$$-1 \in J \text{ and } |\mu_{-1}| < \inf\left\{k^2 - \mu_j + \frac{a^2k^2}{k^2 - \mu_j} \;\middle|\; (j,k) \in (J\times\mathbf{Z})\setminus\{(-1,0)\},\ \mu_j < k^2\right\}$$

holds. For example, if $\sigma(L)$ contains no positive number, $a^2 > |\mu_{-1}| - 1$ is a sufficient condition for (IV), while $-1 \in J$ and

$$a^2 > \max\{|\mu_{-1}| - 1,\ |\mu_{-1}| + 2\mu_m - 2\mu_m^{1/2}(|\mu_{-1}| + \mu_m)^{1/2}\}$$

is a sufficient one in case that there are positive eigenvalues of $L$ and $m := \min\{j \mid j \in J, \mu_j > 0\}$.

We set $\gamma_\pm(x,t) := \varliminf_{y\to\pm\infty} g(x,t,y)/y$ and $\Gamma_\pm(x,t) = \varlimsup_{y\to\pm\infty} g(x,t,y)/y$ for $(x,t) \in \Omega\times(0,2\pi)$, denote by $w^+$ and $w^-$ the positive respectively negative part of $w \in L_2(\Omega)$, take $w(x,t) \equiv 0$ for $w \in L_2(\Omega\times(0,2\pi))$ in the a.e. sense on $\Omega\times(0,2\pi)$ and write Id for $\mathrm{Id}_{L_2(\Omega)}$.

Now, observing $\ker(T - \lambda\,\mathrm{Id}_H) = \{(x,t) \mapsto w(x) \mid w \in \ker(L - \lambda\,\mathrm{Id})\}$ for $\lambda \in \sigma(T)$ and $\alpha(T) = -\mu_{-1}$, if (IV) holds, we obtain by applying [2; Theorem 2] to (OE):

THEOREM 3. *Let* (I)–(IV) *be satisfied and* $\gamma_\pm(x,t)$ *as well as* $\Gamma_\pm(x,t)$ *belong to* $[0, -\mu_{-1}]$ *for* $(x,t) \in \Omega\times(0,2\pi)$ *a.e.. Moreover, assume* $\gamma_+(x,t)w^+(x) +$

$+ \gamma_-(x, t)\, w^-(x) \not\equiv 0$ *for any* $w \in \ker(L) \setminus \{0\}$ *and* $[-\mu_{-1} - \Gamma_+(x, t)]w^+(x) + [-\mu_{-1} - \Gamma_-(x, t)]\, w^-(x) \not\equiv 0$ *for any* $w \in \ker(L - \mu_{-1}\mathrm{Id})$, *then* (TE) *has a generalized solution.*

Earlier results [3, 4, 5] do not work in this situation. In order to give an example for the applicability of Theorem 3, let us consider the very simple model equation

$$\partial_2^2 u(x, t) - \partial_1^2 u(x, t) - 9u(x, t) + a\partial_2 u(x, t) +$$

$$+ 5u(x, t) \exp(u(x, t))/[1 + \exp(u(x, t))] = f(x, t)$$

fo $(x, t) \times (0, \pi) \times \mathbf{R}$ under the boundary condition $u(0, t) = 0 = u(\pi, t)$ for $t \in \mathbf{R}$. Theorem 3 yields the existence of a generalized, $2\pi$-time periodic solution, provided $|a| > 2$ holds.

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