J. OPERATOR THEORY
11(1984), 365–377

ON OPERATORS WHICH ALMOST COMMUTE WITH THE SHIFT

I. D. BERG

INTRODUCTION

The aim of this paper is to determine when weighted shifts on ℓ_2 which almost commute with the canonical left shift can be perturbed so as to commute with a perturbation of the left shift.

Our analysis encompasses weighted right and left shifts as well as their products, including, for example, any multiplication operator. We are interested both in questions of compact perturbations and pertubations of small norm. It turns out that some of the wonted parallelism between compact perturbations and small norm perturbations breaks down, itself a phenomeon which is of quite some interest.

There has been a great deal of interest in the general phenomenon of lifting from the Calkin algebra, most of it tending towards an abstract approach. These abstractions have been interesting; it is possible that in general they are actually unavoidable because of the non-constructive nature of the realization of the Calkin algebra as an operator algebra. However, it is satisfying to find a situation that yields to constructive methods. Our entire analysis is in the context of $\mathscr{B}(\mathscr{H})$ where $\mathscr{H} = \ell_2$.

We are able to present an essentially complete analysis in the case of shifts with real weights. There are phenomena in the complex case which still elude us. We should mention immediately that the perturbations contemplated do not leave either operator as a weighted shift; commutativity of an operator with a weighted shift is a triviality.

We are concerned with shifts with real weights on ℓ_2. We denote the canonical basis vector of ℓ_2 as $\{\delta_i\}$, $i = 1, 2, \ldots$. We define the left shift of index r with weights t_n by

1) $T(\delta_n) = t_n\delta_{n-r}$ for $n > r$

2) $T(\delta_n) = 0$ for $n \leqslant r$.

We will usually suppress the obvious condition 2 when describing a shift. Contrary to the usual convention we will allow weights of 0. We also consider the multiplicative operator, $r = 0$, as an allowable left shift.

The *canonical* left shift S will be the shift of index 1 (nullity 1, deficiency 0) given by $t_n = 1$ for $n > 1$.

Right shifts will be defined correspondingly.

We note that because we are dealing with pairs of operators it is not possible to make the usual assumption that all weights are positive, since the unitary transformation that achieves this for one shift may be unsuitable for the other.

If T is a left shift of index $r \geqslant 0$ then

$$[T, S](\delta_n) = [TS - ST]\delta_n = \begin{cases} (t_{n-1} - t_n)\delta_{n-r-1} & \text{for } n > r+1 \\ 0 & \text{for } n \leqslant r+1. \end{cases}$$

If T is a right shift of index $-r < 0$ then

$$[T, S](\delta_n) = [TS - ST]\delta_n = \begin{cases} (t_{n-1} - t_n)\delta_{n+r-1} & \text{for } n \geqslant 2 \\ -t_1\delta_r & \text{for } n = 1. \end{cases}$$

In either case $[T, S]$ is compact if and only if $|t_n - t_{n-1}| \to 0$ and so it is clear that T need not be a compact perturbation of a constant shift for $[T, S]$ to be compact.

Our results are phrased in terms of a canonical left shift because the kernel is notationally convenient in proofs, but all the results extend by adjunction if right and left are interchanged.

Since any perturbation $T + \Delta T$ of an operator T which produces an infinite dimensional scalar direct summand of $T + \Delta T$ allows a unitary transformation of that perturbed operator which commutes with $S + \Delta S$, a corresponding perturbation of the left shift S, a commutativity attained by matching a scalar direct summand of $S + \Delta S$ with the non-scalar summand of T and vice-versa, it is essentially impossible to hope for meaningful conditions on T alone which prevent liftings to commutativity; we must look for conditions on T and S simultaneously beyond the trivial conditions on $[S, T]$ or equivalently for conditions on T which involve coordinate representation. This is in contradistinction to the problem of making $[T + \Delta T, S] = 0$ where, for example, a non-zero eigenvalue of $(T + \Delta T)^*$ prevents commutativity, and so if $T - 1$ were of index -1, then $T + \Delta T$ could not commute with S.

We will say that S and T can be *lifted* to commuting $\tilde{S}$ and $\tilde{T}$ if $\tilde{S} - S = \Delta S$ and $\tilde{T} - T = \Delta T$ are both compact. That is, we can lift the commuting images of S and T in the Calkin algebra to genuinely commuting operators. We will save words by occasionally saying S and T can be *lifted to commute*. Recall that the Calkin algebra is the quotient algebra of the bounded operators factored by the compact operators, $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

As usual we define the essential norm of T, $\|T\|_e$, as the norm of the projection of T in the Calkin algebra; that is, $\|T\|_e = \inf_{\Delta T \in \mathcal{K}} \|T + \Delta T\|$.

We are constantly confronted with estimating the norms of ΔS and ΔT such that $[S + \Delta S, T + \Delta T] = 0$. With apologies for introducing yet another bit of notation we will define the *joint commutation distance* of T and S, $J(T, S)$ by $J(T, S) = \inf \max(\|\Delta T\|, \|\Delta S\|)$ over all ΔT, ΔS such that $[T + \Delta T, S + \Delta S] = 0$.

We define the *essential joint commutation distance*, $J_e[T, S]$, by $J_e[T, S] = \inf \max(\|\Delta T\|_e, \|\Delta S\|_e)$ over all $\Delta T, \Delta S$ such that $[T + \Delta T, S + \Delta S] = 0$. Note that $J_e[T, S]$ is not equivalent to $\|[T, S]\|_e$; even if $\|[T, S]\|_e = 0$ we will have $J_e[T, S] = 0$ only if T, S lift to commute, and there are examples where T, S do not so lift.

We show that for real weighted shifts for which there is no joint indicial obstruction of the type described in Berg-Olsen [3] there is a commuting lifting from the Calkin algebra achieved by an uncoupling construction of the form there described. Moreover, we show that this construction is qualitatively, not numerically, best possible in achieving the smallest norm perturbation accessible, in that the perturbation achieved by the construction is continuous in the least possible perturbation at 0. This is accomplished by showing that a shift T on which $[T, S]$ is small, yet on which uncoupling cannot yield a small commuting perturbation, necessarily has a relatively large $J[T, S]$ even though T, S may lift to commutativity. This, in turn, is achieved by considering a shift with initial weights which do not appear at ∞. This is more subtle than it appears; as will become evident, a shift with weights at some finite non-initial position bounded away from the weights at ∞ behaves quite differently.

Among those investigating similar lines we should mention K. Davidson. In particular in his investigation of essentially spectral operators [4] he is confronted with perturbing nilpotents and normals so as to commute and so there is some resemblance, though, so far as we can tell, no overlap with our work. We also acknowledge, with thanks, helpful comments of K. Davidson regarding this paper.

This paper follows leads opened in Berg-Olsen [3], and so several ideas originated from collaboration with C. L. Olsen.

§ 1

There are a few technical results which are useful in our analysis and which require some exposition.

The first matter with which we deal is the establishment of the technique of "orbit exchange" used for extracting direct summands which are scalar multiples of finite dimensional cyclic unitary shifts at the expense of a small compact perturbation with, moreover, stretches of original coordinates left unchanged in these summands.

That is, given a weighted shift, one can extract as a direct summand any section flanked both initially and terminally by stretches of the same weights. The perturbation required to do this changes only these initial and terminal segments, leaving the middle section unperturbed. The error incurred can be made as small as desired by using sufficiently long flanking segments, independent of the unchanged middle segment and the entries themselves outside these segments. Therefore this construction can be applied successively to uncouple segment after segment without increasing the norm of this perturbation. The section of shift preceding the construction is coupled to the section succeeding. The same exchange can be used to exchange orbits in direct sums of shifts as long as the exchange takes place where the weights of the two shifts being exchanged are the same.

The actual technique of the exchange procedure is given below. Because we only exchange sections of shift with sections of the same weight there is no loss of generality in assuming the weights are 1.

Suppose we have two sections of one shift, or, for that matter, two sections of distinct shifts:

$$\to \varphi_0 \to \varphi_1 \to \varphi_2 \to \dots \to \varphi_n \to \dots \qquad S_1$$

$$\to \psi_0 \to \psi_1 \to \psi_2 \to \dots \to \psi_n \to \dots \qquad S_2.$$

Consider the new basis for the space spanned by $\bigvee_0^n \varphi_i$ and $\bigvee_0^n \psi_i$ given by $\xi_0, \dots, \xi_n$ and $\omega_0, \dots, \omega_n$ and the new transformation $T_1 \oplus T_2$ given by

$$\xi_0 = \varphi_0,$$

$$\omega_0 = \psi_0,$$

$$\xi_i \xrightarrow{T_1} \alpha_{i+1}\varphi_{i+1} + \beta_{i+1}\psi_{i+1} = \xi_{i+1} \qquad i = 0, \dots, n-1,$$

$$\omega_i \xrightarrow{T_2} \rho_{i+1}[-\beta_{i+1}\varphi_{i+1} + \alpha_{i+1}\psi_{i+1}] = \omega_{i+1},$$

where $\alpha_{i+1}^2 + \beta_{i+1}^2 = 1$ and the α_i slowly tapers from 1 to 0, and the ρ_i slowly revolve over the unit circle from 1 to -1. Then

$$T_1^n\psi_0 = \varphi_n,$$

$$T_2^n\varphi_0 = \psi_n.$$

Moreover $T_1 \oplus T_2$ is approximately equal to $S_1 \oplus S_2$, with an error bounded by a constant times difference in successive α_i plus difference in successive ρ_i. Thus, by using a large enough space for the transformation the error can be made as small as desired. By using this procedure on a finite segment of a shift we pinch off direct summands. We can then work our way in, getting smaller summands. Of course, as the summands become of smaller dimension we must exchange more rapidly and hence acquire more error. Again, where we do not have constant weights we acquire an error in making the weights constant for a stretch.

An actual estimate on the bound we have by choosing α_i and ρ_i evenly spaced shows that if $|t_n - t_{n-1}| < 1/M^2 < 1/256$ for all coordinates involved in an exchange then the perturbation involved in an uncoupling on the stretch involved is bounded by $100/M$.

At the risk of appearing frivolous we write:

LEMMA 1.1. *Orbit exchange pinches off sections of a shift, with the same initial and terminal weights, of length at least* 16, *where successive weights differ by at most* $1/M^2$ *for* $M \geqslant 16$, *with an error bounded by* $100/M$.

We introduced this uncoupling procedure in [1], where we worked out the details, and used it again in [2]. The formal statements were a bit different in these contexts because we had different objectives but the procedure was the same. D. Herrero [5] has since given his own extensive developments and applications of this idea and J. Stampfli [6] has shown how to pinch out a single cycle with a one dimensional perturbation.

The next observation is almost algebraic and states that a perturbation of the shift has a subspace on which the perturbation looks so much like the shift that a commuting operator looks like a polynomial in the shift on this subspace.

LEMMA 1.2 (Polynomial subspace). *Let* S *be the unilateral left shift. Let* $\tilde{S} = S + \triangle S$ *where* $\|\triangle S\| < 1/16$. *Then there exists an infinite dimensional subspace,* $\tilde{\mathscr{H}}$, *invariant under* $\tilde{S}$ *and spanned by an infinite orthonormal sequence* $\{\varphi_i\}$ *such that*

1. $\tilde{S}^n(\varphi_n) = 0$;

2. $\liminf\langle S\varphi_n, \varphi_{n-1}\rangle \geqslant 3/4$.

If $[T, \tilde{S}] = 0$ *then*

3. $\tilde{\mathscr{H}}$ *is an invariant subspace for* T, *and on* $\tilde{\mathscr{H}}$ *we have* $T = \sum_{n=0}^{\infty} \alpha_n \tilde{S}^n$ *with strong convergence;*

4. $\langle T\varphi_n, \varphi_{n-r}\rangle = \langle \sum_{j=0}^{r} \alpha_j \tilde{S}^j \varphi_n, \varphi_{n-r}\rangle$ *for* $r \geqslant 0$.

Proof. The construction practically proves the lemma except for 2, which requires a bit more comment. Because $\|\triangle S\| < 1$ and S has a right inverse of norm 1 we see that $\tilde{S}$ is semi-Fredholm of nullity 1 and deficiency 0, and hence there is a unique φ_1 such that $\tilde{S}(\varphi_1) = 0$. Similarly there exists a unique φ_2 such that $\tilde{S}^2\varphi_2 = 0$ and φ_2 is orthogonal to φ_1 and so on. Because $\tilde{S}$ on $\tilde{\mathscr{H}}$ takes the form of a superdiagonal matrix with no zeros on the first superdiagonal and any commuting T has zeros under the diagonal since $\tilde{S}^n T\varphi_n = 0$, the commutativity relation for an $n \times n$ corner block involves only the $n \times n$ corner blocks of T.

Hence, since the Jordan form of a finite $n \times n$ corner of S is that of the canonical shift, we see that the $n \times n$ corner block of T is an n'th degree polynomial in S. Since higher degree terms in S cannot change the $n \times n$ corner block, we see that the first n degrees of the polynomial are fixed by the $n \times n$ block and in turn fix the first n superdiagonals of T. This establishes 3 and 4.

Now to prove 2 consider $\langle \varphi_n, \psi \rangle = 0$ for any unit $\psi \in \bigvee_{i=1}^{n-1} \varphi_i$. Then

$$\langle \tilde{S}\varphi_n, \tilde{S}\psi \rangle = \langle \varphi_n, \tilde{S}^*\tilde{S}\psi \rangle = \langle \varphi_n, (S + \Delta S)^*(S + \Delta S)\psi \rangle =$$

$$= \langle \varphi_n, [I - P_1 + (\Delta S^*)S + S^*\Delta S + |\Delta S|^2]\psi \rangle$$

where P_1 is projection on δ_1. Now

$$\langle \tilde{S}\varphi_n, \tilde{S}\psi \rangle = -\langle \varphi_n, P_1\psi \rangle + \langle \varphi_n, R\psi \rangle$$

where

$$\|R\| < 1/16 + 1/16 + (1/16)^2.$$

For large enough n we see that $\langle \varphi_n, P_1\psi \rangle < 1/16^2$ for any ψ and so $\langle \tilde{S}\varphi_n, \tilde{S}\psi \rangle < 3/16$.

Now we note that $\tilde{S}\psi$ includes the ball of radius 15/16 in $\bigvee_{i=1}^{n-2} \varphi_i$ and hence the projection of $\tilde{S}\varphi_n$ on $\bigvee_{i=1}^{n-2} \varphi_i$ is of norm at most $(16/15)(3/16) = 1/5$. But $\tilde{S}\varphi_n$ is itself of norm at least 15/16 and is in $\bigvee_{i=1}^{n-1} \varphi_i$ and so $|\langle \tilde{S}\varphi_n, \varphi_{n-1} \rangle|^2 + (1/5)^2 \geqslant 15/16$ and so $|\langle \tilde{S}\varphi_n, \varphi_{n-1} \rangle| \geqslant 3/4$. This establishes 2 and completes the proof of Lemma 1.2.

The next observation provides us with one of the few tools for showing that two operators cannot lift to commutativity. This is the joint indicial criterion developed in Berg-Olsen [3] (Proposition 3). We paraphrase the criterion here.

THEOREM 1.3 (Joint indicial criterion). *Let A be semi-Fredholm of non-zero index. Let $P(x, y)$ be a polynomial in two variables such that x factors $P(x, y)$. Let B, necessarily semi-Fredholm, be such that* 0 *is in the unbounded component of the essential resolvent of $P(A, B)$. Then* $J_e[A, B] \neq 0$.

This joint indicial criterion makes it immediate that if T is a right shift of index $-r$ with real weights bounded away from 0 and S is a left shift with similar weights then T and S cannot lift to commutativity because $(TS^r)^2$ has a positive essential spectrum.

We will give another proof that illustrates our approach to norm obstructions.

2. LIFTINGS

We start this section with the proof that shifts that suffer from our joint indicial obstruction cannot be lifted. We consider the canonical left shift but any left shift with positive weights bounded away from 0 would serve here. It may not be immediately apparent, but the proof we give here does really isolate the same phenomenon as that in Theorem 1.3, and Proposition 2 of [3] can be taken as its precursor.

THEOREM 2.1. *Let T be a right shift of index $-r$ with positive weights so that $1 > \liminf_n (T\delta_n) > \eta > 0$. Let S be the canonical unilateral left shift. Then T and S cannot be lifted to commute. Moreover $J[T, S] \geqslant J_e[T, S] > \eta/3$ for $r = 1$. In general $J[T, S] \geqslant J_e[T, S] > \eta/2^{r-1}\cdot 3$.*

Proof. Assume $\|\tilde{S} - S_e\|_e < \eta/3$ and $\|\tilde{T} - T\|_e < \eta/3$. Consider $\bigvee_{i=1}^{\infty} \varphi_i$ the subspace of successive kernels of $\tilde{S}$. If $[\tilde{T}, \tilde{S}] = 0$ then by Lemma 1.2 $\langle \tilde{T}\varphi_n, \varphi_n\rangle = \alpha$ for all n. But then $\langle \tilde{S}\tilde{T}\varphi_n, \varphi_n\rangle = \langle \alpha\tilde{S}\varphi_n, \varphi_n\rangle = 0$. Now we observe that ST is simply a multiplication operator and moreover $\langle ST\varphi_n, \varphi_n\rangle > \eta$ for large enough n (since for any finite set of δ_j we have φ_n as close to orthogonal to that set as needed). Hence $\langle (S + \Delta S)(T + \Delta T)\varphi_n, \varphi_n\rangle > \eta - 2\eta/3 - \eta^2/9 > 0$ which is impossible.

If $r > 1$ we consider $\langle \tilde{S}^r\tilde{T}\varphi_n, \varphi_n\rangle = 0$ where φ_n is any element in the n'th kernel of $\tilde{S}^r$. Even without invoking the polynomial subspace lemma it is clear that $\tilde{S}^r\tilde{T}\varphi_n$ is in $\bigvee_{j=1}^{n-1} (\tilde{S}^r)^{-j}(0)$, and so $\langle \tilde{S}^r\tilde{T}\varphi_n, \varphi_n\rangle = 0$. Once again $\langle S^rT\varphi_n, \varphi_n\rangle > \eta$ for large enough n and so if $|\Delta S|, |\Delta T| < \eta/3\cdot 2^{r-1}$ then

$$\langle (S + \Delta S)^r(T + \Delta T)\varphi_n, \varphi_n\rangle > \eta - \left(1 + \frac{\eta}{3\cdot 2^{r-1}}\right)\left[\binom{r}{1}\frac{\eta}{3\cdot 2^{r-1}} + \dots\right] >$$

$$> \eta - \left(1 + \frac{\eta}{3\cdot 2^{r}}\right)\left(2^r \frac{\eta}{3\cdot 2^{r-1}}\right) > 0,$$

where this remarkably unenlightening computation is put in merely to show good faith.

This completes the proof of Theorem 2.1.

Let us now present our promised positive results. The first theorem, which extends and explains Proposition 6 of Berg-Olsen [3] shows that if a real weighted left shift and the canonical left shift almost commute they can be lifted to commute, while if they have a small commutator and the initial weights reappear at ∞ they are close to commuting.

THEOREM 2.2. *Let S be the canonical left shift. Let T be a real weighted left shift of multiplicity $r \geqslant 0$ given by $T\delta_n = t_n\delta_{n-r}$. Then if $[T, S]$ is compact then T and S lift to commute. Further, there exists $\delta(r, \varepsilon) > 0$ independent of T such that if*

1) $[T, S] < \delta(r, \varepsilon)$,

2) $\liminf t_n < t_{r+1} < \limsup t_n$,

then $J[T, S] < \varepsilon$.

If $[T, S]$ is compact and 1) and 2) hold then there exist compact ΔT and ΔS such that $\|\Delta T\| < \varepsilon$, $\|\Delta S\| < \varepsilon$ and $[T + \Delta T,\ S + \Delta S] = 0$.

Proof. Assume first that $r = 1$, that is, T is a real weighted shift of index 1, and that $[T, S]$ is compact, that is simply that $\lim_n |t_n - t_{n-1}| = 0$. Then if α is any cluster point of the weights of T we attain arbitrarily long stretches of the weights of T as close as desired to α. Then, using the uncoupling procedure we split off finite dimensional cyclic direct summands, on each of which the weights are almost constant, leaving a shift which genuinely approaches α in weight. Another small perturbation leaves each of these finite cyclic summands actually of constant weight. Now on any stretch of weights where $|t_n - t_{n-1}| \leqslant 1/M^2 < 1/256$ these desired finite dimensional perturbations are of norm at most $100/M$, and since they have both domain and range on disjoint stretches they commute with the corresponding summand of S. We now have our commuting perturbation. We will show later that this construction is best possible in the sense that there may be no way of attaining a small norm commuting perturbation if the shift-like direct summand cannot be made close to α uniformly. Using the same uncoupling on the same vectors split off direct cyclic summands of S itself. Now each of the summands of S commutes with the corresponding summands of T. The remaining summand of T, now a shift with weights approaching α is simply perturbed by the compact perturbation of changing each weight to α thereby allowing it to commute with the corresponding shift summand of S. Now if $|t_2 - t_{n-1}| < 1/256$ and 2) holds then we split off our summands as before choosing $\alpha = t_2$ and acquiring a remaining shift summand within again $100/M$ of the constant weighted shift α and so acquire commutativity at the expense of $\|\Delta S\| < 100/M$ and $\|\Delta T\| < 100/M$. If 1) and 2) hold and $|t_n - t_{n-1}| \to 0$ then ΔS and ΔT are each compact and with the same norm bound. We need consider only small $|t_n - t_{n-1}|$ since where this is large the corresponding large finite dimensional perturbation does not affect compactness and by producing a large δ relieves our theorem of quantitative implications.

Now we consider the case where T is a multiplication operator. Once again the requirement that $[T, S] < \delta$ implies that $\sup |t_n - t_{n-1}| < \delta$ and we require $|t_n - t_{n-1}| \to 0$ for compactness. Once again we choose stretches on which the t_n are as close as desired to a cluster point α and choose basis vectors as if we were to uncouple shift sections with the same weights. We then proceed to match sections of similar weights throughout the operator just as if it were a shift. This leaves us with direct summands on each of which the multiplication operator is almost constant.

Now we use the new basis vectors to uncouple the sections of S corresponding to sections of T and what we have are finite dimensional cyclic direct summands of S and multiplication by constants on the same vectors for T. The remaining shift summand of S matches with, depending whether or not 2) holds, multiplication by a constant α or by a sequence tending to α. This completes the analysis if T is a multiplication operator.

Now suppose T is a shift of multiplicity r. Once again we have $|t_n - t_{n+1}|$ small as before and the same split into constant r-cyclic summands of T (that is, each element hits every r'th index) matched against single cyclic summands of S yields the same results as before. Alternatively, one could choose r'th roots of the weights in such a way that $T = (T')^r$ where T' was a weighted shift of index 1 with weights chosen real or pure imaginary in such a way as to approximately match r'th roots where the weights themselves were close. This same argument works just as well for $P(T)$ observing that $[P(T), S]$ is compact iff $[T, S]$ is compact. In this case we factor $P(T)$ into linear factors and since each is of the form $(T + \lambda)$ the same perturbation splits cyclic direct summands for all factors and for S simultaneously and so we get our commuting perturbations for $P(T)$.

This completes the proof of Theorem 2.2.

We have seen that if T is a left shift and $[T, S]$ is compact then T, S lift to commute. If T is a right shift with 0 in its essential spectrum and $[T, S]$ is compact then a variation of the previous argument shows that T, S lift to commute. If T is a right shift without 0 in its essential spectrum, that is 0 is not a cluster point of its weights, then we saw earlier that T, S cannot lift to commute. That is, with the next theorem we have covered all cases.

THEOREM 2.3. *Let S be the canonical left shift. Let T be a real weighted right shift of finite index r whose essential spectrum includes 0. That is,* $T\delta_n = t_n\delta_{n+r}$ and $\liminf_n |t_n| = 0$.

If $[T, S]$ is compact then T and S lift to commute.

Further there exists $\delta(r, \varepsilon) > 0$ independent of T such that if $[T, S] < \delta(r, \varepsilon)$ then $\mathrm{J}[T, S] < \varepsilon$.

Proof. We first consider the case of multiplicity 1. Because of the zeros of the weights after a small perturbation we can completely uncouple T into finite dimensional shifts of small self-commutator. We then uncouple each of these shifts into a finite sum of scalars times the cyclical shifts. We then perform the same decomposition, using the same vector, on S^*, leaving cycles of S^* matching cycles of T and leaving the necessary remaining direct summand of S^*, call it $\tilde{S}^*$ which is still unitarily equivalent to S^*, matching the vectors on which T is 0. Then, since S^* on each cycle is itself merely a scalar multiple of the cycles of T, we take S, still retaining commutativity on the cyclic summands, and matching $(\tilde{S}^*)^*$ with 0, thereby attaining commutativity. Observing that a small commutator in this case requires that $T(\varphi_1)$ be very small we see that there is no initial piece around the early coordinates to be matched and so a small perturbation of T and S

commute. For shifts of larger multiplicity take a root $T^{1/n}$ with a small self-commutator and with real or imaginary weights and perturb S and $T^{1/n}$ so as to commute.

We have now completed our desired exhaustive analysis of liftings of arbitrary shifts and the canonical left shift so as to commute. That is, let S be the canonical left shift. If T is a right shift with all (real) weights bounded away from 0 for all large enough indices then by our indicial criterion S and T cannot be lifted to commute. For all other shifts T we see S and T can be lifted to commute.

Of course we can exchange right and left consistently (by considering adjoints) in all our results. In the case where lifting is possible we have shown that additional restrictions on the first non-trivial weights of T allow us to deduce that a small commutator of T and S allows nearly commuting $\tilde{T}$ and $\tilde{S}$. The next part of this paper is devoted to showing that this condition on the initial weights is necessary.

3. NORM OBSTRUCTIONS

We now present our results showing that there can exist a norm obstruction which prevents S and T from being close to commuting operators, independent of the norm of the commutator of S and T, and yet allowing S and T to lift to commutativity. The reasonable version of the theorem includes so many parameters as to completely mask the ideas at first reading and so we present first a theorem with many parameters fixed. We choose a left shift of index 1 in this theorem. The phenomenon is actually attainable with a multiplication operator as T but the proof is sufficiently simplified in that case so as not to illustrate clearly its generalization; the shift offers just enough difficulty to demonstrate the method.

THEOREM 3.1. *Let T be a real weighted left shift such that $T\delta_2 = 2\delta_1$ and $\|T\| = 2$. Let $\limsup_i T\delta_i < 1$. Let S be the canonical unilateral left shift. Then* $J[S, T] > 1/64$.

Proof. Let $\|\Delta S\| < 1/64$. Then $\tilde{S} = S + \Delta S$ has a corresponding subspace $\tilde{\mathscr{H}}$ of successive kernels $\bigvee \varphi_i$ as described in Lemma 2.2. Then one can see that $\|\varphi_1 - \delta_1\| < 3/64$ and $\|\varphi_2 - \delta_2\| < 6/64$ and hence if $\|\Delta T\| < 1/64$ then $\tilde{T} = T + \Delta T$ satisfies $\|(T + \Delta T)\varphi_1\| < 2/64 + 1/64$ and $\|(T + \Delta T)\varphi_2 - 2\delta_1\| < 1/64 + 12/64$, and so $\|(T + \Delta T)\varphi_2 - 2\varphi_1\| < 13/64 + 4/64 = 17/64$. If we assume that $\tilde{T}$ commutes with $\tilde{S}$ then by Lemma 2.2 we have a first degree polynomial in $\tilde{S}$, $\alpha\tilde{S} + \beta I$ so that $\langle \tilde{T}\varphi_n, \varphi_{n-1}\rangle = \langle \alpha\tilde{S}\varphi_n, \varphi_{n-1}\rangle$. Since, for large enough n, we have both $\langle \tilde{S}\varphi_n, \varphi_{n-1}\rangle > 3/4$ by our lemma, and $\langle \tilde{T}\varphi_n, \varphi_{n-1}\rangle < 65/64$, since $\|\tilde{T}\|_e < 65/64$, we see that $\alpha < 4/3 \cdot 65/64 = 65/48$. But this implies $\langle \tilde{T}\varphi_2, \varphi_1\rangle = \langle \alpha\tilde{S}\varphi_2, \varphi_1\rangle < 65/48 \cdot 65/64$ and we had earlier seen that $\langle \tilde{T}\varphi_2, \varphi_1\rangle > 2 - 17/64$ a flat contradiction.

Let us consider $\varphi_1 - \delta_1$ and $\varphi_2 - \delta_2$ in more detail. To see that $\|\varphi_1 - \delta_1\| < 2/64$ observe that for unit η we have $0 = \langle \varphi_1, (S^* + \Delta S^*)\eta\rangle$ and since S^* is an isometry from ℓ_2 onto $\delta_1^\perp$ we see that for unit $\xi \in \delta_1^\perp$ we have $|\langle \varphi_1, \xi\rangle| < 1/64$. Hence $\langle \varphi_1, \delta_1\rangle^2 > (1 - (1/64)^2) \geqslant (63/64)^2$ and so $\|\varphi_1 - \delta_1\| < 2/64$.

Similarly $0 = \langle \varphi_2, (S^* + \Delta S^*)^2\eta\rangle$ and so for $\xi \in (\delta_1 \vee \delta_2)$ we see

$$|\langle \varphi_2, \xi\rangle| \leqslant |\varphi_2, ([(\Delta S)^*]^2 + S^*\Delta S^* + (\Delta S^*)S^*)\eta| < 3/64.$$

Since $|\langle \varphi_2, \delta_1\rangle| < 2/64$ we have $|\langle \varphi_2, \delta_2\rangle|^2 > 1 - (3/64)^2 - (2/64)^2 \geqslant (63/64)^2$ and so $\|\varphi_2 - \delta_2\| < 3/64 + 2/64 + 1/64 = 6/64$.

The fact that we have a worse bound on $\|\varphi_2 - \delta_2\|$ then on $\|\varphi_1 - \delta_1\|$ is not simply a computational annoyance. The fact is that no matter how small the fixed perturbation allowed it is perfectly possible that $\delta_n \perp \bigvee_{i=1}^{\infty} \varphi_i$ for some large enough n; indeed δ_n can be in one of the cyclic unitary direct summands of $\tilde{S}$.

THEOREM 3.2. *Let S be the canonical left shift. Let T be a real weighted left shift of multiplicity $r \geqslant 0$. Suppose for some $\eta > 0$ we have $t_{r+1} \notin [-\eta + \liminf t_n, \limsup t_r + \eta]$. There exists $F(\eta, r) > 0$ independent of T such that* $J[S, T] > F(\eta, r)$.

Proof. We note the changes required to generalize Theorem 3.1. First note that if $[\tilde{T}, \tilde{S}] = 0$ then $\alpha\tilde{T} + \beta\tilde{S}$ is an acceptable commuting lifting of $\alpha T + \beta S$ for scalar α, β and so we can assume that T has only positive weights and consider only $t_{r+1} > \limsup t_n + \eta$.

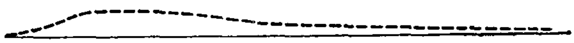
Now for sufficiently small ΔS and ΔT depending only on t_{r+1} we see $\tilde{T}\varphi_{r+1} \approx t_{r+1}\varphi_1$, $\tilde{T}(\varphi_m) \approx 0$ for $m < r + 1$ with the accuracy depending only on $|\Delta T|$. Now observing that $\langle \tilde{T}\varphi_{n+j}, \varphi_n\rangle$ for $j \leqslant r$ is given by the first j terms of a polynomial in $\tilde{S}$ we see that

$$\langle \tilde{T}\varphi_{n+r}, \varphi_n\rangle = \langle a_r\tilde{S}^r + \dots + a_0 I\varphi_{n+r}, \varphi_n\rangle \approx$$

$$\approx \langle t_{r+1}\tilde{S}^r + 0\cdot\tilde{S}^{r-1} + \dots + 0\cdot I\varphi_{n+r}, \varphi_n\rangle$$

with accuracy once again depending only on $|\Delta T||\Delta S|$. Because, with again, as much accuracy as needed $\langle \tilde{S}^r\varphi_{n+r}, \varphi_n\} \approx 1$ for large enough n we have $\langle \tilde{T}\varphi_{n+r}, \varphi_n\rangle \approx \approx t_{r+1} > \limsup t_n + \eta$ which is clearly impossible. That is, our relatively large t_{r+1} prevented $\tilde{T}$ from commuting with $\tilde{S}$ although $[T, S]$ can be made as small as desired keeping the same parameters.

The reader will note that there is no $F(\eta, r)$ if one merely requires that for some m we have $\|T(\delta_m)\| > \limsup\|T(\delta_n)\| + \eta$; indeed for any fixed candidate for $F(\eta, r)$ we can, for large enough m, pull δ_m into one of our cyclic direct summands as in Theorem 2.3 and attain commutativity. Yet he will observe that for any fixed m there would exist a corresponding $F_m(\eta, r)$ and therefore ask why we have put our hypotheses on the first non-trivial weight of T instead of developing $F_m(\eta, r)$. The answer is that J$[S, T]$ is only of interest where $[S, T]$ itself is very small and for small enough $[S, T]$ the suggested condition on t_m implies t_{r+1} fits our hypotheses and we are back in the already covered case. That is, if the weights of T take on, for example, the pattern



the norm of the commutator $[T, S]$ alone controls J$[T, S]$. This is, of course, precisely what we are gainsaying in our theorem.

We can summarize our results. By means of uncoupling we were able to remove all bumps with the same initial and terminal weights from T in a manner which commuted with corresponding removals from S leaving either a residual left shift of T which could commute with a corresponding residual left shift of S, leaving nothing, or necessarily leaving a right shift-like residual which could not commute with the necessary shift-like residual of S. A one-sided bump could be removed with a compact perturbation of T but could not be uncoupled and necessarily left a large perturbation.

We close with a consideration of the difficulties of complex weights. Of course, our methods work with large classes of complex shifts. Rather than present a large class of partial results let us go right to a case that illustrates the fundamental difficulties. Consider T_1 a unitary left shift with weights all 1 on E_n and T_2 a unitarily equivalent unitary left shift but with weight $e^{2\pi ik/n}$ on the same E_n. Does J$[T_1, T_2] \underset{n}{\rightarrow} 0$? We conjecture it does not, principally because the most likely construction, an uncoupling which leaves all entries in orbits which never take an entry far from its original coordinate presentation, would lead to small commuting perturbations of S and S^*, which we know to be impossible. Similarly if S is a left shift and T is the right shift of index 1 with $t_n = e^{2\pi i(\ln n)}$, can S and T be lifted to commute?[1] Theorem 1.3 does not apply. Further ST is unitarily equivalent to a compact perturbation of Q, the bilateral shift with projection on δ_0 removed, and $\dim Q^{-n}(0) \underset{n}{\rightarrow} \infty$. That is, implausible though it might seem at first glance, the weights of T allow ST to be perturbed so as to allow an increasing sequence of kernels, a phenomenon which strikes at the heart of the proof of Theorem 1.2 and Theorem 1.3. Of course this destruction of the hypothesis of a sufficiency theorem still leaves the possibility of what we consider a desirable result, that is, the impossibility of a commuting lifting of S and T. We should mention that what we believe to be this same problem in different guises impinges on other current research, for example, that of K. Davidson and D. Voiculescu, and so it is possible that the solution will come about quite indirectly.

This work was partially supported by an NSF grant.

[1] Yes. K. Davidson (personal communication).

REFERENCES

1. BERG, I. D., On approximation of normal operators by weighted shifts, *Michigan Math.J.*, **21**(1974), 377–383.
2. BERG, I. D., Index theory for perturbations of direct sums of normal operators and weighted shifts, *Canad. J. Math.*, **30**(1978), 1152–1165.
3. BERG, I. D.; OLSEN, C. L., A note on almost commuting operators, *Proc. Roy. Irish Acad.*, **81**(1981), 43–47.
4. DAVIDSON, K., Essentially spectral operators, preprint.
5. HERRERO, D., *Approximation of Hilbert space operators. Vol. 1*, Research Notes in Mathematics, No. 72, Pitman, London, 1982.
6. STAMPFLI, J., One dimensional perturbations of operators, preprint.

I. D. BERG
Department of Mathematics,
University of Illinois,
1409 West Green Street,
Urbana, Illinois 61801,
U.S.A.

Received March 15, 1983.