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UNIFORM ERGODIC THEOREMS FOR IDENTITY PRESERVING SCHWARZ MAPS ON W^*-ALGEBRAS

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1.

A bounded linear operator T on a Banach space E is called *uniformly ergodic* (resp. *strongly ergodic*), if the averages $M_n(T) := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ converge in the uniform operator topology (resp. strong operator topology). The limit P of the sequence $(M_n(T))$ satisfies $P = PT = TP = P^2$ and is called the *ergodic projection* associated with T. It follows that P is a projection of E onto the fixed space $F(T) := \{x \in E : Tx = x\}$ of T. Obviously, every uniformly ergodic operator is strongly ergodic with the same ergodic projection. But in general there is no further connection between these two concepts. For example, every contraction on a Hilbert space is strongly ergodic ([13, Theorem III.7.11]), whereas a contraction is uniformly ergodic iff 1 is a pole of the resolvent $R(\cdot, T)$ ([5, VII; 9, Proposition on p. 223]). An important class of uniformly ergodic operators is formed by the *quasi-compact* operators, i.e. those T for which there exists a sequence (K_n) of compact operators such that $\lim_n \|T^n - K_n\| = 0$ ([5, VIII.8]). But in general, even a uniformly ergodic operator with finite-dimensional fixed space is not quasi-compact (for an example see [9, p. 224]).

In this paper we study identity preserving Schwarz maps on unital C^*-algebras $\mathscr{A}$, i.e. those $T \in \mathscr{L}(\mathscr{A})$ such that $T\mathbf{1} = \mathbf{1}$ and $T(xx^*) \geqslant T(x)T(x)^*$. In [7] it is proved that such a map is uniformly ergodic with finite-dimensional fixed space iff it is quasi-compact. This surprising result has great influence on the peripheral spectrum of T (see [7] for details). In this paper we prove uniform ergodic theorems for identity preserving Schwarz maps T on W^*-algebras $\mathscr{M}$. We give a short outline of our results: In Section two we study the ultrapower of the predual $\mathscr{M}_*$ of $\mathscr{M}$ and the structure of the fixed space $F(T_*)$, if T possesses a preadjoint $T_* \in \mathscr{L}(\mathscr{M}_*)$. The key for our uniform ergodic theorems is Lemma 3.1.

We show that an identity preserving Schwarz map T on a W^*-algebra $\mathcal{M}$ with preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$ is not uniformly ergodic with finite-dimensional fixed space, if

(i) there exists a sequence (z_n) in $\mathcal{M}$ such that $0 \leqslant z_n \leqslant \mathbf{1}$ $(n \in \mathbf{N})$ and $\lim_n z_n = 0$ in the $s^*(\mathcal{M}, \mathcal{M}_*)$-topology and

(ii) there exists a sequence of normal states (ψ_n) such that $\lim_n \|(I - T_*)\psi_n\| = 0$ and $\psi_n(z_n) \geqslant 1/2$ for all $n \in \mathbf{N}$.

From this it follows that every identity preserving Schwarz map T on $\mathcal{M}$ is uniformly ergodic with finite-dimensional fixed space, if every T^*-invariant state is normal (Theorem 3.2) and that an irreducible identity preserving Schwarz map on a W^*-algebra is quasi-compact (Theorem 3.4). If we assume that T possesses a preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$ then the following properties are proved to be equivalent (Corollary 3.3):

(a) T_* is uniformly ergodic with finite-dimensional fixed space.

(b) T is strongly ergodic with finite-dimensional fixed space.

(c) Every T^*-invariant state is normal.

For similar results concerning uniformly ergodic operators on commutative C^*-algebras we refer to [2, Theorem 2], [3, Theorem 4.6] and the recent results in [10]. Strongly ergodic operators on the predual of a W^*-algebra are discussed in [8].

Concerning the theory of C^*-and W^*-algebras we are following the monographs [12, 15]. In particular, we frequently make use of the various topologies on a W^*-algebra $\mathcal{M}$ ([12, 1.8]). In particular, we denote by $s^*(\mathcal{M}, \mathcal{M}_*)$ the topology on $\mathcal{M}$ generated by the semi-norms $x \mapsto [\psi(xx^*) + \psi(x^*x)]^{1/2}$ where ψ runs through $\mathcal{M}_*^+$ (in [15] is called the σ-strong* topology). For topological properties of the predual $\mathcal{M}_*$, in particular the characterization of $\sigma(\mathcal{M}_*, \mathcal{M})$-compact subsets, polar decomposition of normal linear functionals etc. we refer to [15, III]. We always denote by $\mathcal{M}_1$ the closed unit ball of $\mathcal{M}$ and by $\mathcal{M}_1^+$ the positive part of $\mathcal{M}_1$. All C^*-algebras are supposed to be unital and $\mathbf{1}$ denotes the unit.

2.

Let E denote a (complex) Banach space and let $\mathfrak{U}$ denote a free ultrafilter on $\mathbf{N}$. If $l^\infty(E)$ is the Banach space of all bounded sequences with values in E, the space $c_{\mathfrak{U}}(E) := \{(x_n) \in l^\infty(E) : \lim_{\mathfrak{U}} \|x_n\| = 0\}$ is a closed subspace of $l^\infty(E)$. By the ultrapower $\hat{E}$ we understand the quotient space $l^\infty(E)/c_{\mathfrak{U}}(E)$. Moreover, for a bounded linear operator $T \in \mathcal{L}(E)$ we denote by $\hat{T}$ the well defined operator $\hat{T}((x_n) + c_{\mathfrak{U}}(E)) = (Tx_n) + c_{\mathfrak{U}}(E)$. It is clear that by virtue of $x \mapsto (x, x, \ldots) + c_{\mathfrak{U}}(E)$ each x in E defines an element $\hat{x} \in \hat{E}$. This isometric embedding, as well as the operator map $T \mapsto \hat{T}$, is called *canonical*. Recall that in particular the approximative point spectrum $A\sigma(T)$ equals the point spectrum $P\sigma(\hat{T})$ ([13, Theorem V.1.4]).

Let $\mathcal{M}$ be a W^*-algebra with predual $\mathcal{M}_*$, let $\mathfrak{U}$ be a free ultrafilter on $\mathbf{N}$ and let $\hat{\mathcal{M}}$ (resp. $\widehat{\mathcal{M}_*}$) be the ultrapower of $\mathcal{M}$ (resp. $\mathcal{M}_*$) with regard of $\mathfrak{U}$. Then it is easy to see, that $c_{\mathfrak{U}}(\mathcal{M})$ is a two sided ideal. Hence $\hat{\mathcal{M}}$ is a C^*-algebra with unit the canonical image of $\mathbf{1}$, but in general not a W^*-algebra. For $\hat{x} \in \hat{\mathcal{M}}$ and $\hat{\psi} \in \widehat{\mathcal{M}_*}$ let $J: \widehat{\mathcal{M}_*} \to \hat{\mathcal{M}}^*$ be defined by

$$\langle x, J(\hat{\psi}) \rangle := \lim_{\mathfrak{U}} \psi_n(x_n)$$

where $(x_n) \in \hat{x}$ and $(\psi_n) \in \hat{\psi}$. Then J is well defined and is an isometric embedding. In the following we investigate some properties of J and $J(\widehat{\mathcal{M}_*})$. For this we single out the next lemma which will be quite useful for our further investigation. Since the proof is nothing else than an adaptation of [15, Proposition III.4.10] we omit the details. Recall that the Fréchet filter $\mathfrak{F}_0$ on $\mathbf{N}$ is the filter generated by all subsets of $\mathbf{N}$ with finite complement.

LEMMA 2.1. *Let $\mathfrak{F}$ denote a filter on $\mathbf{N}$ which is finer than the Fréchet filter $\mathfrak{F}_0$. If $(\varphi_n) \in l^\infty(\mathcal{M}_*)$ and $(\psi_n) \in l^\infty(\mathcal{M}_*)$ such that $\lim_{\mathfrak{F}} \|\varphi_n - \psi_n\| = 0$, then $\lim_{\mathfrak{F}} \| |\varphi_n| - |\psi_n| \| = 0$.*

Let $\hat{\psi} \in \widehat{\mathcal{M}_*}$. We call $\hat{\psi}$ *positive*, if there exists a sequence $(\psi_n) \in \hat{\psi}$ such that $\psi_n \geqslant 0$ for all $n \in \mathbf{N}$. If $\hat{\psi} \geqslant 0$ and $\|\hat{\psi}\| = 1$ then it is easy to see that there exists a sequence of states in $\hat{\psi}$. In case $\hat{\psi}^* := (\widehat{\psi_n^*})$ and $|\hat{\psi}| := (\widehat{|\psi_n|})$ where $(\psi_n) \in \hat{\psi}$, then $*$ is a well defined isometric involution on $\widehat{\mathcal{M}_*}$ and $|\hat{\psi}|$ is well defined by Lemma 2.1. Obviously, $J(\hat{\psi}^*) = J(\hat{\psi})^*$ and $J(\hat{\psi})$ is positive iff $\hat{\psi}$ is positive.

If $\mathcal{A}^*$ is the dual space of a C^*-algebra, we consider $\mathcal{A}^*$ as the predual of the enveloping W^*-algebra $\mathcal{A}^{**}$ of $\mathcal{A}$ (see [15, III.2] for details). Hence every $\psi \in \mathcal{A}^*$ has a polar decomposition $\psi = u|\psi|$ where $u \in \mathcal{A}^{**}$ is a partial isometry and $|\psi| \in \mathcal{A}^*$ is positive and uniquely determined by the relations $|\psi(x)|^2 \leqslant \|\psi\| |\psi|(xx^*)$, $\|\psi\| = \| |\psi| \|$ ([15, Theorem III. 4.2, Proposition III.4.6]). By duality theory ([14, IV.5]) it is enough to examine the first inequality for all $x \in \mathcal{A}$.

PROPOSITION 2.2. *Let $\mathcal{M}$ be a W^*-algebra with predual $\mathcal{M}_*$, let $\hat{\mathcal{M}}$ respectively $\hat{\mathcal{M}}_*$ be their ultrapowers and let J be as above. Then the following holds:*

(a) *$J(|\hat{\psi}|) = |J(\hat{\psi})|$ for every $\hat{\psi} \in \widehat{\mathcal{M}_*}$;*

(b) *$J(\widehat{\mathcal{M}_*})$ is a translation invariant subspace of $\hat{\mathcal{M}}^*$ and there exists a central projection z in $\hat{\mathcal{M}}^{**}$ such that $J(\widehat{\mathcal{M}_*}) = (\hat{\mathcal{M}}^*)z$. In particular, $J(\widehat{\mathcal{M}_*})$ is the predual of the W^*-algebra $(\hat{\mathcal{M}}^{**})z$.*

Proof. (a) Let $(x_n) \in \hat{x} \in \hat{\mathcal{M}}$ and let $(\psi_n) \in \hat{\psi} \in \widehat{\mathcal{M}_*}$. Then

$$|\langle \hat{x}, J(\hat{\psi})\rangle|^2 = \lim_{\mathfrak{u}} |\psi_n(x_n)|^2 \leqslant \lim_{\mathfrak{u}} \|\psi_n\| \, |\psi_n|(x_n x_n^*) =$$

$$= \|\hat{\psi}\| \, \langle xx^*, J(|\hat{\psi}|)\rangle.$$

Since $\|J(\hat{\psi})\| = \|J(|\hat{\psi}|)\|$ assertion (a) is proved.

(b) It is enough to prove $\hat{x}J(\widehat{\mathcal{M}_*}) \subseteq J(\widehat{\mathcal{M}_*})$ (resp. $J(\widehat{\mathcal{M}_*})\hat{x} \subseteq J(\widehat{\mathcal{M}_*})$ for all $\hat{x} \in \hat{\mathcal{M}}$ ([15, Theorem III.2.7]). If $\hat{\psi} \in \widehat{\mathcal{M}_*}$ and $\hat{x} \in \hat{\mathcal{M}}$ let us define $\hat{x}\hat{\psi}$ (resp. $\hat{\psi}\hat{x}$) by $\hat{x}\hat{\psi} := \widehat{(x_n\psi_n)}$ (resp. $\hat{\psi}\hat{x} := \widehat{(\psi_n x_n)}$) where $(x_n) \in \hat{x}$ and $(\psi_n) \in \hat{\psi}$. Then $\hat{x}\hat{\psi}$ and $\hat{\psi}\hat{x}$ are well defined and $J(\hat{x}\hat{\psi}) = \hat{x}J(\hat{\psi})$ (resp. $J(\hat{\psi}\hat{x}) = J(\hat{\psi})\hat{x}$), which proves (b). Q.E.D.

In the rest of this paper we identify $\widehat{\mathcal{M}_*}$ via J with the translation invariant subspace $J(\widehat{\mathcal{M}_*})$ of $\hat{\mathcal{M}}^*$. From the construction the following is obvious: If T is an identity preserving Schwarz map on $\mathcal{M}$ with preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$, then $\hat{T}$ is an identity preserving Schwarz map on $\hat{\mathcal{M}}$ such that $\hat{T}_* = \hat{T}^*|\widehat{\mathcal{M}_*}$. For examples of Schwarz maps on C^*-algebras we refer to [4].

PROPOSITION 2.3. *Let T be an identity preserving Schwarz map on a W^*-algebra $\mathcal{M}$ with preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$.*

(a) *If $\psi \in F(T_*)$, then ψ^* and $|\psi|$ are in $F(T_*)$. In particular, if ψ is self-adjoint then ψ^+ and ψ^- are in $F(T_*)$.*

(b) *If $F(T_*)$ is infinite-dimensional, then there exists a sequence of states (ψ_n) in $F(T_*)$ such that the support projections $s(\psi_n)$ are mutually orthogonal.*

Before proving the proposition we want to make two remarks:

(1) Let $p \in \mathcal{M}$ be a projection such that $T(1-p) \leqslant (1-p)$, let $\mathcal{M}_p$ be the W^*-algebra $p\mathcal{M}p$ and let $T_p: \mathcal{M}_p \to \mathcal{M}_p$ be the map $(x \mapsto p(Tx)p)$. Then $T_p(p) = p$ and T_p is a Schwarz map since $T_p = PTP$ where P is the completely positive projection map $(x \mapsto pxp)$ on $\mathcal{M}$.

(2) If there exists a faithful family of T_*-invariant states on $\mathcal{M}$, then $F(T)$ is a W^*-subalgebra of $\mathcal{M}$ such that $T(xy) = xT(y)$ and $T(yx) = T(y)x$ for all $x \in F(T)$, $y \in \mathcal{M}$. To see this let Φ be such a family and note that $F(T)$ is a self-adjoint, $\sigma(\mathcal{M}, \mathcal{M}_*)$-closed subspace of $\mathcal{M}$. Since

$$\varphi(xx^*) = \varphi(T(xx^*)) \geqslant \varphi(T(x)T(x)^*) = \varphi(xx^*) \quad (x \in F(T),\ \varphi \in \Phi)$$

we obtain $T(xx^*) = xx^*$. For each state φ let

$$B_\varphi(x, y) := \varphi(T(xy^*) - T(x)T(y)^*) \quad (x, y \in \mathcal{M}).$$

Then B_φ is a positive semidefinite sesquilinear form on $\mathcal{M}$ such that $B_\varphi(x, x) = 0$ for some $x \in \mathcal{M}$ iff $B_\varphi(x, y) = 0$ for all $y \in \mathcal{M}$ by the Cauchy-Schwarz inequality. Thus $xy \in F(T)$ for all $x, y \in F(T)$ and $xT(y) = T(xy)$ resp. $T(y)x = T(yx)$ for all $x \in F(T)$ and $y \in \mathcal{M}$.

Proof. (a) follows from [7, Proposition 2.1 (a)] and [15, Theorem III.4.2].

(b) Let $\mathfrak{S}(T_*) := F(T_*) \cap \mathfrak{S}(\mathcal{M})$ ($\mathfrak{S}(\mathcal{M})$ the state space of $\mathcal{M}$) and let $p := \sup\{s(\psi) : \psi \in \mathfrak{S}(T_*)\}$. Because of $T_*\psi = \psi$ for all $\psi \in \mathfrak{S}(T_*)$ it follows $T(1 - s(\psi)) \leqslant (1 - s(\psi))$ hence $T(1 - p) \leqslant (1 - p)$. Let T_p be the map on $\mathcal{M}_p$ induced by T and note that the family $\mathfrak{S}(T_*)$ is faithful on $\mathcal{M}_p$ and contained in the fixed space of $(T_p)_*$. Thus $F(T_p)$ is a W^*-subalgebra of $\mathcal{M}_p$ such that $\dim(F(T_*)) \leqslant \dim(F(T_p))$ ([7, Proposition 3.1]). Let (p_n) be a sequence of mutually orthogonal projections in $F(T_p)$ and choose a sequence (ψ_n) in $\mathfrak{S}(T_*)$ such that $\psi_n(p_n) \neq 0$. For $n \in \mathbf{N}$ let φ_n be the normal linear functional

$$\varphi_n(x) := \psi_n(p_n)^{-1}\psi_n(p_n x p_n)$$

on $\mathcal{M}$. Because of $p_n \geqslant s(\varphi_n)$ the support projections of the φ_n's are mutually orthogonal and for all $x \in \mathcal{M}$:

$$(T_*\varphi_n)(x) = \psi_n(p_n)^{-1}\langle p_n(Tx)p_n, \psi_n\rangle = \psi_n(p_n)^{-1}\langle p_n p(Tx)pp_n, \psi_n\rangle =$$
$$= \psi_n(p_n)^{-1}\langle T_p(p_n x p_n), \psi_n\rangle = \varphi_n(x).$$

This proves assertion (b). Q.E.D.

REMARKS 2.4. (1) Let $\alpha \in \sigma(T_*) \cap \{\lambda \in \mathbf{C} : |\lambda| = 1\}$. If (φ_n) is a sequence in $\mathcal{M}_*$ such that $\|\varphi_n\| = 1$ $(n \in \mathbf{N})$ and $\lim_n \|(\alpha - T_*)\varphi_n\| = 0$, then $\lim_n \|(I - T_*)|\varphi_n|\| = 0$ which follows from Lemma 2.1 and Proposition 2.3 (a).

(2) If $F(\widehat{T_*})$ is infinite-dimensional, then we may choose a sequence of states in $F(\widehat{T_*})$ such that the corresponding support projections are mutually orthogonal in $\hat{\mathcal{M}}^{**}$. This follows from Proposition 2.2 (b) in combination with Proposition 2.3 (b).

(3) If $2 \leqslant \dim(F(T_*)) < \infty$, then there exists at least two states in $F(T_*)$ such that the corresponding support projections are orthogonal in $\mathcal{M}$. This can be easily seen from the proof of Proposition 2.3 (b).

3.

In this final section we give proofs for the results announced in Section one. Crucial for the proof of our key lemma is the following observation: If T is a contraction on a Banach space E, then T is uniformly ergodic with finite-dimensional fixed space iff $\dim F(\hat{T}) < \infty$ where $\hat{T}$ is the canonical extension of T to some ultrapower $\hat{E}$ of E (see [7, Proposition 3.2]).

LEMMA 3.1. *Let T be an identity preserving Schwarz map on a W^*-algebra $\mathcal{M}$ with preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$ and suppose that T_* is not uniformly ergodic with finite-dimensional fixed space. Then there exists a sequence (z_n) in $\mathcal{M}_1^+$ and a sequence of normal states (ψ_n) such that*

(i) $\lim_n z_n = 0$ *in the* $s^*(\mathcal{M}, \mathcal{M}_*)$*-topology;*

(ii) $\psi_n(z_n) \geqslant 1/2$ *for all* $n \in \mathbf{N}$*;*

(iii) $\lim_n \|(I - T_*)\psi_n\| = 0$.

Proof. Let $\widehat{\mathcal{M}_*}$ be the ultrapower of $\mathcal{M}_*$ with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$. It follows from the assumption that the fixed space of $\widehat{T_*}$ is infinite-dimensional. Hence there exists a sequence of states $(\hat{\psi}_n)$ in $F(\widehat{T_*})$ such that the corresponding support projections $s(\hat{\psi}_n)$ are mutually orthogonal in $\hat{\mathcal{M}}^{**}$ (Proposition 2.3 (b) and Remarks 2.4 (2)). For every $n \in \mathbf{N}$ let $(\psi_{n,k}) \in \hat{\psi}_n$ be a representing sequence of states, let $\varphi := \sum_{n,k} 2^{-(n+k+1)} \varphi_{n,k}$ and let $p := \sup\{s(\psi_{n,k}) : n, k = 1, 2, \ldots\}$ in $\mathcal{M}$. Then φ is a normal state on $\mathcal{M}$ which is faithful on the W^*-algebra $\mathcal{M}_p := p\mathcal{M}p$. Since $1 = \langle \psi_{n,k}, s(\psi_{n,k})\rangle = \psi_{n,k}(p)$ for all n and k, it follows that $\hat{\psi}_n(\hat{p}) = 1$. This implies $\hat{p} \geqslant s(\hat{\psi})$ in $\hat{\mathcal{M}}^{**}$. Since $\hat{\mathcal{M}}_1^+$ is $\sigma(\hat{\mathcal{M}}^{**}, \hat{\mathcal{M}}^*)$ dense in $(\hat{\mathcal{M}}^{**})_1^+$ (Kaplansky density theorem [12, 1.9.1] in combination with [12, 1.8.9 and 1.8.12]) there exists for all $n \geqslant 1$ a net $(\hat{z}_{n,\gamma})$ in $\hat{\mathcal{M}}_1^+$ such that $\sigma(\hat{\mathcal{M}}^{**}, \hat{\mathcal{M}}^*)\text{-}\lim_\gamma \hat{z}_{n,\gamma} = s(\hat{\psi}_n)$. From [12, 1.7.8] and the considerations above we obtain that the net $(\hat{p}\hat{z}_{n,\gamma}\, \hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\hat{\mathcal{M}}^{**}, \hat{\mathcal{M}}^*)$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\hat{\mathcal{M}}_{\hat{p}})_1^+$. In the following we denote by $\hat{\varphi}$ the canonical image of φ in $\hat{\mathcal{M}}^*$.

Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal there exists a sequence (ε_n) of reals, $0 < \varepsilon_n < 1$, such that $\lim_n \varepsilon_n = 0$ and $\langle \hat{\varphi}, s(\hat{\psi}_n)\rangle < \frac{1}{2} \varepsilon_n$. For all $n \in \mathbf{N}$ choose $\hat{z}_n \in (\hat{\mathcal{M}}_{\hat{p}})_1^+$ such that

$$|\langle s(\hat{\psi}_n) - \hat{z}_n, \hat{\varphi}\rangle| < \frac{1}{2} \varepsilon_n$$

$$|\langle s(\hat{\psi}_n) - \hat{z}_n, \hat{\psi}_n\rangle| < \frac{1}{2} \varepsilon_n.$$

Hence $\hat{\varphi}(\hat{z}_n) < \varepsilon_n$ and $\hat{\psi}_n(\hat{z}_n) > 1/2$ for all $n \in \mathbf{N}$. For every n let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(\mathcal{M}_p)_1^+ = p(\mathcal{M}_1^+)p$. Since $T_* \hat{\psi}_n = \hat{\psi}_n$, $\hat{\varphi}(\hat{z}_n) < \varepsilon_n$ and $\hat{\psi}_n(\hat{z}_n) > 1/2$ there exists for all $n \in \mathbf{N}$, $U_n \in \mathfrak{U}$ such that for all $k \in U_n$:

(i) $\varphi(z_{n,k}) < \varepsilon_n$

(ii) $\psi_{n,k}(z_{n,k}) > 1/2$

(iii) $\|(I - T_*)\psi_{n,k}\| < \varepsilon_n$.

Inductively we find a sequence $(z_n) \in (\mathcal{M}_p)_1^+$ and a sequence of states (ψ_n) in $\mathcal{M}_*$ such that for all $n \in \mathbf{N}$:

(i)′ $\lim_n \varphi(z_n) = 0$;

(ii)′ $\psi_n(z_n) > 1/2$;

(iii)′ $\lim_n \|(I - T_*)\psi_n\| = 0$.

Since φ is faithful on $\mathcal{M}_p$, condition (i)′ implies $\lim_n z_n = 0$ in the $s^*(\mathcal{M}_p, (\mathcal{M}_p)_*)$--topology ([15, Proposition III.5.3]). Since $s^*(\mathcal{M}_p, (\mathcal{M}_p)_*) = s^*(\mathcal{M}, \mathcal{M}_*)|\mathcal{M}_p$ the assertion is proved. Q.E.D.

Conversely, one can show that a $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous identity preserving Schwarz map which satisfies (i)—(iii) of Lemma 3.1 cannot be uniformly ergodic. Since we don't need this result we omit the proof.

Without further comments in the rest of this paper we frequently make use of the following facts:

(1) A sequence (ψ_n) in $\mathcal{M}_+^*$ converges in the $\sigma(\mathcal{M}^*, \mathcal{M})$-topology iff it converges in the $\sigma(\mathcal{M}^*, \mathcal{M}^{**})$-topology ([1, Corollary 3]);

(2) If $\varphi \in \mathcal{M}_+^*$ then we can decompose φ into its normal and singular part $\varphi = \varphi^{(n)} + \varphi^{(s)}$, where $0 \leqslant \varphi^{(n)} \in \mathcal{M}_*$, $0 \leqslant \varphi^{(s)} \in (\mathcal{M}_*)^\perp$ and $\|\varphi\| = \|\varphi^{(n)}\| + \|\varphi^{(s)}\|$ ([15, Theorem III.2.14]);

(3) If (ψ_n) is a sequence in $\mathcal{M}_*$ which converges to some $\psi \in \mathcal{M}_*$ and if (z_n) is a sequence in $\mathcal{M}_1$ which converges to zero in the $s^*(\mathcal{M}, \mathcal{M}_*)$-topology, then $\lim_n \psi_k(z_n) = 0$ uniformly in $k \in \mathbf{N}$ ([15, Lemma III.5.5]).

THEOREM 3.2. *Let T be an identity preserving Schwarz map on a W^*-algebra $\mathcal{M}$. If every T^*-invariant state is normal then T is uniformly ergodic with finite--dimensional fixed space.*

Proof. If T is not uniformly ergodic with finite-dimensional fixed space, then likewise T^* on $\mathcal{M}^*$ and T^{**} on the W^*-algebra $\mathcal{A} := \mathcal{M}^{**}$. In that case choose a sequence (z_k) in $\mathcal{A}_1^+$ and a sequence of states (ψ_k) in $\mathcal{M}^*$ $(= \mathcal{A}_*)$ in accordance with Lemma 3.1. With respect to the duality $\langle \mathcal{M}, \mathcal{M}_* \rangle$ we decompose every ψ_k into its normal and singular part $\psi_k = \psi_k^{(n)} + \psi_k^{(s)}$. If ψ is a $\sigma(\mathcal{M}^*, \mathcal{M})$-accumulation point of the sequence (ψ_k), then $T^*\psi = \psi$ (property (iii) of Lemma 3.1), hence is normal. Choose an ultrafilter $\mathfrak{V}$ on $\mathbf{N}$ such that $\psi = \lim_{\mathfrak{V}} \psi_k$ and let $\varphi_1 = \lim_{\mathfrak{V}} \psi_k^{(n)}$

resp. $\varphi_2 = \lim_{\mathfrak{V}} \psi_k^{(s)}$ (always with respect to the $\sigma(\mathcal{M}^*, \mathcal{M})$-topology). Then φ_2 is singular ([15, Proposition III.5.8]). If we decompose $\varphi_1 = \varphi_1^{(n)} + \varphi_1^{(s)}$, then $\psi - \varphi_1^{(n)} = \varphi_1^{(s)} + \varphi_2 \in \mathcal{M}_* \cap (\mathcal{M}^*)^\perp = \{0\}$. Since $\varphi_1^{(s)}$ and φ_2 are both positive, they must be equal to zero. Thus $\lim_{\mathfrak{V}} \psi_k^{(s)}(1) = \lim_{\mathfrak{V}} \|\psi_k^{(s)}\| = 0$. Choosing some subsequence, we may assume $\lim_k \|\psi_k^{(s)}\| = 0$. But then $\lim_k \|(I - T^*)\psi_k^{(n)}\| = 0$ and $\psi_k^{(n)}(z_k) \geqslant 1/4$ for all $k \geqslant k_0$. Since every $\sigma(\mathcal{M}^*, \mathcal{M})$-accumulation point is T^*-invariant, hence normal, the sequence $(\psi_k^{(n)})$ is $\sigma(\mathcal{M}_*, \mathcal{M})$-relatively compact in $\mathcal{M}_*$. Using the theorem of Eberlein ([14, Corollary IV.II.2]) we may assume $\lim_k \psi_k^{(n)} = \psi_0$ for some $\psi_0 \in \mathcal{M}_*$ in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. Since $\sigma(\mathcal{M}_*, \mathcal{M}) = \sigma(\mathcal{M}^*, \mathcal{M})|\mathcal{M}_*$ it follows $\lim_k \psi_k^{(n)} = \psi$ in the $\sigma(\mathcal{M}^*, \mathcal{M}^{**})$-topology $(= \sigma(\mathcal{A}_*, \mathcal{A})$-topology). Thus $\lim_k \psi_m^{(n)}(z_k) = 0$ uniformly in $m = 1, 2, \ldots$ which conflicts with $\psi_m^{(n)}(z_m) \geqslant 1/4$ for all large m. This contradiction proves the theorem. Q.E.D.

COROLLARY 3.3. *Let T and $\mathcal{M}$ be as in Theorem 3.2 and assume that T has a preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$. Then the following are equivalent:*

(a) *T_* is uniformly ergodic with finite-dimensional fixed space;*

(b) *T is strongly ergodic with finite-dimensional fixed space;*

(c) *Every T^*-invariant state is normal;*

(d) *T is quasi-compact.*

Proof. The equivalence of (a) and (d) is proved in [7], whereas "(a) ⇒ (b)" is obvious. If (b) is satisfied, then the averages $M_n(T^*)$ converge to P^* in the $\sigma(\mathcal{M}^*, \mathcal{M})$-topology, where P is the ergodic projection associated with T. Since by assumption $\dim(F(T^*)) = \mathrm{rank}(P^*) = \mathrm{rank}(P) < \infty$, and P^* maps $\mathcal{M}_*$ into itself, and $\mathcal{M}_*$ is $\sigma(\mathcal{M}^*, \mathcal{M})$-dense in $\mathcal{M}^*$, we obtain $F(T^*) = P^*(\mathcal{M}^*) = P^*(\mathcal{M}_*) \subseteq \mathcal{M}_*$. The implication "(c) ⇒ (a)" follows from Theorem 3.2. Q.E.D.

A positive map T on a C^*-algebra $\mathcal{A}$ is called *irreducible* if no (norm) closed face of $\mathcal{A}_+$, distinct from $\mathcal{A}_+$ and $\{0\}$, is invariant under T. For examples of irreducible maps and their spectral theoretical properties we refer to [6]. Note the following: If $0 \leqslant T \in \mathcal{L}(\mathcal{A})$ and $\varphi \in \mathfrak{S}(\mathcal{A})$ such that $T^*\varphi = \varphi$, then the face $\{x \in \mathcal{A}_+ : \varphi(x) = 0\}$ is T-invariant. Therefore, if T is irreducible such a linear form is faithful. Also for every positive $T \in \mathcal{L}(\mathcal{A})$ there exists a state $\varphi \in \mathcal{A}^*$ such that $T^*\varphi = \mathrm{r}(T)\varphi$, $\mathrm{r}(T)$ the spectral radius of T (see, e.g. [6, Theorem 2.1]).

THEOREM 3.4. *Every irreducible, identity preserving Schwarz map on a W^*-algebra is quasi-compact.*

Proof. If T is an irreducible Schwarz map on a W^*-algebra $\mathcal{M}$ then $\dim F(T) = 1$ ([6, Theorem 3.1]). Therefore it remains to show the uniform ergodicity of T. First we claim that $\dim F(T^*) = 1$. Taking this for granted it follows that the fixed space of T^* is generated by a faithful state $\varphi \in \mathcal{M}^*$. If T is not uniformly ergodic then likewise T^{**} on the W^*-algebra $\mathcal{M}^{**}$. Choose a sequence (z_n) in $(\mathcal{M}^{**})_1^+$ and a sequence of states (ψ_n) in $\mathcal{M}^*$ satisfying (i)—(iii) of Lemma 3.1. If ψ is a $\sigma(\mathcal{M}^*, \mathcal{M})$-accumulation point of some subsequence of (ψ_n), then ψ is a T^*-invariant state on $\mathcal{M}$, hence equal to φ. Therefore $\lim_n \psi_n = \varphi$ in the $\sigma(\mathcal{M}^*, \mathcal{M})$-topology which implies $\lim_n \psi_n = \varphi$ in the $\sigma(\mathcal{M}^*, \mathcal{M}^{**})$-topology. But this conflicts with Lemma 3.1(ii) and (3) before Theorem 3.2.

Let us turn to the proof that $\dim F(T^*) = 1$. Useful for this is the following observation: If φ is a faithful state on $\mathcal{M}$ then the normal part $\varphi^{(n)}$ of φ is faithful, too. Indeed, if $0 \neq x \in \mathcal{M}_+$ such that $\varphi^{(n)}(x) = 0$, then there exists a projection $0 \neq p$ in $\mathcal{M}$ such that $\varphi^{(n)}(p) = \varphi^{(s)}(p) = 0$ (use [15, Theorem III.3.8]). Hence $\varphi(p) = 0$ which conflicts with the faithfulness of φ.

If $\dim F(T^*) \geqslant 2$ then there exists states ψ_1 and ψ_2 in $F(T^*)$ such that the corresponding support projections in $\mathcal{M}^{**}$ are orthogonal. Since $\psi_i^{(n)} \leqslant \psi_i$ $(i = 1, 2)$, the support projections $s(\psi_i^{(n)})$ of the $\psi_i^{(n)}$'s in $\mathcal{M}^{**}$ are likewise orthogonal. Let (z_γ) be a net in $\mathcal{M}_1^+$ such that $\sigma(\mathcal{M}^{**}, \mathcal{M}^*)\text{-}\lim_\gamma z_\gamma = s(\psi_1^{(n)})$. Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ where $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be an $\sigma(\mathcal{M}, \mathcal{M}_*)$-accumulation point of (z_γ). Since every $\psi_i^{(n)}$ is normal, it follows $\psi_1^{(n)}(z) = 1$ (hence $z \neq 0$) whereas $\psi_2^{(n)}(z) = 0$. Since this conflicts with the faithfulness of $\psi_2^{(n)}$, we obtain $\dim F(T^*) = 1$ and the theorem is proved. Q.E.D.

The last theorem should be compared with the following: If T is an identity preserving Schwarz map on a W^*-algebra $\mathcal{M}$ such that $\dim F(T) = 1$ and $T^*\varphi = \varphi$ for some faithful normal state on $\mathcal{M}$, then T possesses a preadjoint $T_* \in \mathcal{L}(\mathcal{M}_*)$ which is strongly ergodic ([8]). If $\mathcal{F}$ is a closed face of $\mathcal{M}_*^+$ such that $T_*(\mathcal{F}) \subseteq \mathcal{F}$ then $M_n(T_*)\mathcal{F} \subseteq \mathcal{F}$. If $0 \neq \psi \in \mathcal{F}$, then $\lim_n M_n(T_*)\psi = \psi(\mathbf{1})\varphi \in \mathcal{F}$, which implies $\mathcal{F} = \mathcal{M}_*^+$. From this it follows that there is no non-trivial $\sigma(\mathcal{M}, \mathcal{M}_*)$-closed T-invariant face in $\mathcal{M}_+$. But there are many of such maps with peripheral spectrum equal the unit circle, hence $\dim F(T^*) = \infty$.

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