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ON RESTRICTIONS OF UNBOUNDED SYMMETRIC OPERATORS

KONRAD SCHMÜDGEN

1. INTRODUCTION AND MAIN RESULTS

1.1. Let A_1 and A_2 be densely defined unbounded symmetric operators in separable Hilbert spaces $\mathscr{H}_1$ resp. $\mathscr{H}_2$. Suppose that $A_s\mathscr{D}(A_s) \subseteq \mathscr{D}(A_s)$ for $s \in \{1, 2\}$. We write $A_1 \sim A_2$ if there exist an isometry U of $\mathscr{H}_1$ onto $\mathscr{H}_2$ and a dense linear subspace $\mathscr{D}_{10} \subseteq \mathscr{D}(A_1)$ of $\mathscr{H}_1$ such that $A_1\mathscr{D}_{10} \subseteq \mathscr{D}_{10}$, $\mathscr{D}_{20} := U\mathscr{D}_{10} \subseteq \mathscr{D}(A_2)$ and $A_1\varphi = U^*A_2U\varphi$ for all $\varphi \in \mathscr{D}_{10}$. This clearly implies that $\mathscr{D}_{20}$ is dense in $\mathscr{H}_2$, $A_2\mathscr{D}_{20} \subseteq \mathscr{D}_{20}$ and hence $A_2 \sim A_1$. In other words, the relation $A_1 \sim A_2$ means that A_1 and A_2 have restrictions to dense invariant domains which are unitarily equivalent.

In this paper we show that if $\mathscr{D}(A_s) = \bigcap_{n=1}^{\infty} \mathscr{D}(\overline{A_s^n})$ for $s \in \{1, 2\}$, then $A_1 \sim A_2$ if and only if A_1 and A_2 are both strongly unbounded from above or strongly unbounded from below (for a definition of these notions, see 1.2). In the case where A_s is essentially self-adjoint and $\mathscr{D}(A_s) = \bigcap_{n=1}^{\infty} \mathscr{D}((\overline{A_s})^n)$ for $s \in \{1, 2\}$ this result means that $A_1 \sim A_2$ if and only if A_1 and A_2 are both unbounded from above or unbounded from below. This answers a question raised by the author in [1], p. 335, Problem 10, in the affirmative. For diagonal operators the latter was already proved in [3], Theorem 4.5, by a different method. (Note that an error in the proof or Corollary 4.2 in [3] has been corrected in [4], Section 7.)

Of course, the result mentioned above shows the pathology of unbounded operators (in the spirit of [2]). In case of diagonal operators we used it in [4] to construct non-integrable representations of the Heisenberg commutation relation $PQ - QP = -\mathrm{i}I$ by essentially self-adjoint operators P, Q. The theorems and the corollaries stated below have similar applications, but they seem to be of some interest in its own right as well. In a forthcoming paper they will be the main tool for the construction of $*$-representations of unbounded operator algebras.

1.2. Before stating our results, we introduce some more notation. We let $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbf{N} = \{1, 2, \ldots\}$. Suppose T is a densely defined symmetric operator in a Hilbert space $\mathscr{H}$. Let $\mathscr{D}(T)$ denote the domain of T. $\overline{T}$ will denote the closure of T. In order to describe the growth and the unboundedness of T, it will be convenient to define some numbers. Assume that $T\mathscr{D}(T) \subseteq \mathscr{D}(T)$. For $\varphi \in \mathscr{D}(T)$ and $k \in \mathbf{N}$, define $\rho_0^T(\varphi) := 1$ and $\rho_k^T(\varphi) := \langle T^k\varphi, \varphi\rangle / \langle T^{k-1}\varphi, \varphi\rangle$, where $\frac{0}{0} := 1$ and $\frac{\pm c}{0} := \pm\infty$ if $c > 0$. For a linear subspace $\mathscr{D}$ of $\mathscr{D}(T)$ and $k, n \in \mathbf{N}$, put

$$B_n(\mathscr{D}) := \{\varphi \in \mathscr{D}\,;\ |\langle T^j\varphi, \varphi\rangle| \leqslant 1 \text{ for } j = 0, \ldots, n-1\}$$

and define

$$\lambda_{2k-1}^+(T; \mathscr{D}) := \sup\{\langle T^{2k-1}\varphi, \varphi\rangle;\ \varphi \in B_{2k-1}(\mathscr{D})\},$$

$$\lambda_{2k-1}^-(T; \mathscr{D}) := \inf\{\langle T^{2k-1}\varphi, \varphi\rangle;\ \varphi \in B_{2k-1}(\mathscr{D})\},$$

$$\lambda_n(T; \mathscr{D}) := \sup\{|\langle T^n\varphi, \varphi\rangle|;\ \varphi \in B_n(\mathscr{D})\}$$

and

$$\lambda_{2k}^+(T; \mathscr{D}) := \lambda_{2k}(T; \mathscr{D}).$$

If $\mathscr{D} = \mathscr{D}(T)$, we simply write $\lambda_n^\pm(T)$ for $\lambda_n^\pm(T; \mathscr{D})$ and $\lambda_n(T)$ for $\lambda_n(T; \mathscr{D})$. T is said to be *strongly unbounded from above* (*below*) if $\lambda_{2k-1}^+(T) = +\infty$ ($\lambda_{2k-1}^-(T) = -\infty$) for all $k \in \mathbf{N}$. Recall that T is called *unbounded from above* (*below*) if $\sup\{\langle T\varphi, \varphi\rangle;\ \varphi \in B_1(\mathscr{D}(T))\} = +\infty$ ($\inf\{\langle T\varphi, \varphi\rangle;\ \varphi \in B_1(\mathscr{D}(T))\} = -\infty$). Clearly, for the latter we do not need that $\mathscr{D}(T)$ is invariant under T.

It is plain from the definition and $T\mathscr{D}(T) \subseteq \mathscr{D}(T)$ that $\lambda_{2k-1}^+(T) \leqslant \lambda_{2j-1}^+(T)$ and $\lambda_{2k-1}^-(T) \geqslant \lambda_{2j-1}^-(T)$ if $k \geqslant j$. The following lemma will be needed later.

LEMMA 1. *Let T be as above (i.e., T is symmetric, $\mathscr{D}(T)$ is dense in $\mathscr{H}$ and $T\mathscr{D}(T) \subseteq \mathscr{D}(T)$). Suppose that T is unbounded. Then, $\lambda_n(T) = +\infty$ for all $n \in \mathbf{N}$. Therefore, T is strongly unbounded from above or strongly unbounded from below.*

The first assertion has been proved in [5], pp. 378—380. The second assertion follows immediately from the above inequalities and $\lambda_{2k-1}(T) = \max\{\lambda_{2k-1}^+(T), -\lambda_{2k-1}^-(T)\}$.

1.3. We now state our results. In this subsection, let A_1 and A_2 be densely defined symmetric linear operators in separable complex Hilbert spaces $\mathscr{H}_1$ resp. $\mathscr{H}_2$ such that $A_s\mathscr{D}(A_s) \subseteq \mathscr{D}(A_s)$ and $\mathscr{D}(A_s) = \bigcap_{n=1}^{\infty} \mathscr{D}(\overline{A_s^n})$. Moreover, we assume that A_1 and A_2 are both unbounded.

THEOREM 1. *$A_1 \sim A_2$ if and only if A_1 and A_2 are both strongly unbounded from above or strongly unbounded from below.*

Theorem 1 will be deduced from Theorem 2 which is the main result of this paper.

THEOREM 2. *Suppose that A_1 and A_2 are both strongly unbounded from above. Let $(\gamma_{kr})_{k,r\in\mathbf{N}_0}$ be a given matrix of positive numbers γ_{kr}.*

Then there exist sequences $\{\varphi_n^s,\ n \in \mathbf{N}_0\}$ of vectors $\varphi_n^s \in \mathscr{D}(A_s)$, $n \in \mathbf{N}_0$, for $s \in \{1,2\}$ such that:

(a) $\langle A_1^r\varphi_k^1, \varphi_n^1\rangle = \langle A_2^r\varphi_k^2, \varphi_n^2\rangle = 0$ *for $k, n, r \in \mathbf{N}_0$, $k \neq n$;*

(b) $\langle A_1^r\varphi_k^1, \varphi_k^1\rangle = \langle A_2^r\varphi_k^2, \varphi_k^2\rangle > 0$ *for $k, r \in \mathbf{N}_0$;*

(c) $\rho_{j+1}^{A_s}(\varphi_k^s) \geqslant 8\gamma_{kj}\rho_l^{A_s}(\varphi_n^s)$ *for $2k+j+1 > 2n+l$, $j, k, l, n \in \mathbf{N}_0$, $s \in \{1,2\}$;*

(In particular, $\rho_{j+1}^{A_s}(\varphi_k^s) \geqslant 8\gamma_{kj}\rho_j^{A_s}(\varphi_k^s)$.)

(d) $\mathscr{D}_{s0} := \mathrm{Lin}\{A_s^r\varphi_k^s;\ k, r \in \mathbf{N}_0\}$ *is dense in $\mathscr{H}_s$ for $s \in \{1,2\}$.*

The proof of Theorem 2 will be given in Section 2 and 3. We now retain the assumptions and notations of Theorem 2 and derive some corollaries.

COROLLARY 1. *The operators $A_{10} := A_1 \upharpoonright \mathscr{D}_{10}$ and $A_{20} := A_2 \upharpoonright \mathscr{D}_{20}$ are unitarily equivalent. Therefore, $A_1 \sim A_2$.*

Proof. Let $(\mu_{kr})_{k,r\in\mathbf{N}_0}$ be a complex matrix with finitely many non-zero entries. Define $U(\sum_{k,r} \mu_{kr}A_1^r\varphi_k^1) := \sum_{k,r}\mu_{kr}A_2^r\varphi_k^2$. From (a), (b) and (d) we see that U extends uniquely to an isometry of $\mathscr{H}_1$ onto $\mathscr{H}_2$. Obviously, $U\mathscr{D}_{10} = \mathscr{D}_{20}$ and $A_1\varphi = U^*A_2U\varphi$ for $\varphi \in \mathscr{D}_{10}$. Since $A_1\mathscr{D}_{10} \subseteq \mathscr{D}_{10}$ and $\mathscr{D}_{20} \subseteq \mathscr{D}(A_2)$, we have $A_1 \sim A_2$.

Proof of Theorem 1. Suppose that A_1 and A_2 are both strongly unbounded from above or strongly unbounded from below. Replacing A_1, A_2 by $-A_1, -A_2$ in the latter case, Corollary 1 implies that $A_1 \sim A_2$.

Suppose now that $A_1 \sim A_2$. Let $\mathscr{D}_{10}$ and $\mathscr{D}_{20}$ be dense invariant domains in $\mathscr{H}_1$ resp. $\mathscr{H}_2$ such that $A_1 \upharpoonright \mathscr{D}_{10}$ and $A_2 \upharpoonright \mathscr{D}_{20}$ are unitarily equivalent. Assume that the assertion were false. Changing the role of A_1 and A_2 if necessary, we can find natural numbers k, n such that $\lambda_{2k-1}^{+}(A_1) < +\infty$ and $\lambda_{2n-1}^{-}(A_2) > -\infty$. Let $m = \max\{k, n\}$. Then, $\lambda_{2m-1}^{+}(A_1; \mathscr{D}_{10}) \leqslant \lambda_{2m-1}^{+}(A_1) \leqslant \lambda_{2k-1}^{+}(A_1) < +\infty$. Similarly, $\lambda_{2m-1}^{-}(A_2; \mathscr{D}_{20}) > -\infty$. Since $A_1 \upharpoonright \mathscr{D}_{10}$ and $A_2 \upharpoonright \mathscr{D}_{20}$ are unitarily equivalent, $\lambda_{2m-1}^{\pm}(A_1; \mathscr{D}_{10}) = \lambda_{2m-1}^{\pm}(A_2; \mathscr{D}_{20})$. Therefore, $\lambda_{2m-1}(A_1; \mathscr{D}_{10}) < +\infty$. On the other hand, $T := A_1 \upharpoonright \mathscr{D}_{10}$ satisfies the assumptions of Lemma 1 and hence $\lambda_{2m-1}(T) = \lambda_{2m-1}(A_1; \mathscr{D}_{10}) = +\infty$ which is the desired contradiction. This completes the proof of Theorem 1.

The next two corollaries follow mainly from the growth condition (c). For simplicity we write $\rho_j(\cdot)$ for $\rho_j^{A_s}(\cdot)$, A for A_s and φ_k for φ_k^s.

COROLLARY 2. *Let $k \in \mathbf{N}_0$ and $0 < \varepsilon_k \leqslant 1$. If $8\gamma_{kj} \geqslant 2^{j+2}\varepsilon_k^{-2}$ for $j \in \mathbf{N}_0$, then*

$$(1 - \varepsilon_k)\sum_j |\mu_j|^2 \langle A^{r+2j} \varphi_k, \varphi_k\rangle \leqslant$$

$$\leqslant \langle A^r(\sum_j \mu_j A^j)\varphi_k, (\sum_l \mu_l A^l)\varphi_k\rangle \leqslant (1 + \varepsilon_k)\sum_j |\mu_j|^2 \langle A^{r+2j}\varphi_k, \varphi_k\rangle$$

for each complex polynomial $\sum_j \mu_j A^j$ and $r \in \mathbf{N}_0$.

Proof. Our assumptions and (c) imply that $\rho_j(\varphi_k) \geqslant \rho_l(\varphi_k) \geqslant 1$ and

$$\rho_{r+2j}(\varphi_k) \geqslant \varepsilon_k^{-2} 2^{2j+1} \rho_{r+2j-1}(\varphi_k) \geqslant \varepsilon_k^{-2} 2^{j+l+2} \rho_{r+j+l}(\varphi_k)$$

for $j, l \in \mathbf{N}_0$, $j > l$. Therefore,

$$\langle A^{r+2j}\varphi_k, \varphi_k\rangle / \langle A^{r+j+l}\varphi_k, \varphi_k\rangle =$$

$$= \rho_{r+2j}(\varphi_k) \cdots \rho_{r+j+l+1}(\varphi_k) \geqslant \varepsilon_k^{-2} 2^{j+l+2} \rho_{r+j+l}(\varphi_k) \cdots \rho_{r+2l+1}(\varphi_k) =$$

$$= \varepsilon_k^{-2} 2^{j+l+2} \langle A^{r+j+l}\varphi_k, \varphi_k\rangle / \langle A^{r+2l}\varphi_k, \varphi_k\rangle,$$

that is,

$$\langle A^{r+j+l}\varphi_k, \varphi_k\rangle^2 \leqslant \varepsilon_k^2\, 2^{-(j+l+2)} \langle A^{r+2j}\varphi_k, \varphi_k\rangle \langle A^{r+2l}\varphi_k, \varphi_k\rangle$$

for $j, l \in \mathbf{N}_0$, $j > l$. Using this and the Cauchy-Schwarz inequality, we get

$$\Big|\sum_{j \neq l} \mu_j \overline{\mu_l} \langle A^{r+j+l}\varphi_k, \varphi_k\rangle\Big| \leqslant \frac{\varepsilon_k}{2} \left(\sum_j |\mu_j| \langle A^{r+2j}\varphi_k, \varphi_k\rangle^{1/2}\, 2^{-j/2} \right)^2 \leqslant$$

$$\leqslant \varepsilon_k \sum_j |\mu_j|^2 \langle A^{r+2j}\varphi_k, \varphi_k\rangle.$$

This gives the assertion.

COROLLARY 3. *Suppose $\gamma_{k,j+1} \geqslant \gamma_{kj} \geqslant 2^{j+3}$ and $\gamma_{kj} \geqslant \gamma_{0j}$ for all $k, j \in \mathbf{N}_0$. Let $\mathscr{D}^k := \mathrm{Lin}\{A^r\varphi_k; r \in \mathbf{N}_0\}$ and let $\mathscr{D}_0 := \mathrm{Lin}\{A^r\varphi_k; k, r \in \mathbf{N}_0\}$.*
Then we have

$$\rho_{r+1}(\xi_k) \geqslant \gamma_{kr}\rho_r(\xi_k), \quad \rho_{r+1}(\xi) \geqslant \gamma_{0r}\rho_r(\xi),$$

$$\langle A^{r+1}\xi_k, \xi_k\rangle \geqslant \gamma_{kr}\langle A^r\xi_k, \xi_k\rangle \geqslant \gamma_{kr}\gamma_{k,r-1} \cdots \gamma_{k0}\langle \xi_k, \xi_k\rangle$$

and

$$\langle A^{r+1}\xi, \xi\rangle \geqslant \gamma_{0r}\langle A^r\xi, \xi\rangle \geqslant \gamma_{0r}\gamma_{0,r-1} \cdots \gamma_{00}\langle \xi, \xi\rangle$$

for all $\xi_k \in \mathscr{D}^k$, $k \in \mathbf{N}_0$, $\xi \in \mathscr{D}_0$ and $r \in \mathbf{N}_0$.

Proof. Suppose that $r \in \mathbf{N}$. Let $\xi \in \mathscr{D}_0$. Then there are finitely many complex numbers μ_{kj} such that $\xi = \sum_{j,k} \mu_{kj} A^j \varphi_k$. If $2k + r + 1 + 2j > 2n + r + 2l$ for $j, k, l, n, r \in \mathbf{N}_0$, it follows from the monotonicity of $\gamma_{k,\cdot}$ and (c) that

$$\langle A^{r+1+2j}\varphi_k, \varphi_k\rangle\langle A^{r-1+2l}\varphi_n, \varphi_n\rangle =$$

$$= \langle A^{r+2j}\varphi_k, \varphi_k\rangle\langle A^{r+2l}\varphi_n, \varphi_n\rangle \rho_{r+1+2j}(\varphi_k)/\rho_{r+2l}(\varphi_n) \geqslant \tag{1}$$

$$\geqslant 8\gamma_{kr}\langle A^{r+2j}\varphi_k, \varphi_k\rangle\langle A^{r+2l}\varphi_n, \varphi_n\rangle.$$

Put $\varepsilon_k = 1/4$ for $k \in \mathbf{N}_0$. Since $8\gamma_{kj} \geqslant 2^{j+6} = 2^{j+2}\varepsilon_k^{-2}$ for $j, k \in \mathbf{N}_0$, Corollary 2 applies. Combined with (a) and (1), it gives

$$\langle A^{r+1}\xi, \xi\rangle\langle A^{r-1}\xi, \xi\rangle \geqslant$$

$$\geqslant \frac{9}{16} \sum_{j,k,l,n} |\mu_{kj}|^2 |\mu_{nl}|^2 \langle A^{r+1+2j}\varphi_k, \varphi_k\rangle\langle A^{r-1+2l}\varphi_n, \varphi_n\rangle \geqslant$$

$$\geqslant \sum_{2k+2j+1>2n+2l} 4\gamma_{kr}|\mu_{kj}|^2|\mu_{nl}|^2\langle A^{r+2j}\varphi_k, \varphi_k\rangle\langle A^{r+2l}\varphi_n, \varphi_n\rangle \geqslant$$

$$\geqslant \sum_{j,k,l,n} 2\cdot\gamma_{0r} \ldots \geqslant \gamma_{0r}\langle A^r\xi, \xi\rangle^2,$$

that is, $\rho_{r+1}(\xi) \geqslant \gamma_{0r}\rho_r(\xi)$. In case $\xi = \xi_k \in \mathscr{D}^k$ the same argument shows that $\rho_{r+1}(\xi_k) \geqslant \gamma_{kr}\rho_r(\xi_k)$.

In case $r = 0$ a similar (but somewhat simpler) argument gives $\langle A\xi, \xi\rangle \geqslant \gamma_{00}\langle\xi, \xi\rangle$ and $\langle A\xi_k, \xi_k\rangle \geqslant \gamma_{k0}\langle\xi_k, \xi_k\rangle$, i.e., $\rho_1(\xi) \geqslant \gamma_{00} = \gamma_{00}\rho_0(\xi)$ and $\rho_1(\xi_k) \geqslant \gamma_{k0}\rho_0(\xi_k)$. The other assertions follow immediately from $\rho_j(\xi) \geqslant 1$ for $j \in \mathbf{N}$ and $\xi \in \mathscr{D}_0$ and the definition of $\rho_j(\cdot)$.

1.4. Remarks. 1) We briefly discuss the assumptions

$$A_s\mathscr{D}(A_s) \subseteq \mathscr{D}(A_s) \quad \text{for } s = 1, 2 \tag{2}$$

and

$$\mathscr{D}(A_s) = \bigcap_{n=1}^{\infty} \mathscr{D}(\overline{A_s^n}) \quad \text{for } s = 1, 2. \tag{3}$$

By (2), the mapping $x \to A_s$ induces a $*$-representation π_s of the $*$-algebra $\mathscr{P}(x)$ of all complex polynomials on the dense invariant domain $\mathscr{D}(A_s)$. (3) means that π_s is closed on $\mathscr{D}(A_s)$. Theorem 1 may be rephrased by saying that *two closed $*$-representations π_1 and π_2 of $\mathscr{P}(x)$ in separable Hilbert spaces with unbounded operators*

$A_1 := \pi_1(x)$ and $A_2 := \pi_2(x)$ have unitarily equivalent (densely defined) restrictions if and only if A_1 and A_2 are both strongly unbounded from above or strongly unbounded from below.

2) The conclusion of Theorem 1 does not hold if (3) is not assumed. For a counter-example, take $A_1 = x$ and $A_2 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathscr{D}(A_1) = \mathscr{D}(A_2) = C_0^\infty(\mathbf{R})$ in $\mathscr{H}_1 = \mathscr{H}_2 = L_2(\mathbf{R})$. (All vectors in $\mathscr{D}(A_1)$ are bounded for A_1, but $\mathscr{D}(A_2)$ has no non-zero bounded vectors for A_2.)

3) Let T_1 and T_2 be *closed* symmetric operators such that $\mathscr{D}_\infty(T_s) := \bigcap_{n=1}^{\infty} \mathscr{D}(T_s^n)$ is dense in $\mathscr{H}_s$ for $s = 1, 2$. Then, of course, $A_1 := T_1 \upharpoonright \mathscr{D}_\infty(T_1)$ and $A_2 := T_2 \upharpoonright \mathscr{D}_\infty(T_2)$ satisfy (2) and (3). But (3) is more general, because there are examples satisfying (2) and (3) for which $\mathscr{D}(A_s) \subsetneq \mathscr{D}_\infty(\overline{A_s})$ for $s = 1, 2$.

4) Theorem 2 may be considered as an improvement of Theorem 7.1 in [4].

5) There are closed symmetric operators T such that T^{2k-1} is unbounded from above on $\mathscr{D}_\infty(T)$ for some $k \in \mathbf{N}$, but $T^{2k+1} \leqslant 0$ on $\mathscr{D}_\infty(T)$. Examples of this kind will be constructed elsewhere. However, if T is *self-adjoint*, then it follows easily from the spectral theorem that $T \upharpoonright \mathscr{D}_\infty(T)$ is strongly unbounded from above (below) if and only if $T \upharpoonright \mathscr{D}_\infty(T)$ [or equivalently, T] is unbounded from above (below).

2. PRELIMINARIES TO THE PROOF OF THEOREM 2

2.1 Let T be a densely defined symmetric operator in the Hilbert space $\mathscr{H}$. Suppose $T\mathscr{D}(T) \subseteq \mathscr{D}(T)$. Throughout this subsection we assume that T is strongly unbounded from above. $\mathscr{F}$ will always denote a given finite dimensional subspace of $\mathscr{D}(T)$.

LEMMA 2. (i) $\lambda_n^+(T; \mathscr{D}(T) \ominus \mathscr{F}) = +\infty$ *for all* $n \in \mathbf{N}$.

(ii) *Let* $\gamma_1, \ldots, \gamma_n$ *be given positive numbers. There is a* $\psi \in \mathscr{D}(T) \ominus \mathscr{F}$ *such that* $\rho_r^T(\psi) \geqslant \gamma_r$ *and* $\langle T^r\psi, \psi\rangle > 0$ *for* $r = 1, \ldots, n$.

Proof. (i) Let F be the orthogonal projection on $\mathscr{F}$. Since all operators T^k, $k \in \mathbf{N}$, are bounded on $\mathscr{F}$, there is a $c > 0$ such that $\|T^kF\varphi\| \leqslant c\|\varphi\|$ for all $\varphi \in \mathscr{D}$ and $k = 0, \ldots, n$. The latter implies that

$$|\langle T^k(I-F)\varphi, (I-F)\varphi\rangle - \langle T^k\varphi, \varphi\rangle| \leqslant 2c\|\varphi\|^2.$$

Given $\gamma \in \mathbf{R}_1$, there is a $\xi \in B_n(\mathscr{D}(T))$ such that $\langle T^n\xi, \xi\rangle \geqslant \gamma$, since $\lambda_n^+(T) = +\infty$. Then $\eta := (1+2c)^{-1/2}(I-F)\xi \in B_n(\mathscr{D}(T) \ominus \mathscr{F})$ and $\langle T^n\eta, \eta\rangle \geqslant (\gamma - 2c)/(1+2c)$. This implies $\lambda_n^+(T; \mathscr{D}(T) \ominus \mathscr{F}) = +\infty$.

(ii). Let $\mathscr{F}_1 = \mathscr{F}$ and $\mathscr{F}_j = \mathrm{Lin}\{\mathscr{F}, T^r\psi_i\,; r = 0, \ldots, n;\ i = 1, \ldots, j-1\}$ for $j \in \mathbf{N}$, $j \geqslant 2$. By a repeated application of (i) with $\mathscr{F}$ replaced by $\mathscr{F}_k$ we can find vectors $\psi_k \in B_k(\mathscr{D}(T) \ominus \mathscr{F}_k)$, $k = 1, \ldots, n$, such that $\langle T\psi_1, \psi_1\rangle > \gamma_1 n$ and $\langle T^k\psi_k, \psi_k\rangle > \gamma_k \left(n + \sum_{i=1}^{k-1} \langle T^{k-1}\psi_i, \psi_i\rangle\right)$ for $k \geqslant 2$. Putting $\psi = \psi_1 + \ldots + \psi_n$, we have for $r \in \{1, \ldots, n\}$

$$\langle T^{r-1}\psi, \psi\rangle = \sum_{i=1}^{n} \langle T^{r-1}\psi_i, \psi_i\rangle < n + \sum_{i=1}^{r-1} \langle T^{r-1}\psi_i, \psi_i\rangle <$$

$$< \gamma_r^{-1}\langle T^r\psi_r, \psi_r\rangle \leqslant \gamma_r^{-1}\langle T^r\psi, \psi\rangle.$$

(In case $r = 1$ the second sum is interpreted to be zero.) Hence $\rho_r^T(\psi) \geqslant \gamma_r$ and $\langle T^r\psi, \psi\rangle > 0$.

LEMMA 3. *Let $m \in \mathbf{N}_0$ and $k, n \in \mathbf{N}$. Suppose $a = (a_1 = 0, a_2, \ldots, a_k) \in \mathbf{R}^k$. Let $\varepsilon, \gamma, \delta_{m+1}, \ldots, \delta_{m+n}$ be real numbers satisfying $1 \geqslant \varepsilon > 0$, $\gamma > 0$, $\delta_{m+1} > 1, \ldots$ $\ldots, \delta_{m+n} > 1$.*

Then there are vectors $\psi_1, \ldots, \psi_k \in \mathscr{D}(T) \ominus \mathscr{F}$ such that:

(i) $\psi_i \perp T^r\psi_j$ *for $i \neq j$, $i, j = 1, \ldots, k$ and $r = 0, \ldots, m + n$;*

(ii) $0 < \langle T^r\psi_1, \psi_1\rangle = \ldots = \langle T^r\psi_k, \psi_k\rangle \leqslant \varepsilon$ *for $r \in \mathbf{N}_0$, $r < m$;*

(iii) $\langle T^m\psi_i, \psi_i\rangle = \langle T^m\psi_1, \psi_1\rangle + a_i \geqslant \gamma$ *for* $i = 1, \ldots, k$;

(iv) $\rho_r^T(\psi_{i+1}) \geqslant \delta_r(\rho_r^T(\psi_i) + 1)$ *and* $\rho_r^T(\psi_1) \geqslant \delta_r$ *for* $r = m + 1, \ldots, m + n$ *and $i \in \mathbf{N}$, $i < k$.*

Proof. We proceed by induction on m. Again we write $\rho_j(\cdot)$ for $\rho_j^T(\cdot)$.

First suppose $m = 0$. Let $\mathscr{F}_1 := \mathscr{F}$ and let

$$\mathscr{F}_j := \mathrm{Lin}\{\mathscr{F}, T^r\psi_i\,; r = 0, \ldots, n \text{ and } i = 1, \ldots, j-1\} \quad \text{for } j \geqslant 2.$$

Let $\gamma_{1r} := \delta_r$ and $\gamma_{jr} := \delta_r(\rho_r(\psi_{j-1}) + 1)$ for $j \geqslant 2$ and $r = 1, \ldots, n$. Applying Lemma 2 (ii), with $\mathscr{F}_j$ and $\gamma_{j1}, \ldots, \gamma_{jn}$ for $j = 1, \ldots, k$, we obtain non-zero vectors $\psi_1, \ldots, \psi_k \in \mathscr{D}(T) \ominus \mathscr{F}$ satisfying (i) and (iv). (iii) can be fulfilled if we replace the vectors ψ_i by some suitable scalar multiples of ψ_i.

Now let $m \in \mathbf{N}_0$. Assume that Lemma 3 is valid for m and all possible $k, n, a, \varepsilon, \gamma, \delta_{m+1}, \ldots, \delta_{m+n}$ and $\mathscr{F}$. We prove the assertion in case $m + 1$. Let $k, n, a, \varepsilon, \gamma, \delta_{m+2}, \ldots, \delta_{m+n+1}$ be as above. By the induction hypothesis (with n and k replaced by $n + 1$ resp. $2k - 1$ and $a = 0$) there are vectors $\xi_1, \ldots, \xi_{2k-1} \in \mathscr{D}(T) \ominus \mathscr{F}$ such that (i) and (ii) are satisfied for m and $n + 1$ and moreover

(iii) $\langle T^m\xi_i, \xi_i\rangle = \langle T^m\xi_1, \xi_1\rangle \geqslant 1$ for $i = 1, \ldots, 2k - 1$

and

(iv) $\rho_r(\xi_{i+1}) \geqslant \delta_r'(\rho_r(\xi_i) + 1)$ and $\rho_r(\xi_1) \geqslant \delta_r'$ for $i = 1, \ldots, 2k-1$ and $r = m + 1, \ldots, m + n + 1$

where

(1) $\delta'_r \geqslant 3\delta_r$ and $\delta'_{m+1} \geqslant 6 + (\gamma - 2(|a_2| + \ldots + |a_k|))/\varepsilon$.

Let $M = \langle T^m\xi_1, \xi_1\rangle$ and $M_i = \langle T^{m+1}\xi_i, \xi_i\rangle$. From (iii) and (iv) we know that $M \geqslant 1$ and $M_{2k-1} > M_{2k-2} > \ldots > M_1 > 0$. Define

$$\psi_1 := \sqrt{\frac{\varepsilon}{M}}\,\xi_k \quad \text{and} \quad \psi_i := \sqrt{\frac{\varepsilon}{M}}\,(\sqrt{1-t_i^2}\,\xi_{i-1} + t_i\xi_{k+i-1})$$

for $2 \leqslant i \leqslant k$, where $t_i := (M_k - M_{i-1} + a_iM/\varepsilon)^{1/2}(M_{k+i-1} - M_{i-1})^{-1/2}$. (We check below that $M_k - M_{i-1} + a_iM/\varepsilon > 0$). We verify (i)—(iv) for these vectors.

Recall that we used the induction hypothesis with n replaced by $n+1$. Therefore, (i) is true for $m+1$ and n. For $r = 0, \ldots, m$ and $i = 1, \ldots, k$, we have $\langle T^r\psi_i, \psi_i\rangle = \frac{\varepsilon}{M}\langle T^r\xi_1, \xi_1\rangle$. Since $\frac{\varepsilon}{M} \leqslant 1$ and $\langle T^r\xi_1, \xi_1\rangle \leqslant \varepsilon$ for $r < m$ by the induction hypothesis (ii), the latter gives (ii) in case $m+1$. From the definition of t_i it follows that $\langle T^{m+1}\psi_i, \psi_i\rangle = a_i + \varepsilon M_k/M = \langle T^{m+1}\psi_1, \psi_1\rangle + a_i$ for $2 \leqslant i \leqslant k$. By (1), $\rho_{m+1}(\xi_k) \geqslant \delta'_{m+1} \geqslant (\gamma - a_i)/\varepsilon$ which leads to $a_i + \varepsilon M_k/M \geqslant \gamma$ and completes the proof of (iii).

Finally, we prove (iv). Fix an $i \in \mathbf{N}$, $2 \leqslant i \leqslant k$. We first check that

(2) $2\langle T^r\xi_{i-1}, \xi_{i-1}\rangle \leqslant t_i^2\langle T^r\xi_{k+i-1}, \xi_{k+i-1}\rangle$ for $r = m+1, \ldots, m+n+1$.

Indeed, since $\delta'_{m+1} \geqslant -2a_i/\varepsilon$ and $\delta'_{m+1} \geqslant 6$, we have $M_k/2 \geqslant -a_iM/\varepsilon$ and $M_k/2 \geqslant 3M_{i-1}$. Hence $M_k - M_{i-1} + a_iM/\varepsilon \geqslant 2M_{i-1}$ and $M_{k+i-1}t_i^2 \geqslant 2M_{i-1}$. This is (2) in case $r = m+1$. Since $\delta_l > 1$ by assumption, (iii) and (iv) imply that $\langle T^l\xi_j, \xi_j\rangle > 0$ and that $\rho_l(\xi_j)$ is monotonic w.r.t. j for $l = m+2, \ldots, m+n+1$ and $j = 1, \ldots, 2k-1$. Hence

$$2\langle T^r\xi_{i-1}, \xi_{i-1}\rangle = \rho_r(\xi_{i-1}) \ldots \rho_{m+2}(\xi_{i-1})\, 2M_{i-1} \leqslant$$

$$\leqslant \rho_r(\xi_{k+i-1}) \ldots \rho_{m+2}(\xi_{k+i-1})M_{k+i-1}t_i^2 = t_i^2\langle T^r\xi_{k+i-1}, \xi_{k+i-1}\rangle$$

for $m+2 \leqslant r \leqslant m+n+1$. This proves (2).

Let $r \in \{m+2, \ldots, m+n+1\}$. From (2) and (i) it follows that

$$\rho_r(\psi_i) \leqslant \left(\frac{\varepsilon}{M}\frac{3}{2}t_i^2\langle T^r\xi_{k+i-1}, \xi_{k+i-1}\rangle\right)\left(\frac{\varepsilon}{M}t_i^2\langle T^{r-1}\xi_{k+i-1}, \xi_{k+i-1}\rangle\right)^{-1} =$$

$$= \frac{3}{2}\rho_r(\xi_{k+i-1})$$

and similarly $\rho_r(\psi_i) \geqslant \frac{2}{3}\rho_r(\xi_{k+i-1})$. Moreover, $\rho_r(\psi_1) = \rho_r(\xi_k)$. Now (iv) in case $m+1$ follows immediately from the induction hypothesis (iv) and $\delta_r' \geqslant 3\delta_r$.

REMARK 1. Let A_1 and A_2 be as in Theorem 2 and let $T = A_1 \oplus A_2$ in $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Let $\mathscr{N}_1 \cup \mathscr{N}_2$ be a decomposition of the set $\{1, \ldots, k\}$ into disjoint subsets. Then the vectors ψ_i in Lemma 3 can be chosen such that in addition $\psi_i \in \mathscr{D}(A_1)$ if $i \in \mathscr{N}_1$ and $\psi_i \in \mathscr{D}(A_2)$ if $i \in \mathscr{N}_2$. In the induction proof given above it suffices to take $\xi_k \in \mathscr{D}(A_s)$ if $1 \in \mathscr{N}_s$ and $\xi_{i-1}, \xi_{k+i-1} \in \mathscr{D}(A_s)$ if $i \in \mathscr{N}_s$, $i \geqslant 2$, for $s \in \{1, 2\}$.

2.2. LEMMA 4. *Let A_1 and A_2 be as in Theorem 2. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are finite dimensional subspaces of $\mathscr{D}(A_1)$ resp. $\mathscr{D}(A_2)$. Let $l \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$. Let $b_j \in \mathbf{R}_1$ and $a_{skn} = \overline{a_{snk}} \in \mathbf{C}_1$ be given numbers where $j, k, n = 0, \ldots, l$, $k \neq n$, and $s = 1, 2$. Let ε and γ be positive numbers.*

Then there are vectors $\eta_0^s, \ldots, \eta_l^s \in \mathscr{D}(A_s) \ominus \mathscr{F}_s$, $s = 1, 2$, such that:

(i) $\langle A_s^r \eta_k^s, \eta_n^s \rangle = 0$;

(ii) $\langle A_s^m \eta_k^s, \eta_n^s \rangle = a_{skn}$;

(iii) $0 < \langle A_1^r \eta_j^1, \eta_j^1 \rangle = \langle A_2^r \eta_j^2, \eta_j^2 \rangle \leqslant \varepsilon$;

(iv) $\langle A_1^m \eta_j^1, \eta_j^1 \rangle - \langle A_2^m \eta_j^2, \eta_j^2 \rangle = b_j$;

(v) $\langle A_s^m \eta_j^s, \eta_j^s \rangle \geqslant \gamma$,

for all $r \in \mathbf{N}, r < m$, $j, k, n = 0, \ldots, l$, $k \neq n$ and $s = 1, 2$.

Proof. Let $s \in \{1, 2\}$. We set $a_{skk} = 0$ for $k = 0, \ldots, l$. Let $\mu_{s0}, \ldots, \mu_{sl}$ be the eigenvalues of the Hermitean matrix $\mathscr{A}_s := (a_{skn})_{0 \leqslant k, n \leqslant l}$. From elementary linear algebra we know that there is a unitary matrix $\mathscr{U}_s = (u_{skn})_{0 \leqslant k, n \leqslant l}$ such that $\mathscr{U}_s^* \mathscr{A}_s \mathscr{U}_s$ is the diagonal matrix whose diagonal entries equal $\mu_{s0}, \ldots, \mu_{sl}$. Hence

$$a_{skn} = \sum_{i=0}^{l} u_{ski} \overline{u_{sni}} \mu_{si} \quad \text{for } k, n = 0, \ldots, l. \tag{3}$$

We apply Lemma 3 and Remark 1 in case $T = A_1 \oplus A_2$ and $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. We thus obtain vectors $\xi^1, \psi_0^1, \ldots, \psi_{2l+1}^1 \in \mathscr{D}(A_1) \ominus \mathscr{F}_1$ and $\psi_0^2, \ldots, \psi_{2l+1}^2 \in \mathscr{D}(A_2) \ominus \mathscr{F}_2$ such that:

$$\psi_j^s \perp A_s^r \psi_k^s, A_s^m \psi_k^s; \tag{4}$$

$$0 < \langle A_s^r \psi_i^s, \psi_i^s \rangle = \langle A_s^r \psi_{l+i+1}^s, \psi_{l+i+1}^s \rangle = \langle A_1^r \xi^1, \xi^1 \rangle \leqslant \varepsilon/2; \tag{5}$$

$$\langle A_s^m \psi_i^s, \psi_i^s \rangle = \langle A_1^m \xi^1, \xi^1 \rangle + \mu_{si}; \tag{6}$$

$$\langle A_1^m \psi_{l+i+1}^1, \psi_{l+i+1}^1 \rangle = \langle A_1^m \xi^1, \xi^1 \rangle + b_i > 0; \tag{7$_1$}$$

$$\langle A_2^m \psi_{l+i+1}^2, \psi_{l+i+1}^2 \rangle = \langle A_1^m \xi^1, \xi^1 \rangle \geqslant \gamma, \tag{7$_2$}$$

for all $j, k = 0, \ldots, 2l$, $j \neq k$, $r = 0, \ldots, m-1$, $i = 0, \ldots, l$ and $s = 1, 2$.

Define $\eta_k^s = \psi_{l+k+1}^s + \sum_{i=0}^{l} u_{ski}\psi_i^s$ for $k = 0, \ldots, l$ and $s = 1, 2$. It is obvious that $\eta_k^s \in \mathscr{D}(A_s) \ominus \mathscr{F}_s$. We show that the conditions (i)–(v) are satisfied.

Let $s \in \{1, 2\}$ be fixed. Suppose $k, n \in \{0, \ldots, l\}$, $k \neq n$ and $r \in \{0, \ldots, m\}$. By (4),

$$\langle A_s^r \eta_k^s, \eta_n^s \rangle = \sum_{i=0}^{l} u_{ski}\overline{u_{sni}}\langle A_s^r \psi_i^s, \psi_i^s \rangle. \tag{8}$$

If $r < m$, then $\langle A_s^r \psi_i^s, \psi_i^s \rangle$ does not depend on $i \in \{0, \ldots, l\}$ by (5). Since the rows of $\mathscr{U}_s$ are orthonormal, (8) implies $\langle A_s^r \eta_k^s, \eta_n^s \rangle = 0$. Now let $r = m$. From (8), (6) and (3) we conclude that

$$\langle A_s^m \eta_k^s, \eta_n^s \rangle = \sum_{i=0}^{l} u_{ski}\overline{u_{sni}}(\langle A_1^m \xi^1, \xi^1 \rangle + \mu_{si}) = a_{skn}.$$

Next we prove (iii)–(iv). Let $j \in \{0, \ldots, l\}$ and $r \in \{0, \ldots, m\}$. By (4),

$$\langle A_s^r \eta_j^s, \eta_j^s \rangle = \langle A_s^r \psi_{l+j+1}^s, \psi_{l+j+1}^s \rangle + \sum_{i=0}^{l} u_{sji}\overline{u_{sji}}\langle A_s^r \psi_i^s, \psi_i^s \rangle. \tag{9}$$

Since each row vector of $\mathscr{U}_s$ has norm one, it follows from (9) and (5) that $0 < \langle A_s^r \eta_j^s, \eta_j^s \rangle = 2\langle A_1^r \xi^1, \xi^1 \rangle \leqslant \varepsilon$ for $r < m$. This proves (iii). Suppose $r = m$. Putting $(7)_1$ into (9) and using (6), we get

$$\langle A_1^m \eta_j^1, \eta_j^1 \rangle = 2\langle A_1^m \xi^1, \xi^1 \rangle + b_j + \sum_{i=0}^{l} u_{sji}\overline{u_{sji}}\mu_{1i}.$$

By (3), the sum equals a_{1jj} which is zero by definition. Hence $\langle A_1^m \eta_j^1, \eta_j^1 \rangle = 2\langle A_1^m \xi^1, \xi^1 \rangle + b_j \geqslant \gamma$ by $(7)_1$ and $(7)_2$. Similarly, using $(7)_2$, $\langle A_2^m \eta_j^2, \eta_j^2 \rangle = 2\langle A_1^m \xi^1, \xi^1 \rangle \geqslant \gamma$. Both formulas together imply (iv) and (v).

The proof of Lemma 4 is complete.

3. PROOF OF THEOREM 2

Without loss of generality we assume that $\gamma_{kj} \geqslant 2^j$ for $j, k \in \mathbf{N}_0$. We set $d(i, j) = 2i + j$ for $(i, j) \in \mathbf{N}_0 \times \mathbf{N}_0$. For $m \in \mathbf{N}$, let $\mathbf{N}_m = \left\{(i, m - 2i)\ ;\ i = 0, \ldots \ldots, \left[\frac{m-1}{2}\right]\right\}$, where $\left[\frac{m-1}{2}\right]$ is the entire part of $\frac{m-1}{2}$. For $(i, j) \in \mathbf{N}_0 \times \mathbf{N}$, we then have $(i, j) \in \mathbf{N}_m$ if and only if $d(i, j) = m$. Let $s \in \{1, 2\}$. Since $\mathscr{H}_s$ is assumed to be separable, there is an orthonormal base $\{\zeta_j^s, j \in \mathbf{N}\}$ of $\mathscr{H}_s$ consisting of vectors $\zeta_j^s \in \mathscr{D}(A_s)$. Define $\{\xi_j^s := \zeta_{k_j}^s\ j \in \mathbf{N}\}$ where $\{k_j\}$ is the following sequence: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4 etc. We first show that there exist double sequences $\{\varphi_{kj}^s, (k,j) \in \mathbf{N}_0 \times \mathbf{N}_0\}$, $s \in \{1, 2\}$, of vectors $\varphi_{kj}^s \in \mathscr{D}(A_s)$ such that the following conditions are fulfilled. For all $k, l, n, r \in \mathbf{N}_0$, $i, j \in \mathbf{N}$ and $s \in \{1, 2\}$, we have:

(A.1) If $d(k,j) > d(n,l)$ and $d(k,j) + d(n,l) \geqslant r$, then $\langle A_s^r \varphi_{kj}^s, \varphi_{nl}^s \rangle = 0$;

(A.2) If $d(k,j) = d(n,i) = m$, $k \neq n$ and $m - k - n > r$, then $\langle A_s^r \varphi_{kj}^s, \varphi_{ni}^s \rangle = 0$;

(A.3) If $d(k,j) = d(n,i) = m$ and $k \neq n$, then

$$\langle A_s^{m-k-n} \varphi_{kj}^s, \varphi_{ni}^s \rangle + \langle A_s^{m-k-n} \psi_{kj}^s, \psi_{ni}^s \rangle = 0;$$

(A.4) $\varphi_{j0}^s \perp \mathscr{E}_{sj} := \operatorname{Lin}\{A_s^r(\varphi_{n0}^s + \ldots + \varphi_{n,2(j-n)}^s);\ \ n = 0, \ldots, j-1$ and $r = 0, \ldots, j-n\}$;

(B.1) If $j > r$, then $\langle A_1^r \varphi_{kj}^1, \varphi_{kj}^1 \rangle = \langle A_2^r \varphi_{kj}^2, \varphi_{kj}^2 \rangle > 0$;

(B.2) $\langle A_1^j \varphi_{kj}^1, \varphi_{kj}^1 \rangle + \langle A_1^j \psi_{kj}^1, \psi_{kj}^1 \rangle = \langle A_2^j \varphi_{kj}^2, \varphi_{kj}^2 \rangle + \langle A_2^j \psi_{kj}^2, \psi_{kj}^2 \rangle$;

(C.1) $\|\varphi_{k0}^s\| = 2$ and if $j > r$, then $|\langle A_s^r \varphi_{kj}^s, \varphi_{kj}^s \rangle| \leqslant 2^{-r-2j-6k}$;

(C.2) $\langle A_s^j \varphi_{kj}^s, \varphi_{kj}^s \rangle \geqslant 2|\langle A_s^j \psi_{kj}^s, \psi_{kj}^s \rangle| + 2$;

(C.3) If $d(k,j) > d(n,l)$, then

$$\langle A_s^j \varphi_{kj}^s, \varphi_{kj}^s \rangle \geqslant 36\gamma_{k,j-1} \langle A_s^{j-1} \varphi_{k,j-1}^s, \varphi_{k,j-1}^s \rangle \cdot \langle A_s^l \varphi_{nl}^s, \varphi_{nl}^s \rangle;$$

(D) There are numbers $n_{si} \in \mathbf{N}$ such that $n_{s,i+1} > n_{si} \geqslant i$ and

$$\xi_n^s \in \mathscr{G}_{s,i-1} \quad \text{if } n_{s,i-1} < n \leqslant n_{si}.$$

Here we used the abbreviations $\mathscr{G}_{s0} = \operatorname{Lin}\{\varphi_{00}^s\}$, $\mathscr{G}_{sj} = \operatorname{Lin}\{\varphi_{j0}^s, \mathscr{E}_{sj}\}$, $n_{s0} = 0$ and $\psi_{kj}^s = \varphi_{k0}^s + \ldots + \varphi_{k,j-1}^s$ for $j \in \mathbf{N}$, $k \in \mathbf{N}_0$ and $s \in \{1, 2\}$.

The existence of double sequences having these properties will be proved by induction. Set $\varphi_{00}^s = \xi_1^s$ and $n_{11} = n_{21} = 1$. By Lemma 4, we can find vectors $\varphi_{01}^s \in \mathscr{D}(A_s)$ such that $\varphi_{01}^s \perp \varphi_{00}^s, A_s\varphi_{00}^s$, $\|\varphi_{01}^1\| = \|\varphi_{01}^2\| = 1/2$, $\langle A_1\varphi_{01}^1, \varphi_{01}^1 \rangle - \langle A_2\varphi_{01}^2, \varphi_{01}^2 \rangle = \langle A_2\psi_{01}^2, \psi_{01}^2 \rangle - \langle A_1\psi_{01}^1, \psi_{01}^1 \rangle$ and $\langle A_s\varphi_{01}^s, \varphi_{01}^s \rangle \geqslant 2|\langle A_s\psi_{01}^s, \psi_{01}^s \rangle| + 2 + 36 \cdot 16 \cdot \gamma_{00}$ for $s \in \{1, 2\}$. Now let $m \in \mathbf{N}$, $m \geqslant 2$. Suppose all vectors φ_{kj}^s for which $d(k,j) < m$ with $j \neq 0$ and φ_{k0}^s for which $k < \left[\dfrac{m-1}{2}\right]$ are chosen such that the above conditions are satisfied (more precisely, all conditions involving these vectors are true). Let $l := \left[\dfrac{m-1}{2}\right]$. We want to construct φ_{l0}^s and φ_{kj}^s for $(k,j) \in \mathbf{N}_m$.

We begin with φ_{10}^s. If m is even, then $\left[\frac{m-1}{2}\right] = \left[\frac{m-2}{2}\right]$ and φ_{l0}^s is already defined. Suppose now that m is odd. Let E_{sl} be the orthogonal projection on $\mathscr{E}_{sl}$. Let n_{sl} be the smallest integer $n > n_{s,l-1}$ for which $\xi_n^s \notin \mathscr{E}_{sl}$. Define

$$\varphi_{10}^s = 2(I - E_{sl})\xi_{n_{sl}}^s / \|(I - E_{sl})\xi_{n_{sl}}^s\|.$$

Then, (A.4) and (D) are true in case $j = l$ resp. $i = l$.

We next define φ_{kj}^s for $(k,j) \in \mathbf{N}_m$. Let $\mathscr{F}_s$ be the linear span of all vectors $A_s^r\varphi_{ni}^s$ where $r \in \{0, \ldots, 3m\}$ and $d(n,i) < m$, $n, i \in \mathbf{N}_0$. We apply Lemma 4 in case $\mathscr{F}_s$, $a_{skn} := -\langle A_s^{m-k-n}\psi_{k,m-2k}^s, \psi_{n,m-2n}^s\rangle$, $b_k := -\langle A_1^{m-2k}\psi_{k,m-2k}^1, \psi_{k,m-2k}^1\rangle + \langle A_2^{m-2k}\psi_{k,m-2k}^2, \psi_{k,m-2k}^2\rangle$ for $k, n = 0, \ldots, l$ and $k \neq n$, $\varepsilon := 2^{-2l-3m}$ and

$$\gamma := \max\{2|\langle A_s^j\psi_{kj}^s, \psi_{kj}^s\rangle| + 2, 36\gamma_{k,j-1}\langle A_s^{j-1}\varphi_{k,j-1}^s, \varphi_{k,j-1}^s\rangle \langle A_s^l\varphi_{nl}^s, \varphi_{nl}^s\rangle;$$

$$(k,j) \in \mathbf{N}_m,\ l, n \in \mathbf{N}_0,\ s \in \{1,2\},\ d(k,j) > d(n,l)\}.$$

We then obtain vectors $\eta_k^s \in \mathscr{D}(A_s)$ which are orthogonal to $\mathscr{F}_s$ and satisfy the conditions (i)—(v) of Lemma 4. We define $\varphi_{k,m-2k}^s := A_s^k\eta_k^s$ for $k = 0, \ldots, l$ and check the conditions (A.1)—(A.3), (B.1)—(C.3) for these vectors. $\eta_k^s \perp \mathscr{F}_s$ for $k = 0, \ldots, l$ ensures (A.1) in case $(k,j) \in \mathbf{N}_m$. (A.2) follows from (i) and (A.3) from (ii). (iii) implies (B.1) and (C.1), since $\|\varphi_{l0}^s\| = 2$ by definition. (iv) gives (B.2). (C.2) and (C.3) follow from (v). Thus the existence of the sequences $\{\varphi_{kj}^s\}$ is proved.

Let $s \in \{1, 2\}$. Let t_s be the locally convex topology on $\mathscr{D}(A_s)$ generated by the seminorms $\|\varphi\|_n := \|A_s^n\varphi\| + \|\varphi\|$, $\varphi \in \mathscr{D}(A_s)$, $n \in \mathbf{N}$. Since we assumed that $\mathscr{D}(A_s) = \bigcap_{n=1}^{\infty} \mathscr{D}(\overline{A_s^n})$, the locally convex space $\mathscr{D}(A_s)[t_s]$ is complete. From (C.1) we conclude that $\left\{\sum_{j=0}^{n} \varphi_{kj}^s\right\}$ is a t_s-Cauchy sequence for each $k \in \mathbf{N}_0$. Therefore, the infinite series $\sum_{j=0}^{\infty} \varphi_{kj}^s$ converges in $\mathscr{D}(A_s)[t_s]$ and define a vector $\varphi_k^s \in \mathscr{D}(A_s)$.

We now verify the conditions (a)—(d) for $\{\varphi_k^s\}$.

First we show that $\langle A_s^r\varphi_k^s, \varphi_n^s\rangle = 0$ for $k, n, r \in \mathbf{N}_0$, $k \neq n$, $s \in \{1, 2\}$. Since A_s is symmetric, we can assume that $k > n$.

Case 1. $r > k - n$.

Let $m = r + k + n$. Then $m - 2n > m - 2k = r + n - k > 0$. We have

$$\langle A_s^r\varphi_k^s, \varphi_n^s\rangle = \langle A_s^{m-k-n}\varphi_{k,m-2k}^s, \varphi_{n,m-2n}^s\rangle +$$

$$+ \langle A_s^{m-k-n}\psi_{k,m-2k}^s, \psi_{n,m-2n}^s\rangle + \sum \langle A_s^{m-k-n}\varphi_{kj}^s, \varphi_{nl}^s\rangle,$$

where the sum runs over all pairs $(j, l) \neq (m-2k, m-2n)$ for which $j \geqslant m-2k$ or $l \geqslant m-2n$. We denote this sum by S. By (A.3), $\langle A_s^r \varphi_k^s, \varphi_n^s \rangle = S$. We show that all summands of S are zero. We abbreviate $\alpha := \langle A_s^{m-k-n} \varphi_{kj}^s, \varphi_{nl}^s \rangle$. First let $l = 0$. Since $l = 0 < m-2n$, we then have $j \geqslant m-2k$, that is, $d(k,j) \geqslant m > 2n = d(n,0)$ and $m-k-n \leqslant m \leqslant d(k,j) + d(n,0)$. Thus, by (A.1), $\alpha = 0$. Similarly, $\alpha = 0$ in case $j = 0$. Suppose now $j \neq 0$ and $l \neq 0$. If $d(k,j) \neq d(n,l)$, then $\alpha = 0$ by (A.1), because $m-k-n \leqslant m \leqslant 2(k+n) + j + l = d(k,j) + d(n,l)$. If $d(k,j) = d(n,l) =: m'$, then $m' > m$, since $(j,l) \neq (m-2k, m-2n)$ and $j \geqslant m-2k$ or $l \geqslant m-2n$. In that case, (A.2) yields $\alpha = 0$. Therefore $S = 0$.

Case 2. $r \leqslant k-n$.

Since $r \leqslant k-n$ and $n < k$, $\langle A_s^r \varphi_{k0}^s, \varphi_{n0}^s + \ldots + \varphi_{n,2(k-n)}^s \rangle = 0$ by (A.4). If $l > 2(k-n)$, then $d(n,l) > d(k,0)$ and $d(n,l) + d(k,0) \geqslant 2n + 2(k-n) \geqslant k-n \geqslant r$ which implies $\langle A_s^r \varphi_{k0}^s, \varphi_{nl}^s \rangle = 0$ by (A.1). Hence $\langle A_s^r \varphi_{k0}^s, \varphi_n^s \rangle = 0$. It remains to show that $\alpha := \langle A_s^r \varphi_{kj}^s, \varphi_{nl}^s \rangle = 0$ for $j \in \mathbf{N}$ and $l \in \mathbf{N}_0$. If $l = 0$, then $d(k,j) > d(n,0)$ and $d(k,j) + d(n,0) \geqslant r$ and hence $\alpha = 0$ by (A.1). Suppose now $j, l \in \mathbf{N}$. If $d(k,j) \neq d(n,l)$, then $\alpha = 0$ by (A.1), because $r \leqslant k-n \leqslant d(k,j) + d(n,l)$. For $d(k,j) = d(n,l) =: m$, we have $\alpha = 0$ by (A.2), since $r \leqslant k-n < 2k + j - k - n = m-k-n$. This completes the proof of (a).

Next we prove (b). Using again (A.1), it follows that

$$\langle A_s^r \varphi_k^s, \varphi_k^s \rangle = \langle A_s^r \varphi_{kr}^s, \varphi_{kr}^s \rangle + \langle A_s^r \psi_{kr}^s, \psi_{kr}^s \rangle + \sum_{j > r} \langle A_s^r \varphi_{kj}^s, \varphi_{kj}^s \rangle$$

for $r \in \mathbf{N}$. (B.1), (B.2) and (C.2) imply $\langle A_1^r \varphi_k^1, \varphi_k^1 \rangle = \langle A_2^r \varphi_k^2, \varphi_k^2 \rangle > 0$ for $k \in \mathbf{N}_0$, $r \in \mathbf{N}$. With (B.2) replaced by $\langle \varphi_{k0}^1, \varphi_{k0}^1 \rangle = \langle \varphi_{k0}^2, \varphi_{k0}^2 \rangle$, the same argument gives $\langle \varphi_k^1, \varphi_k^1 \rangle = \langle \varphi_k^2, \varphi_k^2 \rangle > 0$.

Now we pass to (c). Let $s \in \{1, 2\}$. Applying once more (A.1) and then (C.1) and (C.2), we conclude that

$$\begin{aligned} &|\langle A_s^r \varphi_k^s, \varphi_k^s \rangle - \langle A_s^r \varphi_{kr}^s, \varphi_{kr}^s \rangle| = |\langle A_s^r \psi_{kr}^s, \psi_{kr}^r \rangle + \sum_{j > r} \langle A_s^r \varphi_{kj}^s, \varphi_{kj}^s \rangle| \leqslant \\ (1) \qquad &\leqslant |\langle A_s^r \psi_{kr}^s, \psi_{kr}^s \rangle| + 1 \leqslant \frac{1}{2} \langle A_s^r \varphi_{kr}^s, \varphi_{kr}^s \rangle \quad \text{for } k \in \mathbf{N}_0,\ r \in \mathbf{N}. \end{aligned}$$

From (C.1) it follows that

$$(2) \qquad \|\varphi_k^s\| \geqslant 1 \quad \text{for } k \in \mathbf{N}_0.$$

By (1), (2) and (C.2), we have

$$(3) \qquad \langle A_s^r \varphi_k^s, \varphi_k^s \rangle \geqslant 1 \quad \text{for } k, r \in \mathbf{N}_0.$$

Now suppose that $j, k, l, n \in \mathbf{N}_0$ and $2k + j + 1 > 2n + l$. First let $l \geqslant 1$. From (1), (3) and (C.3) we obtain

$$\langle A_s^{j+1} \varphi_k^s, \varphi_k^s \rangle \langle A_s^{l-1} \varphi_n^s, \varphi_n^s \rangle \geqslant \frac{1}{2} \cdot \langle A_s^{j+1} \varphi_{k,j+1}^s, \varphi_{k,j+1}^s \rangle \cdot 1 \geqslant$$

$$\geqslant \frac{1}{2} \cdot 36 \gamma_{kj} \langle A_s^j \varphi_{kj}^s, \varphi_{kj}^s \rangle \langle A_s^l \varphi_{nl}^s, \varphi_{nl}^s \rangle \geqslant$$

$$\geqslant 18 \gamma_{kj} \left(\frac{2}{3} \right)^2 \langle A_s^j \varphi_k^s, \varphi_k^s \rangle \langle A_s^l \varphi_n^s, \varphi_n^s \rangle,$$

i.e., $\rho_{j+1}^{A_s}(\varphi_k^s) \geqslant 8\gamma_{kj} \rho_l^{A_s}(\varphi_n^s)$. Since $\|\varphi_{k0}^s\| = 2$, a similar argument yields $\langle A_s^{j+1} \varphi_k^s, \varphi_k^s \rangle \geqslant$ $\geqslant 8\gamma_{kj} \langle A_s^j \varphi_k^s, \varphi_k^s \rangle$, i.e., $\rho_{j+1}^{A_s}(\varphi_k^s) \geqslant 8\gamma_{kj} \rho_j^{A_s}(\varphi_n^s)$. This completes the proof of (c).

Finally, we prove (d). Fix $s \in \{1, 2\}$. Let $\xi \in \mathscr{G}_{sj}$ for some $j \in \mathbf{N}$. Then, ξ has the form

$$\xi = \sum_{n=0}^{j} \sum_{r=0}^{j-n} \mu_{nr} A_s^r (\varphi_{n0}^s + \ldots + \varphi_{n,2(j-n)}^s),$$

where $\mu_{nr} \in \mathbf{C}$. Let $\tilde{\xi} := \sum_{n=0}^{j} \sum_{r=0}^{j-n} \mu_{nr} A_s^r \varphi_n^s$. From (C.1) we get

$$\|\xi - \tilde{\xi}\| = \left\| \sum_{n=0}^{j} \sum_{r=0}^{j-n} \mu_{nr} A_s^r \left(\sum_{l > 2(j-n)} \varphi_{nl}^s \right) \right\| \leqslant$$

$$\leqslant \sum_{n,r} |\mu_{nr}| \sum_{l > 2(j-n)} 2^{-r-l-3n} \leqslant \sum_{n,r} |\mu_{nr}| \, 2^{-r-n-2j} \leqslant$$

$$\leqslant 2^{-2j+1} \left(\sum_{n,r} |\mu_{nr}|^2 \right)^{1/2} \leqslant 2^{-2j+1} \left(\sum_{n,r} |\mu_{nr}|^2 \langle A_s^{2r} \varphi_n^s, \varphi_n^s \rangle \right)^{1/2}.$$

Here we used (3) and the Cauchy-Schwarz inequality. Since we assumed at the beginning that $\gamma_{kj} \geqslant 2^j$ for $j, k \in \mathbf{N}_0$ and (a)—(c) are already proven, we can apply Corollary 2 in case $\varepsilon_k = 1/2$ and $r = 0$. (Recall that (d) is not used in the proof of Corollary 2.) Hence

$$(4) \qquad \|\xi - \tilde{\xi}\| \leqslant 2^{-2j+1} (2\|\tilde{\xi}\|^2)^{1/2} \leqslant 2^{-2j+2} \|\tilde{\xi}\|.$$

Now fix an $i \in \mathbf{N}$. Let $j \in \mathbf{N}$, $j \geqslant 2$. By the definition of $\{\zeta_j^s\}$, we can find an $n \in \mathbf{N}$, $n \geqslant n_{sj}$, such that $\zeta_i^s = \xi_n^s$. There is an $j' \in \mathbf{N}$, $j' \geqslant j$, so that $n_{s,j'-1} < n \leqslant n_{sj'}$. By (D), $\xi_n^s \in \mathscr{G}_{sj'}$. Since $j' \geqslant j$, it follows from (4) that $\|\zeta_i^s - \tilde{\xi}_n^s\| \leqslant 2^{-2j+3} \|\zeta_i^s\|$.

Since $\tilde{\zeta}_n^s \in \mathscr{D}_{s0}$ and $j \in \mathbf{N}$ was arbitrary, ζ_j^s is in the norm closure of $\mathscr{D}_{s0}$ which implies (d).

Now the proof of Theorem 2 is complete.

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KONRAD SCHMÜDGEN
Sektion Mathematik,
Karl-Marx-Universität Leipzig,
7010 Leipzig,
DDR.

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