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# NON-SELF-ADJOINT CROSSED PRODUCTS. III: INFINITE ALGEBRAS

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## 1. INTRODUCTION

Let $M$ be a von Neumann algebra, let $\alpha$ be a $*$-automorphism of $M$, and let $M\times_{\alpha}\mathbf{Z}$ be the von Neumann algebra crossed product determined by $M$ and $\alpha$. In this work, which is a continuation of [16] and [17], we investigate the structure of a certain non-self-adjoint subalgebra of $M\times_{\alpha}\mathbf{Z}$ which we call a *non-self-adjoint crossed product* and which we denote by $M\times_{\alpha}\mathbf{Z}_{+}$. (Precise definitions are given in the next section.) In [16] and [17] we assumed that $M$ was finite and that $\alpha$ preserves a faithful normal trace. Here, $M$ may be arbitrary and we assume only that $\alpha$ fixes a faithful normal state on $M$. Our primary objective is to describe the invariant subspace structure of $M\times_{\alpha}\mathbf{Z}_{+}$ and to exploit some of its consequences.

We view $M\times_{\alpha}\mathbf{Z}$ in standard form, we identify the underlying Hilbert space with Haagerup's $L^2$-space (see [8] and [27]), $\mathbf{L}^2$, and we identify $M\times_{\alpha}\mathbf{Z}$ with a von Neumann algebra $\mathfrak{L}$ of operators acting on the left of $\mathbf{L}^2$. The commutant of $M\times_{\alpha}\mathbf{Z}$, then, is identified with a von Neumann algebra $\mathfrak{R}$ of operators acting on the right of $\mathbf{L}^2$. In this identification, $M\times_{\alpha}\mathbf{Z}_{+}$ is denoted by $\mathfrak{L}_{+}$. There is a special subspace $\mathbf{H}^2$ of $\mathbf{L}^2$ which stands in the same relation to $\mathbf{L}^2$ as the classical Hardy space on the unit disc stands in relation to $L^2$ of the circle. Our main result, Theorem 3.10, asserts that if $\alpha$ fixes the finite central projections of $M$ elementwise, then every subspace $\mathfrak{M}$ of $\mathbf{L}^2$ that is invariant under $\mathfrak{L}_{+}$ and contains no reducing subspace for $\mathfrak{L}$ may be written $\mathfrak{M}=R_v\mathbf{H}^2$ where $R_v$ is a partial isometry in $\mathfrak{R}$. Conversely, if each subspace of $\mathbf{L}^2$ that is invariant for $\mathfrak{L}_{+}$ and contains no reducing subspace for $\mathfrak{L}$ has the form $R_v\mathbf{H}^2$ for a suitable partial isometry in $\mathfrak{R}$, then $\alpha$ fixes the finite central projections in $M$ elementwise. Thus an exact analogue of Beurling's theorem, as extended by Lax and Halmos, is true for $M\times_{\alpha}\mathbf{Z}_{+}$ precisely when $\alpha$ fixes each finite central projection of $M$.

The algebra $M\times_{\alpha}\mathbf{Z}_{+}$ is an example of what Arveson [1] calls a maximal subdiagonal algebra (cf. [13] and [16] also). Although $M\times_{\alpha}\mathbf{Z}_{+}$ may be maximal as a subdiagonal algebra in $M\times_\alpha \mathbf{Z}$, it need not be maximal among the ultraweakly closed subalgebras of $M\times_\alpha \mathbf{Z}$. In Theorem 4.4 we use our analysis of invariant subspaces to prove that $M\times_\alpha \mathbf{Z}_+$ is maximal among the ultraweakly closed subalgebras of $M\times_\alpha \mathbf{Z}$ precisely when $M$ is a factor. A closely related result was proved by Peligrad and Rubinstein [19, Proposition 4.2] and their proof can be modified to prove ours. However, our proof leads to some interesting refinements of the basic results on invariant subspaces.

When $M$ is finite and $\alpha$ preserves a faithful normal finite trace on $M$, then $M\times_\alpha \mathbf{Z}$ is a finite von Neumann algebra and $M\times_\alpha \mathbf{Z}_+$ is a finite maximal subdiagonal algebra in the sense of [1]. Consequently, the following factorieation theorem is true: Every invertible $k\in M\times_\alpha \mathbf{Z}$ may be factored as $k = u_1a_1 = a_2u_2$ where $u_i$ is unitary in $M\times_\alpha \mathbf{Z}$ and where each $a_i$ is an invertible element in $M\times_\alpha \mathbf{Z}_+$ with inverse also lying in $M\times_\alpha \mathbf{Z}_+$ [1, Theorem 4.2.1]. This result, or more accurately a technical extension of it (cf. [16, Proposition 1.2] and [22, Proposition 1]), was one of the key ingredients in the arguments of [16] and [17]; except for the invariant subspace theorem, all of the major results in [16] and [17] require it. When $M$ is not finite, $M\times_\alpha \mathbf{Z}_+$ is no longer a finite subdiagonal algebra and there is no reason, a priori, to believe that the factorization theorem remains valid. Indeed, Larson [12] has recently exhibited a subdiagonal algebra in which the factorization theorem fails. He does not refer to subdiagonal algebras as such, but his arguments show that if one lets the rationals $\mathbf{Q}$ act on $\ell^\infty(\mathbf{Q})$ through translation and if one forms $\ell^\infty(\mathbf{Q})\times\mathbf{Q}_+$ in the way one builds $M\times_\alpha \mathbf{Z}_+$, then $\ell^\infty(\mathbf{Q})\times\mathbf{Q}_+$ is a maximal subdiagonal algebra in $\ell^\infty(\mathbf{Q})\times\mathbf{Q}$ without the factorization property. In contrast, as we shall show in Corollary 5.2, the factorization theorem is true in $M\times_\alpha \mathbf{Z}_+$ as a consequence of our analysis of invariant subspaces. Thus we find that the invariant subspace structure of $M\times_\alpha \mathbf{Z}_+$ leads directly to *all* of the results about $M\times_\alpha \mathbf{Z}_+$ without the intervention of the general theory from [1].

In [3] Arveson showed that factorization holds in certain subdiagonal algebras of a type $\mathrm{I}_\infty$ factor. This result together with our results in this note are the only positive results known to us concerning factorization in *nonfinite* subdiagonal algebras. So, when contrasted with Larson's discoveries, the problem of identifying those subdiagonal algebras in which the factorization theorem is valid becomes all the more piquant. A recent study of Jørgensen [9] may prove useful here.

The next section is devoted to preliminaries. Section 3 contains our analysis of invariant subspaces and the fourth section is concerned with maximality questions. The fifth and final section is devoted to the factorization theorem.

## 2. PRELIMINARIES: NON-SELF-ADJOINT CROSSED PRODUCTS

Throughout this paper, $M$ will be a von Neumann algebra on a Hilbert space $H$ with a cyclic and separating vector $\xi_0$. For convenience, we normalize $\xi_0$. Put $\varphi(x) = (x\xi_0, \xi_0)$, $x\in M$. Then $\varphi$ is a faithful normal state on $M$. Let $\alpha$ be a $*$-automorphism of $M$ which preserves $\varphi$; $\varphi \circ \alpha = \varphi$. Recall that the *crossed product* $M \times_\alpha \mathbf{Z}$ of $M$ by the automorphism group $\{\alpha^n\}_{n\in \mathbf{Z}}$ is the von Neumann algebra on the Hilbert space $\ell^2(\mathbf{Z}, H)$ generated by the operators $\pi(x)$, $x \in M$, and $u$ defined by the equations

$$(\pi(x)\xi)(n) = \alpha^{-n}(x)\xi(n), \quad \xi \in \ell^2(\mathbf{Z}, H),\ n \in \mathbf{Z},$$

$$(u\xi)(n) = \xi(n-1), \quad \xi \in \ell^2(\mathbf{Z}, H), n \in \mathbf{Z}.$$

Since $\pi$ is a normal faithful $*$-representation of $M$, we may and shall identify $M$ with its image $\pi(M)$ in $M \times_\alpha \mathbf{Z}$. The automorphism group $\{\beta_t\}_{t\in\mathbf{R}}$ of $M \times_\alpha \mathbf{Z}$ dual to $\{\alpha^n\}_{n\in\mathbf{Z}}$ in the sense of Takesaki [26] is implemented by the unitary representation of $\mathbf{R}$, $\{V_t\}_{t\in\mathbf{R}}$, defined by the formula $(V_t\xi)(n) = e^{2\pi i n t}\xi(n)$, $\xi \in \ell^2(\mathbf{Z}, H)$; that is, $\beta_t(a) = V_t a V_t^*$, $a \in M \times_\alpha \mathbf{Z}$.

For every $n \in \mathbf{Z}$, we define a $\sigma$-weakly continuous linear map $\varepsilon_n$ on $M \times_\alpha \mathbf{Z}$ by the integral

$$\varepsilon_n(a) = \int_0^1 e^{-2\pi i n t}\beta_t(a)\,dt, \quad a \in M \times_\alpha \mathbf{Z}.$$

Then it is clear that

$$\varepsilon_n(M \times_\alpha \mathbf{Z}) = \{a \in M \times_\alpha \mathbf{Z} : \beta_t(a) = e^{2\pi i n t}a,\ t \in \mathbf{R}\}.$$

Further, the $\beta_t$, $t \in \mathbf{R}$, are automorphisms of $M \times_\alpha \mathbf{Z}$ and are characterized by the formulae

$$\beta_t(x) = x, \quad x \in M,$$

$$\beta_t(u) = e^{2\pi i t}u, \quad t \in \mathbf{R}.$$

Thus, by [13, 26], we have

$$M = \{y \in M \times_\alpha \mathbf{Z} : \beta_t(x) = x,\ t \in \mathbf{R}\}.$$

Recall that $\varepsilon_0$ is a faithful, normal, $\{\beta_t\}_{t\in\mathbf{R}}$-invariant conditional expectation of $M \times_\alpha \mathbf{Z}$ onto $M$. Put $\tilde{\varphi} = \varphi \circ \varepsilon_0$. Then $\tilde{\varphi}$ is clearly a $\{\beta_t\}_{t\in\mathbf{R}}$-invariant faithful normal state on $M \times_\alpha \mathbf{Z}$. The function $\psi$ is defined by the formula $\psi(0) = \xi_0$ and $\psi(n) = 0$, $n \neq 0$. Then $\psi$ is a cyclic and separating vector for $M \times_\alpha \mathbf{Z}$ on $\ell^2(\mathbf{Z}, H)$ such that $\tilde{\varphi}(x) = (x\psi, \psi)$, $x \in M \times_\alpha \mathbf{Z}$. We now define $M \times_\alpha \mathbf{Z}_+$ to be $\{x \in M \times_\alpha \mathbf{Z} : \varepsilon_n(x) = 0,\ n < 0\}$ and call $M \times_\alpha \mathbf{Z}_+$ the *non-self-adjoint crossed product* determined by $M$ and $\alpha$.

Let $\{\sigma_t\}_{t\in\mathbf{R}}$ be the modular automorphism group on $M \times_\alpha \mathbf{Z}$ associated with $\tilde{\varphi}$ and let $N$ denote the crossed product $(M \times_\alpha \mathbf{Z}) \times_\sigma \mathbf{R}$ of $M \times_\alpha \mathbf{Z}$ by $\{\sigma_t\}_{t\in\mathbf{R}}$. Recall that $N$ is the von Neumann algebra on the Hilbert space $L^2(\mathbf{R}, \ell^2(\mathbf{Z}, H))$ generated by the operators $\rho(x)$, $x \in M \times_\alpha \mathbf{Z}$, and $\lambda(s)$, $s \in \mathbf{R}$, defined by the equations

$$(\rho(x)\xi)(t) = \sigma_{-t}(x)\xi(t), \quad \xi \in L^2(\mathbf{R}, \ell^2(\mathbf{Z}, H)),\ t \in \mathbf{R},$$

$$(\lambda(s)\xi)(t) = \xi(t - s), \quad \xi \in L^2(\mathbf{R}, \ell^2(\mathbf{Z}, H)),\ t \in \mathbf{R}.$$

We identify $M \times_\alpha \mathbf{Z}$ with its image $\rho(M \times_\alpha \mathbf{Z})$ in $N$. Let $\{\theta_s\}_{s \in \mathbf{R}}$ be the dual action of $\{\sigma_t\}_{t \in \mathbf{R}}$ on $N$ which is implemented by the unitary representation of $\mathbf{R}$, $\{S_t\}_{t \in \mathbf{R}}$, defined by the formula

$$(S_t\xi)(s) = e^{ist}\xi(s), \quad \xi \in L^2(\mathbf{R}, \ell^2(\mathbf{Z}, H));$$

that is, $\theta_s(a) = S_t a S_t^*$, $a \in N$. Then we have

$$M \times_\alpha \mathbf{Z} = \{a \in N : \theta_t(a) = a,\ t \in \mathbf{R}\}.$$

Since $N$ is semi-finite (cf. [26]), there exists the faithful normal semi-finite trace $\tau$ on $N$ satisfying the equation $\tau\theta_s = e^{-s}\tau$, $s \in \mathbf{R}$ (see [7, Lemma 5.2]).

According to Haagerup [8] ([27, Chapter 2]), the space $L^p(M \times_\alpha \mathbf{Z})$, $p \in [1, \infty)$ (for simplicity, $\mathbf{L}^p$) is defined as the set of all $\tau$-measurable operators $k$ such that

$$\theta_s(k) = e^{-s/p}k, \quad s \in \mathbf{R}.$$

The algebraic structure in $\mathbf{L}^p$ is inherited from the regular ring of $\tau$-measurable operators. As in [8], we define the operators $L_x$ and $R_x$ on $\mathbf{L}^2$ by the equations

$$L_x k = xk, \quad R_x k = kx, \quad x \in M \times_\alpha \mathbf{Z},\ k \in \mathbf{L}^2.$$

If $S$ is a subset of $M \times_\alpha \mathbf{Z}$, we will write $L(S)$ (resp. $R(S)$) for $\{L_x : x \in S\}$ (resp. $\{R_x : x \in S\}$). In particular, we put $\mathfrak{L} = L(M \times_\alpha \mathbf{Z})$, $\mathfrak{R} = R(M \times_\alpha \mathbf{Z})$, $\mathfrak{L}_+ = L(M \times_\alpha \mathbf{Z}_+)$ and $\mathfrak{R}_+ = R(M \times_\alpha \mathbf{Z}_+)$. It is clear that $\mathfrak{L} = \{L(M), L_u\}''$ and $\mathfrak{R} = \{R(M), R_u\}''$. Further, $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) is a $\sigma$-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) which is generated by $L(M)$ and $L_u$ (resp. $R(M)$ and $R_u$). The involution $\mathbf{J} : k \in \mathbf{L}^2 \to k^* \in \mathbf{L}^2$ and $\mathbf{L}^2_+$ together with $\mathfrak{L}$ form a standard form $\{\mathfrak{L}, \mathbf{L}^2, \mathbf{J}, \mathbf{L}^2_+\}$ in the sense of Haagerup [5] (cf. [27, Theorem 36]). By the uniqueness of standard forms, the positive cone $P^\natural$ in $\ell^2(\mathbf{Z}, H)$ is identified with $\mathbf{L}^2_+$. Through this identification, we denote the operator in $\mathbf{L}^1_+$ corresponding to $\tilde{\varphi}$ by $h_0$, that is, $h_0$ is the Radon-Nikodym derivative of the dual weight $\omega$ of $\tilde{\varphi}$ with respect to $\tau$ in the sense of Pedersen-Takesaki ([18]). So we have $\omega(x) = \tau(h_0 x)$, $x \in N$ and $h_0^{1/2}$ is a separating and cyclic vector for $M \times_\alpha \mathbf{Z}$ in $\mathbf{L}^2$. Thus we may identify $\{M \times_\alpha \mathbf{Z}, \ell^2(\mathbf{Z}, H), \psi\}$ with $\{\mathfrak{L}, \mathbf{L}^2, h_0^{1/2}\}$; this identification will be made throughout the paper.

Since $\tilde{\varphi} \circ \beta_t = \tilde{\varphi}$, $t \in \mathbf{R}$, there exists a unitary group $\{W_t\}_{t \in \mathbf{R}}$ on $\mathbf{L}^2$ such that $W_t(ah_0^{1/2}) = \beta_t(a)h_0^{1/2}$, $a \in M \times_\alpha \mathbf{Z}$, $t \in \mathbf{R}$. It is elementary to check that the spectral resolution of $\{W_t\}_{t \in \mathbf{R}}$ is given by the formula

$$W_t = \sum_{n=-\infty}^{\infty} e^{2\pi i n t} E_n,$$

where $E_n$ is the projection of $\mathbf{L}^2$ onto the closed linear span $u^n[Mh_0^{1/2}]_2$ of $u^n M h_0^{1/2}$ in $\mathbf{L}^2$. It is equally easy to check that the projection $E_n$ can be calculated as the (Bochner) integral

$$E_n(x) = \int_0^1 e^{-2\pi i n t} W_t(x)\, dt, \quad x \in \mathbf{L}^2.$$

It is clear that $E_n(ah_0^{1/2}) = \varepsilon_n(a)h_0^{1/2}$, $a \in M \times_\alpha \mathbf{Z}$. The following theorem summarizes the basic properties of the structures that we have been discussing.

THEOREM 2.1 (cf. [13, Theorem 3.15] or [20, Theorem 2]). *The algebra $M \times_\alpha \mathbf{Z}_+$ is a maximal subdiagonal algebra in $M \times_\alpha \mathbf{Z}$ with respect to the expectation $\varepsilon_0$. The diagonal of $M \times_\alpha \mathbf{Z}$ equals $M$. In addition, the map $x \to L_x$, (resp. $R_x$), $x \in M \times_\alpha \mathbf{Z}_+$, is a σ-weakly continuous, isometric isomorphism (resp. anti-isomorphism) of $M \times_\alpha \mathbf{Z}_+$ onto $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$).*

## 3. INVARIANT SUBSPACES AND THE BLH THEOREM

In this section, we investigate the structure of the subspaces of $\mathbf{L}^2$ which are invariant under $\mathfrak{L}_+$ or $\mathfrak{R}_+$.

DEFINITION 3.1. Let $\mathfrak{M}$ be a closed subspace of $\mathbf{L}^2$. We shall say that $\mathfrak{M}$ is: *left-invariant*, if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$; *left-reducing*, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; *left-pure*, if $\mathfrak{M}$ contains no non-trivial left-reducing subspace; and *left-full*, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $\mathbf{L}^2$. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left- and right-invariant will be called *two-sided invariant*.

The following proposition shows that the analysis of the invariant subspace structure of $\mathfrak{L}_+$ may be reduced, in part, to known results about the invariant subspaces of $L_u$. The proof is straightforward because $\mathfrak{L}_+$ is the σ-weakly closed subalgebra generated by $L(M)$ and $L_u$, and so will be omitted.

PROPOSITION 3.2. *Let* $\mathfrak{M}$ *be a left-invariant subspace in* $\mathbf{L}^2$. *Then*

(1) $\mathfrak{M}$ *reduces* $L(M)$;

(2) $\mathfrak{M}$ *reduces* $\mathfrak{L}$ *if and only if* $\mathfrak{M}$ *reduces* $L_u$;

(3) $\mathfrak{M}$ *is left-pure if and only if* $\bigwedge_{n>0} L_u^n\mathfrak{M} = \{0\}$;

*and*

(4) $\mathfrak{M}$ *is left-full if and only if* $\bigvee_{n<0} L_u^n\mathfrak{M} = \mathbf{L}^2$.

PROPOSITION 3.3. *Let* $\mathfrak{M}$ *be a left- (resp. right-) invariant subspace of* $\mathbf{L}^2$. *Then* $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, *where* $\mathfrak{M}_1$ *is a left- (resp. right-) reducing subspace and* $\mathfrak{M}_2$ *is a left- (resp. right-) pure left- (resp. right-) invariant subspace.*

As in the proof of [21, Theorem 4.1], we have the following proposition.

PROPOSITION 3.4. *Every left- (resp. right-) reducing subspace* $\mathfrak{M}$ *in* $\mathbf{L}^2$ *has the form* $R_e\mathbf{L}^2$ *(resp.* $L_e\mathbf{L}^2$*) for some projection* $e$ *in* $M\times_\alpha \mathbf{Z}$. *In particular, if* $\mathfrak{M}$ *is two-sided reducing, then there exists a central projection* $e$ *of* $M\times_\alpha\mathbf{Z}$ *such that* $\mathfrak{M} = R_e\mathbf{L}^2 = L_e\mathbf{L}^2$.

We next define $\mathbf{H}^2$ to be the closed linear span $[(M\times_\alpha \mathbf{Z}_+)h_0^{1/2}]_2$ of $(M\times_\alpha\mathbf{Z}_+)h_0^{1/2}$ in $\mathbf{L}^2$, and we put $\mathbf{H}_0^2 = L_u\mathbf{H}^2$. $\mathbf{H}^2$ is called the *noncommutative Hardy space* determined by $M$ and $\alpha$. The following proposition presents the basic properties of $\mathbf{H}^2$.

PROPOSITION 3.5. (1) $\mathbf{L}^2 = \mathbf{H}^2 \oplus \mathbf{J}\mathbf{H}_0^2 = \mathbf{H}_0^2 \oplus \mathbf{J}\mathbf{H}^2$.

(2) $R_u^n\mathbf{H}^2 = L_u^n\mathbf{H}^2$ *for every* $n\in\mathbf{Z}$.

(3) $\mathbf{H}^2$ *is a left-pure, left-full, right-pure, right-full, two-sided invariant subspace of* $\mathbf{L}^2$.

(4) $\mathbf{H}^2 = \sum_{n\geq 0}^{\oplus} L_u^n[Mh_0^{1/2}]_2$.

(5) $\mathbf{H}^2 = \{x\in\mathbf{L}^2 : E_n(x) = 0,\ n<0\}$.

(6) $\mathbf{H}^2 = [h_0^{1/2}(M\times_\alpha\mathbf{Z}_+)]_2$.

*Proof.* (1) Assertion (1) follows from the proof of [1, Theorem 2.2.1].

(2) Since $L_u\mathbf{H}^2 = \mathbf{H}_0^2$ and $\mathbf{J}L_u\mathbf{J} = R_u^*$, we have

$$\mathbf{H}^2 = \mathbf{L}^2 \ominus \mathbf{J}\mathbf{H}_0^2 = \mathbf{L}^2 \ominus \mathbf{J}L_u\mathbf{H}^2 = \mathbf{L}^2\ominus R_u^*\mathbf{J}\mathbf{H}^2 =$$

$$= R_u^*(\mathbf{L}^2\ominus\mathbf{J}\mathbf{H}^2) = R_u^*\mathbf{H}_0^2.$$

Thus we have $R_u\mathbf{H}^2 = \mathbf{H}_0^2 = L_u\mathbf{H}^2$. If $n\geq 1$, and if $R_u^n\mathbf{H}^2 = L_u^n\mathbf{H}^2$, then

$$R_u^{n+1}\mathbf{H}^2 = R_u(R_u^n\mathbf{H}^2) = R_u(L_u^n\mathbf{H}^2) = L_u^n(R_u\mathbf{H}^2) =$$

$$= L_u^n(L_u\mathbf{H}^2) = L_u^{n+1}\mathbf{H}^2.$$

Thus $L_u^n\mathbf{H}^2 = R_u^n\mathbf{H}^2$ for all $n\geq 0$. But, if $n<0$, then using (1) once more and the fact that

$$R_u^{*n}\mathbf{H}_0^2 = R_u^{*(n-1)}\mathbf{H}^2 = L_u^{*(n-1)}\mathbf{H}^2 = L_u^{*n}\mathbf{H}_0^2,$$

we find that

$$R_u^n \mathbf{H}^2 = \mathbf{J} R_u^n \mathbf{H}^2 = \mathbf{J}(L_u^{*n}(\mathbf{J}\mathbf{H}^2)) = \mathbf{J}(L_u^{*n}(\mathbf{L}^2 \ominus \mathbf{H}_0^2)) =$$

$$= \mathbf{J}(\mathbf{L}^2 \ominus L_u^{*n}\mathbf{H}_0^2) = \mathbf{J}(\mathbf{L}^2 \ominus R_u^{*n}\mathbf{H}_0^2) =$$

$$= \mathbf{J}(R_u^{*n}(\mathbf{J}\mathbf{H}^2)) = L_u^n \mathbf{H}^2.$$

(3) Let $v$ be a unitary element in $M$. Since $\mathbf{H}^2$ is clearly left-invariant, $L_v^* \mathbf{H}_0^2 = \mathbf{H}_0^2 = L_v \mathbf{H}_0^2$, by (2). Thus we have

$$R_v \mathbf{H}^2 = R_v(\mathbf{L}^2 \ominus \mathbf{J}\mathbf{H}_0^2) = \mathbf{L}^2 \ominus R_v \mathbf{J}\mathbf{H}_0^2 =$$

$$= \mathbf{L}^2 \ominus \mathbf{J} L_v^* \mathbf{H}_0^2 = \mathbf{L}^2 \ominus \mathbf{J}\mathbf{H}_0^2 = \mathbf{H}^2.$$

Since $M$ is generated by unitary elements in $M$, $\mathbf{H}^2$ is a right-invariant subspace of $\mathbf{L}^2$. It is clear that $\mathbf{H}^2$ is left-full and right-full since $h_0^{1/2} \in \mathbf{H}^2$. Let $x \in \bigcap_{n \geqslant 0} L_u^n \mathbf{H}^2 = \bigcap_{n>0} R_u^n \mathbf{H}^2$. Then, for every $n > 0$, there exists an element $y_n \in \mathbf{H}^2$ such that $x = L_u^n y_n$. For every $y \in \mathbf{J}\mathbf{H}^2$, we have

$$(x, L_u^n y) = \tau(y^* u^{*n} u^{n+1} y_{n+1}) = \tau(y^* u y_{n+1}) =$$

$$= (u y_{n+1}, y) = 0,$$

because $u y_{n+1} \in \mathbf{H}_0^2$ and $y \in \mathbf{J}\mathbf{H}^2$. Since $\bigcup_{n \geqslant 0} L_u^n \mathbf{J}\mathbf{H}^2$ is dense in $\mathbf{L}^2$, we have $x = 0$. Thus $\mathbf{H}^2$ is left-pure and right-pure.

(4) is clear since $\mathbf{H}^2$ is left pure.

(5) is clear by the definition of $\mathbf{H}^2$ and $E_n$.

(6) Since $[M h_0^{1/2}]_2 = [h_0^{1/2} M]_2$, (6) follows from (2) and (4).

This completes the proof.

In this paper we are interested in certain wandering subspaces for the bilateral shifts $L_u$ and $R_u$. As in [16], we have the following proposition.

PROPOSITION 3.6 ([16, Theorem 3.2]). *For $i = 1, 2$, let $\mathfrak{M}_i$ be a left-pure, left-invariant subspace in $\mathbf{L}^2$, let $q_i$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M}_i$, and let $p_i$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M}_i \ominus L_u \mathfrak{M}_i$, $i = 1, 2$. Then each $p_i$ lies in $L(M)'$, and $p_2 \preccurlyeq p_1$ in $L(M)'$ if and only if there is a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ such that $q_2 = R_v q_1 R_v^*$. In this event, $\mathfrak{M}_2 = R_v \mathfrak{M}_1$. In particular, if $\mathfrak{M}_1 = \mathbf{H}^2$ and $p_2 \preccurlyeq p_1$ in $L(M)'$, then there exists a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ such that $\mathfrak{M}_2 = R_v \mathbf{H}^2$ and $vv^* \in M$.*

*Proof.* Each $q_i$ lies in $L(M)'$ by Proposition 3.2; and since $L_u$ normalizes $L(M)$, and therefore $L(M)'$, it follows that $p_i = q_i - L_u q_i L_u^*$ lies in $L(M)'$. If

$p_2 \preccurlyeq p_1$ in $L(M)'$, then there is a partial isometry $w$ in $L(M)'$ such that $p_2 = ww^*$ and $w^*w \leqslant p_1$. As in the proof of [16, Theorem 3.2], put $R_v = \sum_{k=-\infty}^{\infty} L_u^k w L_u^{*k}$. Then $R_v$ is a partial isometry in $\mathfrak{R}$ such that $q_2 = R_v q_1 R_v^*$. The cnnverse is trivial by [14, Theorem 3.2].

If $\mathfrak{M}_1 = \mathbf{H}^2$ and $p_2 \preccurlyeq p_1$, then we have

$$R_v^* R_v = \left(\sum_{n=-\infty}^{\infty} L_u^n w L_u^{*n}\right)^* \left(\sum_{n=-\infty}^{\infty} L_u^n w L_u^{*n}\right) =$$

$$= \sum_{n=-\infty}^{\infty} L_u^n w^* w L_u^{*n}.$$

Hence we have

$$R_v^* R_v q_1 = \left(\sum_{n=-\infty}^{\infty} L_u^n w^* w L_u^{*n}\right)\left(\sum_{n=0}^{\infty} L_u^n p_1 L_u^{*n}\right) \leqslant$$

$$\leqslant \sum_{n=0}^{\infty} L_u^n w^* w L_u^{*n} \leqslant q_1.$$

This implies that $R_v^* R_v \mathbf{H}^2 \subseteq \mathbf{H}^2$ and so $R_v^* R_v \in \mathfrak{R}_+$ by [1, Theorem 2.2.1]. Therefore $R_{vv^*} \in R(M)$ and so $vv^* \in M$. This completes the proof.

Let $\mathfrak{M}$ be a left-invariant subspace of $\mathbf{L}^2$. By Proposition 3.3, $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\mathfrak{M}_1$ is a left-reducing subspace of $\mathbf{L}^2$ and $\mathfrak{M}_2$ is a left-pure left-invariant subspace of $\mathbf{L}^2$. Let $q_i$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M}_i$, and let $p_2$ be the projection of $\mathfrak{M}_2 \ominus L_u \mathfrak{M}_2$. Let $P$ be the projection of $\mathbf{L}^2$ onto $\mathbf{H}^2$, and let $p_0$ be the projection of $\mathbf{L}^2$ onto $\mathbf{H}^2 \ominus L_u \mathbf{H}^2$. Then the following corollary is a consequence of Proposition 3.6.

COROLLARY 3.7. *If $p_2 \preccurlyeq p_0$ in $L(M)'$, then there exist a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ and a projection $q_0$ in $M \times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v \mathbf{H}^2 \oplus R_{q_0} \mathbf{L}^2$ and $R_v R_v^* R_{q_0} = 0$.*

*Proof.* By Propositions 3.4 and 3.6, there is a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ and there is a projection $q_0$ in $M \times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v \mathbf{H}^2 \oplus R_{q_0} \mathbf{L}^2$. Thus $R_v P R_v^* R_{q_0} = 0$. Since $\mathbf{H}^2$ is left-full by Proposition 3.5(3), $L_u^n P L_u^{*n}$ converges strongly to 1 ($n \to -\infty$). So we have

$$R_v L_u^n P L_u^{*n} R_v^* R_{q_0} = L_u^n (R_v P R_v^* R_{q_0}) L_u^{*n} = 0,$$

and $R_v R_v^* R_{q_0} = 0$. This completes the proof.

We shall say that the Beurling-Lax-Halmos (hereafter abbreviated the BLH theorem) is valid if every left-pure, left-invariant subspace $\mathfrak{M}$ of $\mathbf{L}^2$ has the form

$R_v\mathbf{H}^2$ for some partial isometry $v$ in $M \times_\alpha \mathbf{Z}$. In [17], we showed that if $M$ is finite, then the BLH theorem is valid if and only if $\alpha$ fixes the center $\mathfrak{Z}(M)$ of $M$ elementwise. Our aim in this section is to find necessary and sufficient conditions for the BLH theorem to be valid in the present, more general setting.

LEMMA 3.8. *Suppose that $M$ is a properly infinite von Neumann algebra. If $p$ is a projection in $L(M)'$, then $p \preccurlyeq p_0$ in $L(M)'$.*

*Proof.* We recall that $p_\theta$ is the projection of $\mathbf{L}^2$ onto $[Mh_0^{1/2}]_2$ $(= \mathbf{H}^2 \ominus L_u\mathbf{H}^2)$. Thus $p_0 = E_0$. Let $c(p_0)$ be the central support projection of $p_0$ in $L(M)'$. Since the center $\mathfrak{Z}(L(M)')$ of $L(M)'$ equals $L(\mathfrak{Z}(M))$, there exists a central projection $z$ in $M$ such that $L_z = c(p_0)$. Since $p_0$ is the projection of $\mathbf{L}^2$ onto $[Mh_0^{1/2}]_2$, we have

$$xh_0^{1/2} = p_0xh_0^{1/2} = c(p_0)xh_0^{1/2} = L_zxh_0^{1/2} = zxh_0^{1/2}, \quad x \in M.$$

Since $h_0^{1/2}$ is a separating vector in $\mathbf{L}^2$, we have $x = zx$, $x \in M$ and so $z = 1$. Thus $c(p_0) = 1$. Since $(L(M)p_0)' = p_0L(M)'p_0$ (cf. [4, Chapter 1, §2, Proposition 1]), $\{L(M)p_0, [Mh_0^{1/2}]_2\}$ is a standard form and so $p_0$ is properly infinite. Since $M$ is $\sigma$-finite and the central support projection $c(p)$ is dominated by 1, we have $p \preccurlyeq p_0$ in $L(M)'$, by [4, Chapter III, § 8, Corollaire 5]. This completes the proof.

PROPOSITION 3.9. *If $M$ is properly infinite, then the BLH theorem is valid.*

*Proof.* Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\mathbf{L}^2$, let $p$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M} \ominus L_u\mathfrak{M}$. Since $p$ lies in $L(M)'$ and $M$ is properly infinite, $p \preccurlyeq p_0$ in $L(M)'$ by Lemma 3.8. By Proposition 3.6, there exists a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v\mathbf{H}^2$. This completes the proof.

By [4, Chapter 1, § 6, Corollaire 1], any von Neumann algebra $M$ is uniquely decomposed into a direct sum of two algebras, one of which is finite and the other of which is properly infinite; that is, $M = Mz \oplus M(1 - z)$, where $Mz$ is finite, $M(1 - z)$ is properly infinite, and $z$ is the maximal, finite, central projection of $M$. Since, necessarily, $\alpha(z) = z$, $z$ is a central projection of $M \times_\alpha \mathbf{Z}$, and so $M \times_\alpha \mathbf{Z} = (M \times_\alpha \mathbf{Z})z \oplus (M \times_\alpha \mathbf{Z})(1 - z)$. It is clear that $(M \times_\alpha \mathbf{Z})z = Mz \times_{\alpha_1} \mathbf{Z}$ and $(M \times_\alpha \mathbf{Z})(1 - z) = M(1 - z) \times_{\alpha_2} \mathbf{Z}$, where $\alpha_1$ and $\alpha_2$ are the restrictions of $\alpha$ to $Mz$ and $M(1 - z)$, respectively.

THEOREM 3.10. *The following assertions are equivalent:*

(1) *$\alpha$ fixes each finite central projection of $M$.*

(2) *Every left-pure, left-invariant subspace of $\mathbf{L}^2$ is of the form $R_v\mathbf{H}^2$ where $v$ is a partial isometry in $M \times_\alpha \mathbf{Z}$.*

(3) *Every left-invariant subspace of $\mathbf{H}^2$ is of the form $R_v\mathbf{H}^2$, where $v$ is a partial isometry in $M \times_\alpha \mathbf{Z}$.*

*Proof.* (1) $\Rightarrow$ (2). Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\mathbf{L}^2$, and let $z$ be the maximal, finite, central projection of $M$. Then $L_z\mathfrak{M}$ and $L_{1-z}\mathfrak{M}$ are left-pure, left-invariant subspaces of $\mathbf{L}^2$. Since $Mz$ is finite, there exists a partial isometry $v_1$ in $(M\times_\alpha \mathbf{Z})z$ such that $L_z\mathfrak{M} = R_{v_1}\mathbf{H}^2$, by [16, Theorem 3.2] and [17, Theorem 3.2]. On the other hand, since $M(1-z)$ is properly infinite, there exists a partial isometry $v_2$ in $(M\times_\alpha \mathbf{Z})(1-z)$ such that $L_{1-z}\mathfrak{M} = R_{v_2}\mathbf{H}^2$, by Proposition 3.9. Thus, putting $v = v_1 + v_2$, we see that $v$ is a partial isometry in $M\times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v\mathbf{H}^2$. Therefore, the BLH theorem is valid.

(2) $\Rightarrow$ (3). Since $\mathbf{H}^2$ is left-pure by Proposition 3.5 (3), it is clear that (2) implies (3).

(3) $\Rightarrow$ (1). Let $z$ be the maximal, finite, central projection of $M$. If $z=0$, then we are done. Suppose that $z\neq 0$. Consider the finite von Neumann algebra $Mz$. Since $\alpha$ fixes $z$, $(M\times_\alpha \mathbf{Z})z = (Mz)\times_\alpha \mathbf{Z}$, and we may restrict our attention to $(Mz)\times_\alpha \mathbf{Z}$. But then, the assertion follows from Theorem 3.2 of [17]. This completes the proof.

COROLLARY 3.11. *If $\alpha$ fixes each finite central projection, and if $\mathfrak{M}$ is a left-pure, left-full, left-invariant subspace of $\mathbf{L}^2$, then there is an* **isometry** *$v$ in $M\times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v\mathbf{H}^2$.*

*Proof.* By Theorem 3.10, there exists a partial isometry $v$ in $M\times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v\mathbf{H}^2$. Since $R_v$ and $L_u$ commute, we find that

$$R_v\mathbf{L}^2 \supset R_v(\bigvee_{n\in\mathbf{Z}} L_u^n\mathbf{H}^2) = \bigvee_{n\in\mathbf{Z}} L_u^n R_v\mathbf{H}^2 = \bigvee_{n\in\mathbf{Z}} L_u^n\mathfrak{M} = \mathbf{L}^2,$$

that is, $R_v$ is a co-isometry and so $v$ is an isometry in $M\times_\alpha \mathbf{Z}$. This completes the proof.

## 4. MAXIMALITY OF $M\times_\alpha \mathbf{Z}$

Our main objective in this section is to prove the following theorem which determines when $M\times_\alpha \mathbf{Z}_+$ is a maximal $\sigma$-weakly closed subalgebra of $M\times_\alpha \mathbf{Z}$. Recall that Theorem 2.1 tells us that $M\times_\alpha \mathbf{Z}_+$ is maximal as a subdiagonal algebra. In [16], we proved that, when $M$ is finite, $M$ is a factor if and only if $M\times_\alpha \mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M\times_\alpha \mathbf{Z}$. In this section, we generalize this result to cover the case when $M$ is an arbitrary ($\sigma$-finite) von Neumann algebra. To do this we require the following lemmas.

LEMMA 4.1. *Let $M$ be a properly infinite von Neumann algebra. If $B$ is a proper, $\sigma$-weakly closed subalgebra of $M\times_\alpha \mathbf{Z}$ containing $M\times_\alpha \mathbf{Z}_+$, then $[Bh_0^{1/2}]_2\neq \mathbf{L}^2$.*

*Proof.* Since $B$ is a proper $\sigma$-weakly closed subalgebra of $M\times_\alpha \mathbf{Z}$, there exists a nonzero element $x\in \mathbf{L}^1$ such that $\tau(y^*x)=0$ for every $y\in B$, by the Hahn-Banach theorem. Let $x=|x^*|v$ be the polar decomposition of $x$. Since $|x^*|^{1/2}\in \mathbf{L}^2$, we may consider the right-invariant subspace $[|x^*|^{1/2}B]_2$ of $\mathbf{L}^2$. If $[|x^*|^{1/2}B]_2$ were right-reducing, then it is clear that $[|x^*|^{1/2}B]_2 = [|x^*|^{1/2}(M\times_\alpha \mathbf{Z})]_2$. Since $v\in M\times_\alpha\mathbf{Z}$ and $1\in B$, we would have, for each $b\in B$,

$$(|x^*|^{1/2}v,\ |x^*|^{1/2}b) = \tau(b^*|x^*|^{1/2}|x^*|^{1/2}v) = \tau(b^*x) = 0.$$

Since $[|x^*|^{1/2}B]_2=[|x^*|^{1/2}(M\times_\alpha\mathbf{Z})]_2$, we would have $|x^*|^{1/2}v = 0$ and so $x$ would be zero — a contradiction. By Propositions 3.3 and 3.9, there is a partial isometry $w$ in $M\times_\alpha\mathbf{Z}$ and a projection $p$ in $M\times_\alpha\mathbf{Z}$ such that $[|x^*|^{1/2}B]_2 = L_w\mathbf{H}^2\oplus L_p\mathbf{L}^2$. Since $w^*w\in M$ by Proposition 3.6 and $L_wL_w^*L_p = 0$ by Corollary 3.7, we have

$$L_w^*[|x^*|^{1/2}B]_2 = L_w^*L_w\mathbf{H}^2 = L_{w^*w}\mathbf{H}^2 \subset \mathbf{H}^2.$$

Thus there exists a nonzero element $c\in\mathbf{H}^2$ such that $L_w^*|x^*|^{1/2} = L_w^*L_wc$. Proposition 3.5 (1) implies that for every $a\in B$ and $b\in\mathbf{H}_0^\infty$ $(=u(M\times_\alpha\mathbf{Z}_+))$, we have

$$(ah_0^{1/2},\ (bw^*wc)^*) = \tau(bw^*wcah_0^{1/2}) = \tau(bw^*|x^*|^{1/2}ah_0^{1/2}) =$$

$$= \tau(h_0^{1/2}bw^*|x^*|^{1/2}a) = (w^*|x^*|^{1/2}a,\ b^*h_0^{1/2}) = 0,$$

because $w^*|x^*|^{1/2}\in\mathbf{H}^2$ and $b^*h_0^{1/2}\in\mathbf{J}\mathbf{H}_0^2$. Since $(\mathbf{H}_0^\infty w^*wc)^*\neq\{0\}$, we have $[Bh_0^{1/2}]_2\neq\mathbf{L}^2$. This completes the proof.

The following lemmas may be found in [16]. Since the proofs there do not require that $M$ is finite, they apply here and so we shall not reproduce them.

LEMMA 4.2 (cf. [16, Lemma 4.2]). *If $M$ is a factor and if $B$ is a $\{\beta_t\}_{t\in\mathbf{R}}$-invariant, $\sigma$-weakly closed subalgebra of $M\times_\alpha\mathbf{Z}$ containing $M\times_\alpha\mathbf{Z}_+$, then $B = M\times_\alpha\mathbf{Z}_+$ or $B = M\times_\alpha\mathbf{Z}$.*

LEMMA 4.3 (cf. [16, Theorem 2.3]). *Suppose that $M$ is a factor and $M\times_\alpha\mathbf{Z}$ is not a factor. Then $\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap(M\times_\alpha\mathbf{Z}_+)$ is a maximal $\sigma$-weakly closed subalgebra of $\mathfrak{Z}(M\times_\alpha\mathbf{Z})$, where $\mathfrak{Z}(M\times_\alpha\mathbf{Z})$ is the center of $M\times_\alpha\mathbf{Z}$.*

THEOREM 4.4. *The following assertions are equivalent:*

(1) *$M$ is a factor;*

*and*

(2) *$M\times_\alpha\mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M\times_\alpha\mathbf{Z}$.*

*Proof.* The implication (2) $\Rightarrow$ (1) is proved just as in the proof of Theorem 4.1 of [16]. So we concentrate on the implication (1) $\Rightarrow$ (2). If $M$ is a finite factor, then $M\times_\alpha\mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M\times_\alpha\mathbf{Z}$ by [16, Theorem 4.1]. Therefore we suppose that $M$ is a properly infinite factor. Let $B$ be a proper $\sigma$-weakly closed subalgebra of $M\times_\alpha\mathbf{Z}$ containing $M\times_\alpha\mathbf{Z}_+$, form the two-sided invariant subspace $[Bh_0^{1/2}]_2$, and note that $[Bh_0^{1/2}]_2\neq\mathbf{L}^2$ by Lemma 4.1. Note, too, that $[Bh_0^{1/2}]_2$ does not reduce either $\mathfrak{L}_+$ or $\mathfrak{R}_+$ because it contains the cyclic and separating vector $h_0^{1/2}$. Since $\mathbf{H}^2\subset[Bh_0^{1/2}]_2$, $[Bh_0^{1/2}]_2$ is obviously left-full.

To show that it is left-pure, we begin by replacing $B$ by a potentially bigger subalgebra. We set $\tilde{B}=\{x\in M\times_\alpha \mathbf{Z} : L_x[Bh_0^{1/2}]_2\subset [Bh_0^{1/2}]_2\}$. Since $[Bh_0^{1/2}]_2$ is not left-reducing, it is clear that $\tilde{B}$ is a proper, $\sigma$-weakly closed subalgebra of $M\times_\alpha \mathbf{Z}$ containing $B$ and $[\tilde{B}h_0^{1/2}]_2=[Bh_0^{1/2}]_2$.

To show that $[Bh_0^{1/2}]_2$ is left-pure, let $P_\infty$ be the projection of $\mathbf{L}^2$ onto $\bigcap_{n>0} L_u^n[Bh_0^{1/2}]_2$. Since $[Bh_0^{1/2}]_2\neq \mathbf{L}^2$, $P_\infty\neq 1$. It is clear that $P_\infty$ lies in $\mathfrak{L}'=\mathfrak{R}$. Since, however, $[Bh_0^{1/2}]_2$ is right-invariant, so is $\bigcap_{n>0} L_u^n[Bh_0^{1/2}]_2$. This implies that $P_\infty$ commutes with $R(M)$ and $R_uP_\infty R_u^*\leqslant P_\infty$. Thus $P_\infty\in R(M)'\cap\mathfrak{R}$.

Next we shall prove that $R_uP_\infty R_u^*=P_\infty$. Suppose that $R_uP_\infty R_u^*<P_\infty$. Then $P_\infty\mathbf{L}^2\ (=\bigcap_{n>0} L_u^n[Bh_0^{1/2}]_2)$ is right-invariant and not right-reducing. Put $q_1=P_\infty-R_uP_\infty R_u^*$. Then $q_1$ is the projection of $\mathbf{L}^2$ onto $P_\infty\mathbf{L}^2\ominus R_uP_\infty\mathbf{L}^2$. Let $P$ (resp. $p_0$) be the projection of $\mathbf{L}^2$ onto $\mathbf{H}^2$ (resp. $\mathbf{H}^2\ominus R_u\mathbf{H}^2$). Let $P'_\infty$ be the projection of $\mathbf{L}^2$ onto $\bigcap_{n>0} R_u^nP_\infty\mathbf{L}^2$. Since $P_\infty\in R(M)'\cap\mathfrak{R}$, it is clear that $P'_\infty\in\mathfrak{R}'\cap\mathfrak{R}=\mathfrak{Z}(\mathfrak{L})$. Hence there exists a central projection $p'_\infty$ in $M\times_\alpha\mathbf{Z}$ such that $L_{p'_\infty}=P'_\infty$. If $M\times_\alpha\mathbf{Z}$ is a factor, $P'_\infty$ must be zero, since $p'_\infty\neq 1$, and so $P_\infty\mathbf{L}^2$ is right-pure. In the contrary case, since $L_{p'_\infty}[Bh_0^{1/2}]_2\subset L_{p'_\infty}\mathbf{L}^2\subset[Bh_0^{1/2}]_2$, $p'_\infty$ lies in $\tilde{B}$. Thus $P'_\infty\in\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap\tilde{B}$. Since $\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap(M\times_\alpha\mathbf{Z}_+)$ is maximal as a $\sigma$-weakly closed subalgebra of $\mathfrak{Z}(M\times_\alpha\mathbf{Z})$ by Lemma 4.3, we find that either $\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap\tilde{B}=\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap\cap(M\times_\alpha\mathbf{Z}_+)$, in which case $p'_\infty=0$, or $\mathfrak{Z}(M\times_\alpha\mathbf{Z})\cap\tilde{B}=\mathfrak{Z}(M\times_\alpha\mathbf{Z})$. But, if $\mathfrak{Z}(M\times_\alpha\mathbf{Z})$ were contained in $\tilde{B}$, then the $\sigma$-weakly closed subalgebra $D$ generated by $M\times_\alpha\mathbf{Z}_+$ and $\mathfrak{Z}(M\times_\alpha\mathbf{Z})$ would be a $\{\beta_t\}_{t\in\mathbf{R}}$-invariant subalgebra of $M\times_\alpha\mathbf{Z}$ satisfying the relations $M\times_\alpha\mathbf{Z}_+\subsetneqq D\subset\tilde{B}\subsetneqq M\times_\alpha\mathbf{Z}$. Since this is not possible by Lemma 4.2, we conclude once more that $P'_\infty=0$. Thus $P_\infty\mathbf{L}^2$ is right-pure.

We now consider the following two cases:

(i) $q_1$ is infinite in $R(M)'$;

and

(ii) $q_1$ is finite in $R(M)'$.

*Case* (i). Suppose that $q_1$ is infinite. Since $p_0$ is infinite and $R(M)'$ is a factor, $p_0\sim q_1$ by [4, Chapter III, § 8, Corollaire 5]. By Proposition 3.6, there exists a partial isometry $w$ in $M\times_\alpha\mathbf{Z}$ such that $P_\infty\mathbf{L}^2=L_w\mathbf{H}^2$ and $L_w^*L_w=1$, because $\mathbf{H}^2$ is right-full. Hence we have

$$P_\infty\mathbf{L}^2=P_\infty L_w^*L_w\mathbf{L}^2=P_\infty L_w^*\mathbf{L}^2=L_w^*P_\infty\mathbf{L}^2=$$

$$=L_w^*L_w\mathbf{H}^2=\mathbf{H}^2.$$

Since $\mathbf{H}^2$ is right-pure, $P_\infty=0$, which contradicts the assumption that $R_uP_\infty R_u^*<P_\infty$. Thus $P_\infty\in\mathfrak{Z}(\mathfrak{L})$.

*Case* (ii). Suppose that $q_1$ is finite in $R(M)'$. Since $p_0$ is infinite and $R(M)'$ is a factor, $q_1\prec p_0$ by [4, Chapter III, § 8, Corollaire 5]. Thus there exists a partial isometry $w$ in $M\times_\alpha \mathbf{Z}$ such that $P_\infty \mathbf{L}^2 = L_w \mathbf{H}^2$. Since $P_\infty = L_w P L_w^*$ by Proposition 3.6, it is clear that $\bigvee_{n\in\mathbf{Z}} R_u^n P_\infty R_u^{*n} = L_w L_w^*$. Since $P_\infty \in \mathfrak{R} \cap R(M)'$, we have $L_w L_w^* \in \mathfrak{Z}(\mathfrak{L})$. We now prove that $L_w L_w^*$ is a finite projection in $\mathfrak{L}$. Suppose that $L_w L_w^* \sim L_q = L_w L_w^*$, where $q$ is a projection in $M\times_\alpha \mathbf{Z}$. Then there exists a partial isometry $w_1 \in M\times_\alpha \mathbf{Z}$ such that $L_w L_w^* = L_{w_1}^* L_{w_1}$ and $L_q = L_{w_1} L_{w_1}^*$. Since $q_1 = P_\infty - R_u P_\infty R_u^* \in \mathfrak{R} \cap R(M)'$, we have

$$(L_{w_1} q_1)^*(L_{w_1} q_1) = q_1 L_{w_1}^* L_{w_1} q_1 = q_1 L_w L_w^* q_1 = q_1$$

and

$$(L_{w_1} q_1)(L_{w_1} q_1)^* = L_{w_1} q_1 L_{w_1}^* = L_{w_1} L_{w_1}^* q_1 = L_q q_1 .$$

Thus we have $q_1 \sim L_q q_1 \leqslant q_1$ in $R(M)'$. Since $q_1$ is finite in $R(M)'$, $q_1 = L_q q_1$. Hence we have

$$L_w L_w^* = \sum_{n\in\mathbf{Z}} R_u^n q_1 R_u^{*n} = \sum_{n\in\mathbf{Z}} R_u^n L_q q_1 R_u^{*n} = L_q L_w L_w^* .$$

This implies that $L_q \geqslant L_w L_w^*$ and so $L_q = L_w L_w^*$. Consequently, $L_w L_w^*$ is a finite central projection in $\mathfrak{L}$. On the other hand, we assert that $L_w^* L_w \leqslant L_w L_w^*$. For, since $P_\infty L_w^* L_w \mathbf{L}^2 = L_w^* L_w \mathbf{H}^2$, we have $P_\infty L_w^* L_w = L_w^* L_w P$. Thus, $R_u^n P_\infty R_u^{*n} L_w^* L_w = L_w^* L_w R_u^n P R_u^{*n}$. Since $R_u^n P_\infty R_u^{*n} \to L_w L_w^*$ and $R_u^n P R_u^{*n} \to 1$ $(n \to -\infty)$, we have $L_w L_w^* L_w^* L_w = L_w^* L_w$ and so $L_w^* L_w \leqslant L_w L_w^*$. Since $L_w L_w^*$ is finite, $L_w^* L_w = L_w L_w^*$. Thus we have

$$P_\infty \mathbf{L}^2 = P_\infty L_w L_w^* \mathbf{L}^2 = P_\infty L_w^* L_w \mathbf{L}^2 = P_\infty L_w^* \mathbf{L}^2 \subset L_w^* L_w \mathbf{H}^2 \subset \mathbf{H}^2,$$

because $L_w^* L_w \in L(M)$. Thus, since $\mathbf{H}^2$ is right-pure, $P_\infty = 0$. This is a contradiction.

Therefore, in both cases, $P_\infty \in \mathfrak{Z}(\mathfrak{L})$. Thus there exists a central projection $p_\infty$ in $M\times_\alpha \mathbf{Z}$ such that $L_{p_\infty} = P_\infty$. Since $L_{p_\infty}[Bh_0^{1/2}]_2 \subset L_{p_\infty}\mathbf{L}^2 \subset [Bh_0^{1/2}]_2$, $p_\infty$ lies in $\tilde{B}$. Thus $p_\infty \in \mathfrak{Z}(M\times_\alpha \mathbf{Z}) \cap \tilde{B}$. Since $\mathfrak{Z}(M\times_\alpha \mathbf{Z}) \cap (M\times_\alpha \mathbf{Z}_+)$ is maximal as a $\sigma$-weakly closed subalgebra of $\mathfrak{Z}(M\times_\alpha \mathbf{Z})$ by Lemma 4.3, we find that either $\mathfrak{Z}(M\times_\alpha \mathbf{Z}) \cap \tilde{B} = \mathfrak{Z}(M\times_\alpha \mathbf{Z}) \cap (M\times_\alpha \mathbf{Z}_+)$, in which case $p_\infty = 0$, or $\mathfrak{Z}(M\times_\alpha \mathbf{Z}) \cap \tilde{B} = \mathfrak{Z}(M\times_\alpha \mathbf{Z})$. But, if $\mathfrak{Z}(M\times_\alpha \mathbf{Z})$ were contained in $\tilde{B}$, then the $\sigma$-weakly closed subalgebra $D$ generated by $M\times_\alpha \mathbf{Z}_+$ and $\mathfrak{Z}(M\times_\alpha \mathbf{Z})$ would be a $\{\beta_t\}_{t\in\mathbf{R}}$-invariant subalgebra of $M\times_\alpha \mathbf{Z}$ satisfying the relations $M\times_\alpha \mathbf{Z}_+ \subsetneqq D \subset \tilde{B} \subsetneqq M\times_\alpha \mathbf{Z}$. Since this is not possible by Lemma 4.2, we conclude once more that $p_\infty = 0$. Thus $[Bh_0^{1/2}]_2$ is left-pure as we wished to prove.

Let $q_0$ be the projection of $\mathbf{L}^2$ onto $[Bh_0^{1/2}]_2 \ominus L_v[Bh_0^{1/2}]_2$; then we consider the following two cases:

(a) $q_0$ is infinite in $L(M)'$;

and

(b) $q_0$ is finite in $L(M)'$.

*Case* (a). Suppose that $q_0$ is infinite in $L(M)'$. Since $p_0$ is infinite in $L(M)'$, $p_0 \sim q_0$. Since $[Bh_0^{1/2}]_2$ is left-pure and left-full, there exists a unitary operator $v \in M \times_\alpha \mathbf{Z}$ such that $[Bh_0^{1/2}]_2 = R_v \mathbf{H}^2$. This implies that

$$[Bh_0^{1/2}]_2 = R_v^* R_v [L(B)\mathbf{H}^2]_2 = R_v^* [R_v L(B)\mathbf{H}^2]_2 =$$

$$= R_v^* [L(B) R_v \mathbf{H}^2]_2 = R_v^* [L(B)[Bh_0^{1/2}]_2]_2 =$$

$$= R_v^* [Bh_0^{1/2}]_2 = \mathbf{H}^2.$$

Therefore $B = M \times_\alpha \mathbf{Z}_+$, by [1, Theorem 2.2.1]. So in this case, $M \times_\alpha \mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M \times_\alpha \mathbf{Z}$.

*Case* (b). Suppose that $q_0$ is finite in $L(M)'$. Since $L(M)'$ is $\sigma$-finite, there exists a maximal family $\{r_n\}_{n=1}^{\infty}$ of mutually orthogonal, equivalent, finite projections in $L(M)'$ such that $q_0 \sim r_n < p_0$. Then $p_0 - \sum_{n=1}^{\infty} r_n$ is a finite projection in $L(M)'$. Put $r_0 = p_0 - \sum_{n=1}^{\infty} r_n$. Since $L(M)'$ is a factor, $r_0 \prec q_0$ or $q_0 \prec r_0$. If $q_0 \prec r_0$, then we have a contradiction by the maximality of $\{r_n\}_{n=1}^{\infty}$. Thus, $r_0 \prec q_0$. Since $q_0 \sim r_n$ $(n \geqslant 1)$, there exists a partial isometry $v_n \in L(M)'$ such that $v_n^* v_n = r_n < p_0$ and $v_n v_n^* = q_0$. Since $[Bh_0^{1/2}]_2$ is left-full, there exists an isometry $w_n \in M \times_\alpha \mathbf{Z}$ such that $R_{w_n} = \sum_{k \in \mathbf{Z}} L_u^k v_n L_u^{*k}$, $[Bh_0^{1/2}]_2 = R_{w_n} \mathbf{H}^2$ and $R_{w_n}^* R_{w_n} \in R(M)$. Thus, for every $n \geqslant 1$,

$$R_{w_n}^* R_{w_n} [Bh_0^{1/2}]_2 = R_{w_n}^* R_{w_n} [L(B)\mathbf{H}^2]_2 = [R_{w_n}^* R_{w_n} L(B)\mathbf{H}^2]_2 =$$

$$= [R_{w_n}^* L(B) R_{w_n} \mathbf{H}^2]_2 = [R_{w_n}^* L(B)[Bh_0^{1/2}]_2]_2 =$$

$$= [R_{w_n}^* R_{w_n} \mathbf{H}^2]_2 = R_{w_n}^* R_{w_n} \mathbf{H}^2.$$

On the other hand, since $r_0 \prec q_0$, there exists a partial isometry $v_0$ in $L(M)'$ such that $v_0^* v_0 = r_0$ and $v_0 v_0^* < q_0$. It follows that there exists a partial isometry $w_0 \in M \times_\alpha \mathbf{Z}$ such that $R_{w_0} = \sum_{k \in \mathbf{Z}} L_u^k v_0 L_u^{*k}$, $R_{w_0}^* [Bh_0^{1/2}]_2 = R_{w_0}^* R_{w_0} \mathbf{H}^2$ and $R_{w_0}^* R_{w_0} \in R(M)$. Thus we have $R_{w_0}^* R_{w_0} [Bh_0^{1/2}]_2 = R_{w_0}^* R_{w_0} \mathbf{H}^2$. Since $R_{w_n}^* R_{w_n} = \sum_{k \in \mathbf{Z}} L_u^k r_n L_u^{*k}$ and $R_{w_0}^* R_{w_0} = \sum_{k \in \mathbf{Z}} L_u^k r_0 L_u^{*k}$, we have

$$\sum_{n=0}^{\infty} R_{w_n}^* R_{w_n} = \sum_{n=0}^{\infty} \sum_{k \in \mathbf{Z}} L_u^k r_n L_u^{*k} = \sum_{k \in \mathbf{Z}} L_u^k p_0 L_u^{*k} = 1.$$

Consequently, $[Bh_0^{1/2}]_2 = \sum_{n=0}^{\infty} R_{w_n}^* R_{w_n} [Bh_0^{1/2}]_2 = \sum_{n=0}^{\infty} R_{w_n}^* R_{w_n} \mathbf{H}^2 = \mathbf{H}^2$, and we conclude that $B = M \times_\alpha \mathbf{Z}_+$. Thus $M \times_\alpha \mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M \times_\alpha \mathbf{Z}$, and the proof is complete.

There are several corollaries and modifications of Theorem 4.4, which are worth developing.

COROLLARY 4.5. *Suppose that $M$ is a factor. If $e$ is a projection in $M \times_\alpha \mathbf{Z}$ such that $R_e \mathbf{L}^2$ is right-invariant, then $e$ lies in $\mathfrak{Z}(M \times_\alpha \mathbf{Z})$, so that $R_e \mathbf{L}^2$ is two-sided reducing.*

*Proof.* It is clear that $R_e \in R(M)' \cap \mathfrak{R}$ and $R_u R_e R_u^* \leqslant R_e$. Put $\mathfrak{M} = \bigcap_{n \in \mathbf{Z}} R_u^n R_e \mathbf{L}^2$. Suppose that $R_u R_e R_u^* < R_e$. Let $Q$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M}$. Then, we clearly have $Q \in \mathfrak{Z}(\mathfrak{L})$. Thus, there exists a projection $q \in \mathfrak{Z}(M \times_\alpha \mathbf{Z})$ such that $R_q = Q$. But also $R_q R_e \mathbf{L}^2 \subset R_q \mathbf{L}^2 \subset R_e \mathbf{L}^2$. Hence, since $M \times_\alpha \mathbf{Z}_+$ is maximal as a $\sigma$-weakly closed subalgebra of $M \times_\alpha \mathbf{Z}$ by Theorem 4.4, $q \in (M \times_\alpha \mathbf{Z}_+) \cap \mathfrak{Z}(M \times_\alpha \mathbf{Z})$. Since $q \neq 1$, $q = 0$; that is, $R_e \mathbf{L}^2$ is right-pure. Then, as in the proof of Theorem 4.4, $R_e \in \mathfrak{Z}(\mathfrak{L})$. This completes the proof.

Next we investigate the form of two-sided invariant subspaces of $\mathbf{L}^2$.

THEOREM 4.6. *If $M$ is a factor, then every two-sided invariant subspace which is not left-reducing is left-pure, left-full, right-pure and right-full.*

*Proof.* If $M$ is a finite factor, then we are done, by [16, Theorem 4.1]. So we suppose that $M$ is a properly infinite factor. Let $\mathfrak{M}$ be a two-sided invariant subspace which is not left-reducing and let $P_\infty$ be the projection of $\mathbf{L}^2$ onto $\bigcap_{n>0} L_u^n \mathfrak{M}$. Then $P_\infty \mathbf{L}^2$ is left-reducing and right-invariant. By Corollary 4.5, $P_\infty$ lies in $\mathfrak{Z}(\mathfrak{L})$. Thus, there exists a projection $p_\infty$ in $\mathfrak{Z}(M \times_\alpha \mathbf{Z})$ such that $L_{p_\infty} = P_\infty$. But also, $L_{p_\infty} \mathfrak{M} \subset L_{p_\infty} \mathbf{L}^2 \subset \mathfrak{M}$. Hence, by Theorem 4.4, $p_\infty \in M \times_\alpha \mathbf{Z}_+$. Since $\mathfrak{Z}(M \times_\alpha \mathbf{Z}) \cap$ $\cap (M \times_\alpha \mathbf{Z}_+)$ is isomorphic to $H^\infty$ of the unit disc by Lemma 4.3 (cf. [16, Theorem 2.3]) and $P_\infty \neq 1$, we conclude that $P_\infty = 0$; that is, that $\mathfrak{M}$ is left-pure. To show that $\mathfrak{M}$ is left-full, let $P_{-\infty}$ be the projection onto $\bigvee_{n \in \mathbf{Z}} L_u^n \mathfrak{M}$. Then, as before, $P_{-\infty}$ lies in $\mathfrak{Z}(\mathfrak{L})$, but this time $P_{-\infty}$ is not zero. Thus there exists a projection $p_{-\infty}$ in $\mathfrak{Z}(M \times_\alpha \mathbf{Z})$ such that $L_{p_{-\infty}} = P_{-\infty}$. Also, $L_{p_{-\infty}} \mathfrak{M} = \mathfrak{M}$ because $\mathfrak{M} \subset \bigvee_{n \in \mathbf{Z}} L_u^n \mathfrak{M} = L_{p_{-\infty}} \mathbf{L}^2$. Thus, by maximality once more, $p_{-\infty}$ lies in $M \times_\alpha \mathbf{Z}_+$, $P_{-\infty} = 1$, and $\mathfrak{M}$ is left-full. Since $\mathfrak{M}$ is not left-reducing, $\mathfrak{M}$ is not right-reducing by Corollary 4.5. The proof that $\mathfrak{M}$ is right-pure and right-full is similar, so the proof is complete.

Theorem 4.6 yields the following corollary.

COROLLARY 4.7. *If $M$ is a factor, then every two-sided invariant subspace $\mathfrak{M}$ which is not left-reducing is of the form $R_v\mathbf{H}^2 = L_w\mathbf{H}^2$ where $v$ is an isometry in $M\times_\alpha \mathbf{Z}$ and $w$ in a co-isometry in $M\times_\alpha \mathbf{Z}$. In particular, if $M$ is a type* III*-factor or a finite factor, every two-sided invariant subspace which is not left-reducing is of the form $R_v\mathbf{H}^2 = L_w\mathbf{H}^2$ where $v$ and $w$ are unitary operators in $M\times_\alpha \mathbf{Z}$.*

*Proof.* If $M$ is a finite factor, then we are done by [16, Theorem 4.1]. Suppose that $M$ is properly infinite. Since $\mathfrak{M}$ is left-pure, left-full, right-pure and right-full by Theorem 4.6, the first assertion is valid by Corollary 3.11. Let $q_0$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M} \ominus L_u\mathfrak{M}$. Note that the projection of $\mathbf{L}^2$ onto $\mathbf{H}^2 \ominus L_u\mathbf{H}^2$ is $E_0$. If $M$ is a type III-factor, then $L(M)'$ is a type III-factor. Since $q_0$ and $E_0$ are infinite projections of $L(M)'$, $q_0 \sim E_0$, by [4, Chapter III, § 5 Corollaire 5]. Thus, we can choose a unitary operator $v$ in $M\times_\alpha \mathbf{Z}$ such that $\mathfrak{M} = R_v\mathbf{H}^2$. This completes the proof.

REMARKS 4.8. We note in passing that if every two-sided invariant subspace which is not left-reducing is of the form $R_v\mathbf{H}^2 = L_w\mathbf{H}^2$ where $v$ and $w$ are unitary operators in $M\times_\alpha \mathbf{Z}$, then the proof that (3) implies (1) in [16, Theorem 4.1] applies here to show that $M$ is a factor. It is attractive to conjecture that if $M$ is $\mathrm{II}_\infty$ or $\mathrm{I}_\infty$-factor, then every two-sided invariant subspace which is not left-reducing is of the form $R_v\mathbf{H}^2 = L_w\mathbf{H}^2$ where $v$ and $w$ are unitary operators in $M\times_\alpha \mathbf{Z}$. The problem boils down to showing that if $M$ is a $\mathrm{I}_\infty$ or $\mathrm{II}_\infty$ factor and if $\mathfrak{M}$ is a two-sided invariant subspace which is not left-reducing, then the projection of $\mathbf{L}^2$ onto $\mathfrak{M} \ominus L_u\mathfrak{M}$, which lies in $R(M)'$, must be infinite in $R(M)'$. While this is the case in every example we know of, we are unable to show it in general.

## 5. A FACTORIZATION THEOREM

In this section we prove the factorization theorem discussed in the introduction. We begin with a somewhat technical generalization which is the natural analogue, in the present setting, of [16, Proposition 1.2] and [22, Proposition 1]. The factorization theorem is an immediate consequence.

THEOREM 5.1. *Let $k \in M\times_\alpha \mathbf{Z}$. If there exist elements $k_1$ and $k_2 \in \mathbf{L}^2$ such that $L_k k_1 = R_k k_2 = h_0^{1/2}$ and such that $k_1^*$ and $k_2^*$ have dense ranges, then there are unitary operators $u_1$ and $u_2$ in $M\times_\alpha \mathbf{Z}$ and operators $a_1$ and $a_2$ in $M\times_\alpha \mathbf{Z}_+$ such that $k = u_1a_1 = a_2u_2$.*

*Proof.* Let $z$ be the maximal, finite, central projection of $M$. Then $(M\times_\alpha \mathbf{Z}_+)^z$ is a finite maximal subdiagonal algebra in $(M\times_\alpha \mathbf{Z})z$, and so by [16, Proposition 1.2], the conclusion holds in $(M\times_\alpha \mathbf{Z})z$. Therefore, we may suppose that $M$ is properly infinite. We prove that $k = u_1a_1$; the other representation is verified in a similar fashion. Put $\mathfrak{M} = [L_k\mathbf{H}^2]_2$. It is clear that $\mathfrak{M}$ is right-invariant. We first prove that $\mathfrak{M}$

is right-pure, that is, $\bigcap_{n>0} R_u^n[L_k\mathbf{H}^2]_2 = \{0\}$. Let $x \in \bigcap_{n>0} R_u^n[L_k\mathbf{H}^2]_2$. Since $[L_k\mathbf{H}^2]_2 = [L_k(M \times_\alpha \mathbf{Z}_+)h_0^{1/2}]_2$, there exists a sequence $\{x_m\}_{m=0}^\infty$ in $M \times_\alpha \mathbf{Z}_+$, which depends upon $n$, such that $\|R_u^n L_k x_m h_0^{1/2} - x\|_2 \to 0$ $(m \to \infty)$. For every $y \in M \times_\alpha \mathbf{Z}_+$ and $j < 0$, we have

$$\tau(u^j y k_2 x) = \lim_{m\to\infty} \tau(u^j y k_2 R_u^n L_k(x_m h_0^{1/2})) =$$

$$= \lim_{m\to\infty} \tau(u^j y k_2 k x_m h_0^{1/2} u^n) = \lim_{m\to\infty} \tau(u^{j+n} y h_0^{1/2} x_m h_0^{1/2}) =$$

$$= \lim_{m\to\infty} \tau(x_m h_0^{1/2} u^{j+n} y h_0^{1/2}) = \lim_{m\to\infty} (u^{j+n} y h_0^{1/2}, \mathbf{J}(x_m h_0^{1/2})).$$

If $n + j > 0$, then $\tau(u^j y k_2 x) = 0$, by Proposition 3.5 (1), because $u^{j+n} y h_0^{1/2} \in \mathbf{H}_0^2$ and $\mathbf{J} x_m h_0^{1/2} \in \mathbf{JH}^2$. Since $\bigcup_{j<0} u^j(M \times_\alpha \mathbf{Z}_+)$ is $\sigma$-weakly dense in $M \times_\alpha \mathbf{Z}$, we have $k_2 x = 0$ and so $x^* k_2^* = 0$. Thus $x^*$ vanishes on the intersection of the domain of $x^*$ and the range of $k_2^*$, which is strongly dense in the sense of Segal. Consequently, $x^*$ vanishes by [25, Corollary 5.1], $x = 0$, and $\mathfrak{M}$ is right-pure.

On the other hand, since $[L_k\mathbf{L}^2]_2 = \bigvee_{n<0} R_u^n[L_k\mathbf{H}^2]_2$, $[L_k\mathbf{L}^2]_2$ is right-reducing and so there exists a projection $e$ in $M \times_\alpha \mathbf{Z}$ such that $[L_k\mathbf{L}^2]_2 = L_e\mathbf{L}^2$ by Proposition 3.4. Hence we have $L_k = L_e L_k$. Since $k_1 \in \mathbf{L}^2$, we have $h_0^{1/2} = k k_1 = L_k k_1 = L_e L_k k_1 = e h_0^{1/2}$ and so $e = 1$, because $h_0^{1/2}$ is a separating vector. Thus $\mathfrak{M}$ is right-full.

Let $q_0$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{M} \ominus R_u\mathfrak{M}$. Since the projection of $\mathbf{L}^2$ onto $\mathbf{H}^2 \ominus R_u\mathbf{H}^2$ is $E_0$, we have $q_0 \prec E_0$ in $R(M)'$ by Lemma 3.8. Thus there exists a partial isometry $v_0$ in $R(M)'$ such that $v_0^* v_0 \leqslant E_0$ and $v_0 v_0^* = q_0$. Put $V = \sum_{n=-\infty}^{\infty} R_u^n v_0 R_u^{*n}$. Then it is clear that $V \in \mathfrak{R}' = \mathfrak{L}$ and $\mathfrak{M} = V\mathbf{H}^2$. Hence there exists a partial isometry $v$ in $M \times_\alpha \mathbf{Z}$ such that $V = L_v$. We then have

$$L_v^* L_v = \sum_{n=-\infty}^{\infty} R_u^n v_0^* v_0 R_u^{*n} \leqslant \sum_{n=-\infty}^{\infty} R_u^n E_0 R_u^{*n} = 1,$$

$$L_v L_v^* = \sum_{n=-\infty}^{\infty} R_u^n v_0 v_0^* R_u^{*n} = \sum_{n=-\infty}^{\infty} R_u^n q_0 R_u^{*n} = 1,$$

because $\mathfrak{M}$ is right-full. Thus $L_v$ is a co-isometry and $L_v^* L_v \mathbf{H}^2 \subset \mathbf{H}^2$. This implies that $L_v^* L_v \in \mathfrak{L}_+$. Since $L_v^* L_v = L_{v^*v}$ is self-adjoint, we have $L_{v^*v} \in \mathfrak{L}_+ \cap \mathfrak{L}_+^* = L(M)$. Put $a = v^* k$ and $v^* v = r$. Then $[L_a\mathbf{H}^2]_2 = L_{v^*v}\mathbf{H}^2 = L_r\mathbf{H}^2 \subset \mathbf{H}^2$. By [1, Theorem 2.2.1], $a \in M \times_\alpha \mathbf{Z}_+$. Set $r_0 = v_0^* v_0$ and define an operator $T$ in $R(M)'$ by $T = r_0 L_a E_0$. Put $P_n = \sum_{m=n}^{\infty} R_u^m E_0 R_u^{*m} = \sum_{m=n}^{\infty} E_m$. Since $a \in M \times_\alpha \mathbf{Z}_+$,

$$r_0 L_a E_0 = r_0 L_a (P_0 - P_1) = r_0 L_a P_0 - r_0 L_a P_1 =$$

$$= r_0 P_0 L_a P_0 - r_0 P_1 L_a P_1 = r_0 L_a P_0.$$

Therefore, we have

$$[T\mathbf{L}^2]_2 = [r_0 L_a E_0 \mathbf{L}^2]_2 = [r_0 L_a P_0 \mathbf{L}^2]_2 =$$

$$= [r_0 L_a \mathbf{H}^2]_2 = [r_0 L_r \mathbf{H}^2]_2 = r_0 \mathbf{L}^2.$$

On the other hand, it is clear that $T = r_0 L_a E_0 = E_0 L_a E_0 = L_{\varepsilon_0(a)} E_0$.

Put $b = k_2 v$. Then $ba = k_2 v v^* k = k_2 k = h_0^{1/2}$ and $b \in \mathbf{L}^2$. Let $\eta$ be the projection of $kh_0^{1/2}$ onto $R_u\mathfrak{M}$ and set $\zeta = kh_0^{1/2} - \eta$. Then it is clear that $\zeta \in \mathfrak{M} \ominus R_u\mathfrak{M}$ and $v^*\zeta = v_0^*\zeta \in r_0\mathbf{L}^2 = L_r(\mathbf{H}^2 \ominus R_u\mathbf{H}^2) = L_r[Mh_0^{1/2}]_2 \subset [Mh_0^{1/2}]_2 = [h_0^{1/2}M]_2$. Since $\eta \in R_u\mathfrak{M} = [kh_0^{1/2}(M \times_\alpha \mathbf{Z}_+)u]_2$, there exists a sequence $\{b_n\}_{n=1}^{\infty}$ in $(M \times_\alpha \mathbf{Z}_+)u$ such that $\lim_{n\to\infty} \|\eta - kh_0^{1/2}b_n\|_2 = 0$. We then have

$$v^*\zeta = v^*(kh_0^{1/2} - \eta) = \lim_{n\to\infty}(ah_0^{1/2} - ah_0^{1/2}b_n).$$

Since $v^*\zeta \in [Mh_0^{1/2}]_2$ and $ah_0^{1/2}b_n \in \mathbf{H}_0^2$, we have

$$v^*\zeta = E_0(v^*\zeta) = \lim_{n\to\infty} E_0(ah_0^{1/2} - ah_0^{1/2}b_n) =$$

$$= E_0(ah_0^{1/2}) = \varepsilon_0(a)h_0^{1/2}.$$

Therefore, we have, for every $d \in M$,

$$\tau(E_0(b)\varepsilon_0(a)h_0^{1/2}d) = \tau(b\varepsilon_0(a)h_0^{1/2}d) = \tau(k_2 v v^*\zeta d) =$$

$$= \tau(k_2\zeta d) = \lim_{n\to\infty}\tau(k_2(kh_0^{1/2} - kh_0^{1/2}b_n)d) =$$

$$= \lim_{n\to\infty}\tau(h_0 d - h_0 b_n d) = \tau(h_0 d) = \tau(h_0^{1/2}h_0^{1/2}d),$$

because $h_0^{1/2}b_n d \in \mathbf{H}_0^2$. Thus $\tau((E_0(b)\varepsilon_0(a) - h_0^{1/2})x) = 0$ for $x \in E_0\mathbf{L}^2 = [Mh_0^{1/2}]_2$. Since $E_0(b)\varepsilon_0(a) - h_0^{1/2} \in E_0\mathbf{L}^2$, we have $E_0(b)\varepsilon_0(a) = h_0^{1/2}$.

Suppose that $r_0 L_a E_0 x = \varepsilon_0(a)E_0(x) = 0$. Then $h_0^{1/2}(E_0 x) = E_0(b)\varepsilon_0(a)E_0 x = 0$. By [11, Lemma 2.1], we conclude that $E_0 x = 0$. Hence $\operatorname{Ker} T = (1 - E_0)\mathbf{L}^2$ and so $[T^*\mathbf{L}^2]_2 = (\ker T)^\perp = E_0\mathbf{L}^2$. Since $[T\mathbf{L}^2]_2 = r_0\mathbf{L}^2$, $r_0 \sim E_0$ in $R(M)'$. Thus there exists a partial isometry $w_0$ in $R(M)'$ such that $w_0^* w_0 = E_0$ and $w_0 w_0^* = r_0$. Put $L_w = \sum_{n=-\infty}^{\infty} R_u^n w_0 R_u^{*n}$. It is clear that $L_w$ is an isometry of $\mathfrak{L}$ such that $L_w L_w^* = L_r$ and $L_w^* L_r \mathbf{H}^2 = \mathbf{H}^2$. Thus $L_{w^*} L_{v^*}$ is a unitary operator in $\mathfrak{L}$. Set $L_c = L_w^* L_a =$ $= L_{w^* v^*} L_k$. Since $[L_c\mathbf{H}^2]_2 = L_w^*[L_a\mathbf{H}^2]_2 = L_{w^*}L_r\mathbf{H}^2 = \mathbf{H}^2$, we have $L_c \in \mathfrak{L}_+$. Put $u_1 = vw$. Then $k = u_1 c$. This completes the proof.

COROLLARY 5.2. *Let $k$ be an invertible element of $M \times_\alpha \mathbf{Z}$. Then there are unitary operators $u_1$ and $u_2$ in $M \times_\alpha \mathbf{Z}$ and invertible operators $a_1$ and $a_2$ in $M \times_\alpha \mathbf{Z}_+$ such that $k = u_1 a_1 = a_2 u_2$ and $a_1^{-1}, a_2^{-1} \in M \times_\alpha \mathbf{Z}_+$.*

*Proof.* Put $k_1 = k^{-1}h_0^{1/2}$ and $k_2 = h_0^{1/2}k^{-1}$. Then $L_k k_1 = R_k k_2 = h_0^{1/2}$. It is clear that $k_1^*$ and $k_2^*$ have dense ranges. By Theorem 5.1, there is a unitary $u_1$ in $M \times_\alpha \mathbf{Z}$ and an operator $a_1$ in $M \times_\alpha \mathbf{Z}_+$ such that $k = u_1 a_1$. It suffices to prove that $a_1^{-1} \in M \times_\alpha \mathbf{Z}_+$. From the proof of Theorem 5.1, we see that $\mathbf{H}^2 = [L_{a_1}\mathbf{H}^2]_2 = L_{a_1}\mathbf{H}^2$, because $L_{a_1}$ is invertible. Therefore $L_{a_1^{-1}}\mathbf{H}^2 = \mathbf{H}^2$ and so $a_1^{-1} \in M \times_\alpha \mathbf{Z}_+$. The factoring $k = a_2 u_2$ is proved similarly, and the proof is complete.

COROLLARY 5.3. *Every invertible positive operator in $M \times_\alpha \mathbf{Z}$ can be factored in the form $a^*a$, where $a$ belongs to $(M \times_\alpha \mathbf{Z}_+) \cap (M \times_\alpha \mathbf{Z}_+)^{-1}$.*

*Proof.* Let $k$ be an invertible positive operator in $M \times_\alpha \mathbf{Z}_+$. By Corollary 4.2, there exist an operator $a \in M \times_\alpha \mathbf{Z}_+$ and a unitary operator $u_1$ in $M \times_\alpha \mathbf{Z}$ such that $k^{1/2} = u_1 a$ and $a^{-1} \in M \times_\alpha \mathbf{Z}_+$. Thus $k = (k^{1/2})^2 = a^*a$. This completes the proof.

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