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ON POINT INTERACTIONS IN ONE DIMENSION

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1. INTRODUCTION

The problem of studying the spectrum of a second order differential operator of the Schrödinger type $-\Delta + V$ with V a potential describing "point interactions", in the sense that V is supported by a discrete set, arises in models for nuclear physics, many-body theory, solid state physics as well as in acoustics and optics, see e.g. [1, 12, 14, 15, 16, 17, 22, 34, 36–38, 45, 57, 60]. The advantage of such interactions is that explicit computations are possible. Moreover it is possible to develop local short range interactions "around point interactions", a procedure which has been mathematically justified recently in the three-dimensional case by scaling techniques, see [2–5], [19], [20, 21]. For further mathematical work on the definition and approximation of point interactions see also [6–8, 10, 11, 13, 14, 17, 18, 26, 27, 28, 35, 49, 50, 59, 61, 62]. The approximation of the Schrödinger operators describing point interactions by scaled local potentials is closely connected with the study of the low energy limit of such Hamiltonians and in fact expansions around the zero energy limit have been obtained for physical quantities like energy eigenvalues and resonances, as well as scattering amplitudes [3–5], [19], [20, 21].

The purpose of this paper is to extend and continue the above results in the one dimensional case. In some cases we get stronger results and in all cases the extension is not immediate, since the scaling behaviour is different in one and three dimensions.

Section 2 contains in particular the proof of the norm resolvent convergence as $\varepsilon \downarrow 0$ of Hamiltonians $H_{\varepsilon, N}$ with scaled short range interactions V_j:

$$H_{\varepsilon, N} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \varepsilon^{-2} \sum_{j=1}^{N} \lambda_j(\varepsilon)\, V_j\left(\frac{1}{\varepsilon}(x - x_j)\right), \quad \varepsilon > 0,$$

in $L^2(\mathbf{R}, \mathrm{d}x)$ to point interactions $\alpha_j \delta_{x_j}$ with centers at $x_1, \ldots, x_N$ and strengths $\alpha_j \equiv \lambda_j'(0) \int_{\mathbf{R}} \mathrm{d}x V_j(x)$.

The detailed behaviour of eigenvalues and resonances in the limit $\varepsilon \downarrow 0$ is discussed in Section 3. The case of infinitely many centers, and in particular of centers equally spaced to form a one dimensional infinite crystal, is treated in Section 4. Also in this case we show the norm convergence of the resolvent of

$$H_\varepsilon = -\frac{d^2}{dx^2} + \varepsilon^{-2} \sum_{j\in \mathbf{Z}} \lambda_j(\varepsilon)\, V_j\left(\frac{1}{\varepsilon}(x - x_j)\right), \quad \varepsilon > 0$$

to the one of an Hamiltonian describing the interaction with point interactions $\alpha_j \delta_{x_j}(x)$, with $\alpha_j \equiv \lambda_j'(0) \int dx\, V_j(x),\, j \in \mathbf{Z}$.

We confine to the Appendix results on operators defined as sums of quadratic forms and on their eigenvalues and resonances which are valid in arbitrary dimension and as such, are not only useful in connection to our present paper. In particular we study operators in Hilbert space of the form $H_0 + \sum_{j=1}^{N} E_j^* F_j$, with $H_0 \geqslant 0$ and self-adjoint and E_j, F_j closed and infinitesimally bounded with respect to $H_0^{1/2}$.

In particular results on the invariance of the essential spectrum as well as a perturbation theory for the bound states and resonances are given.

2. NORM-RESOLVENT CONVERGENCE TO POINT INTERACTIONS IN THE CASE OF FINITELY MANY CENTERS

In this section we translate the three dimensional convergence results of scaled short range Hamiltonians to the point interaction Hamiltonian (cf. [2, 3, 5]) into one dimension. In the Hilbert space $L^2(\mathbf{R})$ let

$$(2.1) \qquad H_0 = -\frac{d^2}{dx^2}, \quad \mathscr{D}(H_0) = H^{2,2}(\mathbf{R}),$$

then we have:

LEMMA 2.1. *Let* $V \in L^1(\mathbf{R}) + L^\infty_\varepsilon(\mathbf{R})$. *Then* V *is form-compact with respect to* H_0 i.e. $|V|^{1/2}(H_0 + E)^{-1/2}$ *is compact for all* $E > 0$.

Proof. That V is infinitesimally form bounded with respect to H_0 parallels the proof of Theorem I.21 of [52]. If $V \in L^1(\mathbf{R})$, $|V|^{1/2}(H_0 + E)^{-1}|V|^{1/2}$ is a Hilbert-Schmidt operator and hence compact since

$$\| |V|^{1/2}(H_0 + E)^{-1}|V|^{1/2}\|_2^2 = (4E)^{-1} \int dx dy |V(x)| e^{-2\sqrt{E}\,|x-y|}\, |V(y)| < \infty, \quad E > 0.$$

The general case follows by a limiting argument. ▨

Next we introduce

$$G_k = (H_0 - k^2)^{-1}, \quad \operatorname{Im} k > 0 \tag{2.2}$$

and for $V_j \in L^1(\mathbf{R})$, $1 \leqslant j \leqslant N$ we write

$$v_j(x) = |V_j(x)|^{1/2}, \quad u_j(x) = |V_j(x)|^{1/2} \operatorname{sign} V_j(x), \quad V_j = v_j u_j \tag{2.3}$$

(for simplicity we assume V_j to be real-valued throughout the paper). Then we have:

LEMMA 2.2. a) *$u_j G_k v_l$ are Hilbert-Schmidt if* $\operatorname{Im} k \geqslant 0,\ k \neq 0$.

b) *$u_j G_k v_l$ are trace class if* $\operatorname{Im} k > 0$.

c) *If in addition* $\int dx(1 + |x|^{1+\delta})|V_j(x)| < \infty$ *for some* $\delta > 0,\ 1 \leqslant j \leqslant N$ *then* $u_j G_k v_l$ *are trace class for all* $\operatorname{Im} k \geqslant 0,\ k \neq 0$.

Proof. a) follows from

$$\frac{1}{4|k|^2} \int dx dy |V_j(x)| e^{-2 \operatorname{Im} k |x-y|} |V_l(y)| < \infty \quad \text{if } V_j,\ V_l \in L^1(\mathbf{R}),\ \operatorname{Im} k \geqslant 0.$$

b) follows from Problem 161 in [48].

c) is proved in [56], p. 72. ▨

Let $x_1, \ldots, x_N \in \mathbf{R}$, $x_j \neq x_l$ if $j \neq l$ and introduce (cf. the Appendix for notations)

$$\tilde{B}_\varepsilon(k) : (L^2(\mathbf{R}))^N \to (L^2(\mathbf{R}))^N, \quad (\tilde{B}_\varepsilon(k)(f_1; \ldots; f_N))_j = \sum_{l=1}^{N} \tilde{B}_{\varepsilon, jl}(k) f_l \tag{2.4}$$

$$\tilde{B}_{\varepsilon, jl}(k) = \lambda_j(\varepsilon) \tilde{u}_j G_k \tilde{v}_l, \quad \tilde{u}_j(x) = u_j\left(x - \frac{1}{\varepsilon} x_j\right), \quad \tilde{v}_l(y) = v_l\left(y - \frac{1}{\varepsilon} x_l\right), \tag{2.5}$$

where $\varepsilon > 0$ and λ_j are real-valued in a real neighborhood of zero and analytic near the origin with $\lambda_j(0) = 0$, $1 \leqslant j \leqslant N$.

Define the Hamiltonian $H_N(\varepsilon)$:

$$H_N(\varepsilon) = H_0 \dotplus \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j\left(\cdot - \frac{1}{\varepsilon} x_j\right), \quad V_j \in L^1(\mathbf{R}),\ 1 \leqslant j \leqslant N,\ \varepsilon > 0 \tag{2.6}$$

(using quadratic forms). Then, applying Lemma A.1 we have

$$(H_N(\varepsilon) - k^2)^{-1} = G_k - \sum_{j,l=1}^{N} (G_k \tilde{v}_j)(1 + \tilde{B}_\varepsilon(k))_{jl}^{-1} \lambda_l(\varepsilon)(\tilde{u}_l G_k), \quad \operatorname{Im} k > 0. \tag{2.7}$$

Next we introduce

$$H_{\varepsilon, N} = \varepsilon^{-2} U_\varepsilon H_N(\varepsilon) U_\varepsilon^{-1} = H_0 \dot{+} \varepsilon^{-2} \sum_{j=1}^{N} \lambda_j(\varepsilon)\, V_j\left(\frac{1}{\varepsilon}(\cdot - x_j)\right), \quad \varepsilon > 0 \tag{2.8}$$

where U_ε denotes the unitary scaling group in $L^2(\mathbf{R})$

$$(U_\varepsilon f)(x) = \varepsilon^{-1/2} f(x/\varepsilon), \quad \varepsilon > 0, f \in L^2(\mathbf{R}). \tag{2.9}$$

Noting

$$\varepsilon^2 U_\varepsilon G_k U_\varepsilon^{-1} = G_{k/\varepsilon} \tag{2.10}$$

we obtain

$$(H_{\varepsilon, N} - k^2)^{-1} = \varepsilon^2 U_\varepsilon (H_N(\varepsilon) - (\varepsilon k)^2)^{-1} U_\varepsilon^{-1} =$$

$$= G_k - \varepsilon^{-2} \sum_{j,l=1}^{N} G_k U_\varepsilon \tilde{v}_j (1 + \tilde{B}_\varepsilon(\varepsilon k))_{jl}^{-1} \lambda_l(\varepsilon) \tilde{u}_l U_\varepsilon^{-1} G_k, \quad \operatorname{Im} k > 0 \tag{2.11}$$

$$= G_k - \varepsilon^{-1} \sum_{j,l=1}^{N} A_{\varepsilon, j}(k) (1 + B_\varepsilon(k))_{jl}^{-1} \lambda_l(\varepsilon) C_{\varepsilon, l}(k), \quad \operatorname{Im} k > 0 \tag{2.12}$$

where $A_{\varepsilon,j}(k)$, $B_{\varepsilon,jl}(k)$, and $C_{\varepsilon,l}(k)$ are Hilbert-Schmidt operators with kernels

$$A_{\varepsilon,j}(k, x, y) = g_k(x - x_j - \varepsilon y) v_j(y), \quad g_k(z) = \frac{i}{2k} e^{ik|z|}, \tag{2.13}$$

$$B_{\varepsilon, jl}(k, x, y) = \varepsilon^{-1} \lambda_j(\varepsilon) u_j(x) g_k(\varepsilon(x - y) + x_j - x_l)\, v_l(y) \tag{2.14}$$

$$C_{\varepsilon, l}(k, x, y) = u_l(x)\, g_k(\varepsilon x + x_l - y). \tag{2.15}$$

Define the rank-one operators $A_j(k)$, $B_{jl}(k)$, and $C_l(k)$ through their kernels

$$A_j(k, x, y) = g_k(x - x_j) v_j(y), \tag{2.16}$$

$$B_{jl}(k, x, y) = \lambda_j'(0)\, g_k(x_j - x_l)\, u_j(x)\, v_l(y), \tag{2.17}$$

$$C_l(k, x, y) = u_l(x)\, g_k(x_l - y), \tag{2.18}$$

then we have:

LEMMA 2.3. *Let* $\varepsilon, \operatorname{Im} k > 0$. *Then* $A_{\varepsilon, j}(k), B_{\varepsilon, jl}(k), C_{\varepsilon, l}(k)$ *converge to* $A_j(k)$, $B_{jl}(k)$, $C_l(k)$ *in Hilbert-Schmidt norm as* $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \|A_{\varepsilon, j}(k) - A_j(k)\|_2 = 0, \quad \lim_{\varepsilon \to 0} \|B_{\varepsilon, jl}(k) - B_{jl}(k)\|_2 = 0,$$
$$\lim_{\varepsilon \to 0} \|C_{\varepsilon, l}(k) - C_l(k)\|_2 = 0, \quad 1 \leqslant j, l \leqslant N. \tag{2.19}$$

Proof. We follow [3,5]. Since $\operatorname{w-lim}_{\varepsilon\to 0} A_{\varepsilon,j}(k) = A_j(k)$, $\operatorname{w-lim}_{\varepsilon\to 0} B_{\varepsilon,jl}(k) = B_{jl}(k)$, $\operatorname{w-lim}_{\varepsilon\to 0} C_{\varepsilon,l}(k) = C_l(k)$ we only need to prove ([56], p. 41)

$$\lim_{\varepsilon\to 0}\|A_{\varepsilon,j}(k)\|_2 = \|A_j(k)\|_2, \quad \lim_{\varepsilon\to 0}\|B_{\varepsilon,jl}(k)\|_2 = \|B_{jl}(k)\|_2, \quad \lim_{\varepsilon\to 0}\|C_{\varepsilon,l}(k)\|_2 = \|C_l(k)\|_2$$

which is obviously true. ▨

Next we introduce $Q_{\{\alpha_j\},N}$, the form associated with the Hamiltonian describing N point interactions of strength $\alpha_j \in \mathbf{R}$ centered at $x_1, \ldots, x_N$:

$$(2.20)\quad Q_{\{\alpha_j\},N}(f,g) = (f',\, g') + \sum_{j=1}^{N} \alpha_j \overline{f}(x_j) g(x_j), \quad \mathscr{D}(Q_{\{\alpha_j\},N}) = H^{1,2}(\mathbf{R}).$$

The operator $H_{\{\alpha_j\},N}$ corresponding to $Q_{\{\alpha_j\},N}$ is given by

$$(2.21)\quad H_{\{\alpha_j\},N} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathscr{D}(H_{\{\alpha_j\},N}) =$$

$$= \{f \mid f \in H^{1,2}(\mathbf{R}) \cap H^{2,2}(\mathbf{R}\setminus\{x_1, \ldots, x_N\}); f'(x_{j+}) - f'(x_{j-}) = \alpha_j f(x_j), \quad 1 \leqslant j \leqslant N\},$$

and its resolvent reads [62]

$$(2.22)\quad (H_{\{\alpha_j\},N} - k^2)^{-1} = G_k - \sum_{j,l=1}^{N} (T(k)^{-1})_{jl}\, (\overline{g_k(\cdot - x_l)},\, \cdot) g_k(\cdot - x_j), \quad \operatorname{Im} k > 0$$

where

$$(2.23)\quad T_{jl}(k) = \frac{1}{\alpha_j}\delta_{jl} + g_k(x_j - x_l).$$

Regarding the spectrum of $H_{\{\alpha_j\},N}$ we have:

LEMMA 2.4. *$H_{\{\alpha_j\},N}$ has at most N eigenvalues which are all negative. The remaining part of the spectrum is purely absolutely continuous and covers the positive real line*

$$(2.24)\quad \begin{gathered} \sigma_{\mathrm{ess}}(H_{\{\alpha_j\},N}) = \sigma_{\mathrm{ac}}(H_{\{\alpha_j\},N}) = [0,\, \infty), \quad \sigma_{\mathrm{sc}}(H_{\{\alpha_j\},N}) = \emptyset, \\ \sigma_{\mathrm{p}}(H_{\{\alpha_j\},N}) \subset (-\infty, 0), \quad \dim \operatorname{Ran}(E_{H_{\{\alpha_j\},N}}(-\infty,\, 0)) \leqslant N. \end{gathered}$$

Proof. Weyl's theorem and (2.22) prove $\sigma_{\mathrm{ess}}(H_{\{\alpha_j\},N}) = \sigma_{\mathrm{ess}}(H_0) = [0, \infty)$. Moreover both $H_{\{\alpha_j\},N}$ and H_0 are self-adjoint extensions of $\dot{H}_N$.

$$\dot{H}_N = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathscr{D}(\dot{H}_N) = \{f \mid f \in H^{2,2}(\mathbf{R});\, f(x_j) = 0;\ 1 \leqslant j \leqslant N\}$$

where $\dot{H}_N$ has deficiency indices (N, N). Thus $\sigma_p(H_{\{\alpha_j\},N}) \cap (-\infty, 0)$ consists of at most N points since $\sigma(H_0) \cap (-\infty, 0) = \emptyset$ ([58], p. 232). The explicit formula (2.22) shows in addition that $\sigma_p(H_{\{\alpha_j\},N}) \cap (0, \infty) = \emptyset$. Since $kT(k)$ is entire if $k \in \mathbf{C}$, the analytic Fredholm theorem applied to $(1-(-kT(k)+1))$ shows that $(kT(k))^{-1}$ exists for $k \in \mathbf{C} \setminus D$ where D is a discrete set i.e. a set without finite limit points (for $k > 0$ large enough $(kT(k))^{-1}$ clearly exists). Since $D \cap (0, \infty)$ is finite and $\sigma_{sc}(H_{\{\alpha_j\},N}) \subset D$ we get $\sigma_{sc}(H_{\{\alpha_j\},N}) = \emptyset$. ▨

After these preliminaries we can state the main result of this section.

THEOREM 2.1. *Let* $H_{\varepsilon,N} = H_0 \dotplus \varepsilon^{-2} \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j\left(\frac{1}{\varepsilon}(\cdot - x_j)\right)$, $\varepsilon > 0$, *and* $H_{\{\alpha_j\},N}$ *be the Hamiltonian describing N point interactions of stregth α_j centered at x_j, $j = 1, \ldots, N$. Then, as $\varepsilon \to 0$, $H_{\varepsilon,N}$ converges to $H_{\{\alpha_j\},N}$ in norm resolvent sense*

$$\text{(2.25)} \quad \operatorname*{n-lim}_{\varepsilon \to 0} (H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1}, \quad k^2 \notin \sigma(H_{\varepsilon,N}) \cap \sigma(H_{\{\alpha_j\},N})$$

with

$$\text{(2.26)} \qquad \alpha_j = \lambda_j'(0) \int_{\mathbf{R}} \mathrm{d}x\, V_j(x).$$

Proof. From (2.12) and Lemma 2.3 we conclude

$$\text{(2.27)} \quad \operatorname*{n-lim}_{\varepsilon \to 0} (H_{\varepsilon,N} - k^2)^{-1} = G_k - \sum_{j,l=1}^{N} A_j(k)\,(1 + B(k))_{jl}^{-1} \lambda_l'(0) C_l(k).$$

Now

$$B_{jl}(k) = \lambda_j'(0)\, g_k(x_j - x_l)(v_l, \cdot) u_j$$

and thus

$$\text{(2.28)} \quad (1 + B(k))_{jl}^{-1} = 1\delta_{jl} - \lambda_j'(0) \sum_{m=1}^{N} g_k(x_j - x_m)(\hat{T}(k))_{ml}^{-1}(v_l, \cdot) u_j$$

where

$$\text{(2.29)} \qquad \hat{T}_{jl}(k) = \delta_{jl} + \lambda_j'(0)\,(v_j, u_j)\, g_k(x_j - x_l).$$

Inserting

$$(\hat{T}(k))_{jl}^{-1}(v_l, u_l)\, \lambda_l'(0) = (T(k))_{jl}^{-1}$$

and (2.28) into (2.27) we get

$$\text{(2.30)} \quad \operatorname*{n-lim}_{\varepsilon \to 0} (H_{\varepsilon,N} - k^2)^{-1} = G_k - \sum_{j,l=1}^{N} (T(k))_{jl}^{-1} (\overline{g_k(\cdot - x_l)}, \cdot)\, g_k(\cdot - x_j). \qquad ▨$$

REMARKS 2.1. a) If $\lambda'_{j_0}(0)\int_{\mathbf{R}} dx\, V_{j_0}(x) = 0$ for some j_0 then the point interaction centered at x_{j_0} has strength zero and thus disappears in $H_{\{\alpha_j\}, N}$. In particular if and only if $\lambda'_j(0)(v_j, u_j) = 0$ for all $1 \leqslant j \leqslant N$ then H_ε converges in norm resolvent sense to the free Hamiltonian H_0 as $\varepsilon \to 0$. This is in sharp contrast to the corresponding situation in three dimensions. There one carefully has to study the spectral point zero of the underlying Hamiltonian and to distinguish between cases where zero energy resonances and/or zero energy bound states occur [2, 3, 19, 21].

b) Of course $\lambda_j(\varepsilon)$ need not be real-valued. The proof is completely identical if complex point interactions are considered.

c) For a completely different proof (including the case of infinitely many centers) based on local partitioning ([39]) see Section 4.

3. CONVERGENCE OF EIGENVALUES AND RESONANCES IN THE CASE OF FINITELY MANY CENTERS

We first state some results regarding the continuous spectra and then discuss the negative point spectra and the resonances.

Lemmas 2.1 and A.2 applied to $H(\varepsilon)$ and H_ε immediately yield

$$(3.1) \qquad \sigma_{\text{ess}}(H_{\varepsilon, N}) = \sigma_{\text{ess}}(H_N(\varepsilon)) = [0, \infty) \quad \text{for all } \varepsilon > 0.$$

By Lemma 2.4 the same result holds in the limit $\varepsilon \to 0$ since $(H_{\{\alpha_j\}, N} + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1}$ is a finite rank operator

$$(3.2) \qquad \sigma_{\text{ess}}(H_{\{\alpha_j\}, N}) = [0, \infty).$$

Now we turn to a discussion of the discrete spectra.

Application of Lemma A3 to $H_{\varepsilon, N}$ shows that $H_{\varepsilon,N}$ has an eigenvalue $E_\varepsilon < 0$ if and only if $U_\varepsilon \tilde{B}_\varepsilon(\varepsilon k_\varepsilon) U_\varepsilon^{-1}$ has an eigenvalue -1 in $(L^2(\mathbf{R}))^N$ i.e.

$$(3.3) \qquad \sum_{l=1}^{N} U_\varepsilon \tilde{B}_{\varepsilon, jl}(\varepsilon k_\varepsilon) U_\varepsilon^{-1} \tilde{\varphi}_{\varepsilon, l} = -\tilde{\varphi}_{\varepsilon, j}, \quad E_\varepsilon = k_\varepsilon^2,\ \tilde{\varphi}_{\varepsilon, j} \in L^2(\mathbf{R}),\ 1 \leqslant j \leqslant N$$

and also the corresponding multiplicity remains preserved. A change of variables shows that (3.3) is equivalent to -1 is an eigenvalue of $B_\varepsilon(k_\varepsilon)$ i.e.

$$(3.4) \qquad \sum_{l=1}^{N} B_{\varepsilon, jl}(k_\varepsilon) \varphi_{\varepsilon, l} = -\varphi_{\varepsilon, j}, \quad \varphi_{\varepsilon, j} \in L^2(\mathbf{R}),\ 1 \leqslant j \leqslant N.$$

In order to get a feeling what could happen in the general case we first discuss the case $N = 1$ in more detail.

There we have

$$H_1(\varepsilon) = H_0 \dot{+} \lambda_1(\varepsilon)\, V_1\left(\cdot - \frac{1}{\varepsilon} x_1\right), \quad \varepsilon > 0. \tag{3.5}$$

Clearly $H_1(\varepsilon)$ is unitarily equivalent to $H(\varepsilon)$

$$H(\varepsilon) = H_0 \dot{+} \lambda_1(\varepsilon)\, V(\cdot). \tag{3.6}$$

From

$$H_{\varepsilon,1} = \varepsilon^{-2} U_\varepsilon H_1(\varepsilon) U_\varepsilon^{-1}$$

any eigenvalue E_ε of $H_{\varepsilon,1}$ is determined by

$$E_\varepsilon = \varepsilon^{-2} E(\varepsilon) \tag{3.7}$$

where $E(\varepsilon)$ is an eigenvalue of $H_1(\varepsilon)$ and thus of $H(\varepsilon)$. Applying the detailed analysis of Klaus [29] and Simon [53] (see also [9, 30, 33, 42, 54]) on bound states of one dimensional Schrödinger operators we have:

LEMMA 3.1. a) *Let* $\int dx (1 + |x|)\, |V_1(x)| < \infty$ *and* $\lambda_1'(0) \int dx\, V_1(x) < 0$. *Then, for* $\varepsilon > 0$ *small enough,* $\sigma_p(H_{\varepsilon,1}) \cap (-\infty, 0)$ *consists precisely of one simple eigenvalue* E_ε *obeying*

$$\begin{aligned} k_\varepsilon &= \mathrm{i}\sqrt{-E_\varepsilon} = \\ &= -\frac{\mathrm{i}}{2} \lambda_1'(0) \int dx V_1(x) - \frac{\mathrm{i}}{4} \lambda_1''(0)\varepsilon \int dx V_1(x) - \\ &\quad - \frac{\mathrm{i}}{4} \lambda_1'(0)^2 \varepsilon \int dx dy\, V_1(x) |x-y|\, V_1(y) + o(\varepsilon). \end{aligned} \tag{3.8}$$

b) *Let* $\int dx (1 + |x|) |V_1(x)| < \infty$ *and* $\lambda_1'(0) \int dx V_1(x) = 0$. *If* $\lambda_1'(0) \neq 0$ *or* $\lambda_1'(0) = 0$ *but* $\lambda_1''(0) \int dx V_1(x) < 0$ *then, for* $\varepsilon > 0$ *small enough,* $\sigma_p(H_{\varepsilon,1}) \cap (-\infty, 0)$ *consists precisely of one simple eigenvalue* E_ε *with*

$$\begin{aligned} k_\varepsilon &= \mathrm{i}\sqrt{-E_\varepsilon} = \\ &= -\frac{\mathrm{i}}{4} \lambda_1''(0)\varepsilon \int dx V_1(x) - \frac{\mathrm{i}}{4} \lambda_1'(0)^2 \varepsilon \int dx dy V_1(x) |x-y|\, V_1(y) + o(\varepsilon). \end{aligned} \tag{3.9}$$

c) *Let* $\int dx(1+|x|)\,|V_1(x)| < \infty$. *If* $\lambda_1'(0)\int dx\,V_1(x) > 0$ *or* $\lambda_1'(0) = 0$ *and* $\lambda_1''(0)\int dx\,V_1(x) > 0$ *then, for* $\varepsilon > 0$ *small enough,* $\sigma_p(H_{\varepsilon,1}) \cap (-\infty, 0) = \emptyset$.

d) *If in addition* $\int dx\,e^{a|x|}|V_1(x)| < \infty$ *for some* $a > 0$ *then* k_ε *in* (3.8) *and* (3.9) *is analytic in* ε *at* $\varepsilon = 0$.

e) *If only* $\int dx|V_1(x)| < \infty$ *and* $\lambda_1'(0)\int dx\,V_1(x) < 0$ *then, for* ε *small enough, there exists a simple ground state* E_ε^0 *of* $H_{\varepsilon,1}$ *obeying*

$$k_\varepsilon^0 = i\sqrt{-E_\varepsilon^0} = -\frac{i}{2}\lambda_1'(0)\int dx\,V_1(x) + o(1). \tag{3.10}$$

Proof. The results of [9, 29] show that under conditions a) the operator $H(\varepsilon)$ for ε small enough has a unique negative and simple bound state $E(\varepsilon)$ with

$$k(\varepsilon) = i\sqrt{-E(\varepsilon)} =$$

$$= -\frac{i}{2}\varepsilon\lambda_1'(0)\int dx\,V_1(x) - \frac{i}{4}\varepsilon^2\lambda_1''(0)\int dx\,V_1(x) -$$

$$-\frac{i}{4}\varepsilon^2\lambda_1'(0)^2\int dx\,dy\,V_1(x)\,|x-y|\,V_1(y) + o(\varepsilon^2).$$

(3.8) now simply follows by (3.7) since $k_\varepsilon = \varepsilon^{-1}k(\varepsilon)$. b) — e) are proved similarly [9,29]. ▨

Now we turn to $N \geqslant 2$ and state

THEOREM 3.1. *Suppose that* V_j, $1 \leqslant j \leqslant N$ *have compact support.*

a) *If* $\text{n-lim}_{\varepsilon\to 0}(H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1}$ *has* $1 \leqslant M \leqslant N$ *simple eigenvalues* $E_m = k_m^2 < 0$, $1 \leqslant m \leqslant M$ *then, for* $\varepsilon > 0$ *small enough,* $\sigma(H_{\varepsilon,N}) \cap (-\infty, 0)$ *consists precisely of* M *negative and simple eigenvalues* $E_{\varepsilon,m}$ *which are analytic in* ε *at* $\varepsilon = 0$

$$k_{\varepsilon,m} = i\sqrt{-E_{\varepsilon,m}} = k_m + O(\varepsilon), \quad 1 \leqslant m \leqslant M. \tag{3.11}$$

If $E_m = k_m^2$, $1 \leqslant m \leqslant M$ *are simple but* $E_{M+l} = k_{M+l}^2$, $1 \leqslant l \leqslant L$ *are degenerate eigenvalues of* $H_{\{\alpha_j\},N}$ *with multiplicities* $n_l \geqslant 2$ *respectively then, for* ε *small enough,* $(\sigma H_{\varepsilon,N}) \cap (-\infty, 0)$ *contains at most* $M + \sum_{l=1}^{L} n_l \leqslant N$ *distinct negative eigenvalues* *which are analytic (if $1 \leqslant m \leqslant M$) resp. have a Puiseux expansion ($m \geqslant M+1$) at $\varepsilon = 0$.*

b) *If* $\text{n-}\lim_{\varepsilon \to 0}(H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1} \neq G_k$ *and* $H_{\{\alpha_j\},N}$ *has no eigenvalues then, for $\varepsilon > 0$ small enough, $H_{\varepsilon,N}$ has no negative eigenvalues.*

c) *If* $\text{n-}\lim_{\varepsilon \to 0}(H_{\varepsilon,N} - k^2)^{-1} = G_k$, *or equivalently if* $\lambda_j'(0)\int dx V_j(x) = 0, 1 \leqslant j \leqslant N$, *then as $\varepsilon \to 0$ all negative eigenvalues of $H_{\varepsilon,N}$ tend to zero* i.e. *are absorbed into the continuous spectrum as $\varepsilon \to 0$.*

Proof. By (3.4) any negative eigenvalue $E_\varepsilon = k_\varepsilon^2$ of $H_{\varepsilon,N}$ is determined through solutions of

$$\sum_{l=1}^{N} B_{\varepsilon,jl}(k_\varepsilon)\varphi_{\varepsilon,l} = -\varphi_{\varepsilon,j}, \quad 1 \leqslant j \leqslant N.$$

Introducing an operator $A_\varepsilon(k)$ in $(L^2(\mathbf{R}))^N$ by

$$A_{\varepsilon,jl}(k) = \varepsilon^{-1}\lambda_j(\varepsilon)\frac{i}{2k}e^{ik|x_j - x_l|}(v_l, \cdot)u_j \tag{3.12}$$

we infer that

$$\|B_\varepsilon(k) - A_\varepsilon(k)\| = O(\varepsilon) \tag{3.13}$$

uniformly in k if k varies in compact subsets of $\text{Im}\,k > -a$.

From (3.13) and

$$\det(1 + B_\varepsilon(k)) = \det(1 + B_\varepsilon(k) - A_\varepsilon(k))\det(1 + (1 + B_\varepsilon(k) - A_\varepsilon(k))^{-1}A_\varepsilon(k)) \tag{3.14}$$

one concludes that $k^2 < 0$ is an eigenvalue of $H_{\varepsilon,N}$ if and only if

$$\det(1 + (1 + B_\varepsilon(k) - A_\varepsilon(k))^{-1}A_\varepsilon(k)) = 0. \tag{3.15}$$

Since $A_\varepsilon(k)$ has finite rank and is analytic in ε and k for $|\varepsilon|$ small and $\text{Im}\,k > -a$, $k \neq 0$, $\det(1 + (1 + B_\varepsilon(k) - A_\varepsilon(k))^{-1}A_\varepsilon(k))$ is analytic with respect to ε and k in the same domain.

Moreover

$$\det(1 + A_0(k_m)) = 0, \quad \left(\frac{\partial}{\partial k}\det(1 + A_0(k))\right)\Big|_{k=k_m} \neq 0, \quad 1 \leqslant m \leqslant M \tag{3.16}$$

proves by the implicit function theorem that in a neighborhood of $(0, k_m)$, equation (3.15) has precisely $1 \leqslant m \leqslant M$ simple zeros $k_{\varepsilon,m}$ which are analytic in ε

$$k_{\varepsilon,m} = k_m + O(\varepsilon), \quad 1 \leqslant m \leqslant M.$$

Using Lemma A.4 a), $E_{\varepsilon,m} = k_{\varepsilon,m}^2$ are simple eigenvalues of $H_{\varepsilon,N}$.

If $E_{M+l} = k_{M+l}^2$, $l \geqslant 1$ is an eigenvalue of $H_{\{\alpha_j\},N}$ with multiplicity $n_l \geqslant 2$ then

$$\det(1 + A_0(k_{M+l})) = 0, \quad \left(\frac{\partial^r}{\partial k^r} \det(1 + A_0(k))\right)\Big|_{k=k_{M+l}} = 0, \quad 1 \leqslant r \leqslant n_l - 1,$$

(3.17)

$$\left(\frac{\partial^{n_l}}{\partial k^{n_l}} \det(1 + A_0(k))\right)\Big|_{k=k_{M+l}} \neq 0$$

shows the existence of positive integers $\xi_1, \ldots, \xi_s$ with $\sum_{i=1}^{s} \xi_i = n_l$ and the existence of multivalued functions $k_{1,l}(\varepsilon), \ldots, k_{s,l}(\varepsilon)$ (not necessarily distinct) with convergent Puiseux series

$$k_{i,l}(\varepsilon) = k_{M+l} + \sum_{j=1}^{\infty} \alpha_{j,l}^{(i)} \varepsilon^{j/\xi_i}, \quad 1 \leqslant i \leqslant s.$$

In fact using results of [44] one can show that $1 \leqslant \xi_i \leqslant s$, $1 \leqslant i \leqslant s$. Since any solution k_ε of (3.15) obeys:

$$k_\varepsilon = k_n + o(1), \quad 1 \leqslant n \leqslant M + \sum_{l=1}^{L} n_l$$

as $\varepsilon \to 0_+$, part a) is proved.

Parts b) and c) follow from a). ▨

REMARKS 3.1. a) The observation that in case c) of Theorem 3.1 the operator $A_\varepsilon(k)$ is nilpotent might be the starting point for a more detailed analysis than we have done.

b) In sharp contrast to the three dimensional case there are no eigenvalues of $H_{\varepsilon,N}$ running to infinity as $\varepsilon \to 0$.

Concerning resonances (cf. the Appendix for their definition) we state:

THEOREM 3.2. *Assume* V_j, $1 \leqslant j \leqslant N$ *have compact support. Let* $\underset{\varepsilon \to 0}{\text{n-lim}}(H_{\varepsilon,N} - k^2)^{-1} = (H_{\{\alpha_j\},N} - k^2)^{-1} \neq G_k$ *and suppose that* k_0, $\operatorname{Im} k_0 < 0$ *is a resonance of* $H_{\{\alpha_j\},N}$. *Then, for* $\varepsilon > 0$ *small enough, there exists a multivalued function* $k(\varepsilon)$, $\operatorname{Im} k(\varepsilon) < 0$ *such that* $k(\varepsilon)$ *has a convergent Puiseux expansion in* ε *at* $\varepsilon = 0$ *with* $k(0) = k_0$ *and* $k(\varepsilon)$ *is a resonance of* $H_{\varepsilon,N}$.

Proof. Follows by the proof of part a) of Theorem 3.1. ▨

Finally we state a partial converse to Theorems 3.1 and 3.2.

THEOREM 3.3. *Let* V_j, $1 \leqslant j \leqslant N$ *have compact support.*

a) *If* $\varkappa(\varepsilon)^2 < 0$ *are eigenvalues of* $H_{\varepsilon,N}$ *for* $\varepsilon > 0$ *small enough and* $k_0^2 < 0$ *is a (finite) accumulation point of* $\varkappa(\varepsilon_n)$ *for some* $\varepsilon_n \to 0_+$, *then* k_0^2 *is a discrete eigen-*

value of $H_{\{\alpha_j\},N}$. *Moreover there are eigenvalues* $k(\varepsilon)^2 < 0$ *of* $H_{\varepsilon,N}$ *such that* $k(\varepsilon)$ *have a Puiseux expansion at* $\varepsilon = 0$ *with* $k(0) = k_0$.

b) *If* $\varkappa(\varepsilon)$, $\operatorname{Im}\varkappa(\varepsilon) < 0$ *are resonances of* $H_{\varepsilon,N}$ *for* $\varepsilon > 0$ *small enough and* k_0, $\operatorname{Im} k_0 < 0$ *is a (finite) accumulation point of* $\varkappa(\varepsilon_n)$ *for some* $\varepsilon_n \to 0_+$, *then* k_0 *is a resonance of* $H_{\{\alpha_j\},N}$. *In addition there is a multivalued function* $k(\varepsilon)$, $\operatorname{Im} k(\varepsilon) < 0$ *such that* $k(\varepsilon)$ *is a resonance of* $H_{\varepsilon,N}$ *and* $k(\varepsilon)$ *has a Puiseux expansion at* $\varepsilon = 0$ *with* $k(0) = k_0$.

Proof. Since $\det(1 + B_\varepsilon(k))$ is analytic for $|\varepsilon|$ small enough and $k \neq 0$ we get $\det(1 + B(k_0)) = 0$ in both cases. Thus k_0^2 is a discrete eigenvalue of $H_{\{\alpha_j\},N}$ in case a), whereas k_0 is a resonance of $H_{\{\alpha_j\},N}$ in case b) (cf. the definition of resonances for point interactions in [4]). Since $\det(1 + B(k))$ is not identically zero near $k = k_0$ we get assertions a) and b) from the proof of Theorem 3.1 a). ▨

REMARK 3.2. Using the Taylor expansion of $B_\varepsilon(k)$ in ε and k around $\varepsilon = 0$ and $k = k_0$ one can get explicitly the coefficients in the above Puiseux expansions, similarly as in [19]. ▨

4. NORM RESOLVENT CONVERGENCE TO POINT INTERACTIONS IN THE CASE OF ARBITRARILY MANY CENTERS

We finally show how the results of Section 2 generalize to arbitrarily many centers.

Let $\{x_j\}_{j\in\mathbf{Z}}$ be a discrete subset of the real line with $|x_j - x_l| \geqslant a > 0$ and $x_j < x_{j+1}$, $j, l \in \mathbf{Z}$. Suppose that V_j, $j \in \mathbf{Z}$ are real measurable functions on $\mathbf{R}$ and assume the existence of some function $W \in L^1(\mathbf{R})$ such that for all $j \in \mathbf{Z}$, $|V_j| \leqslant W$ a.e. . Define

$$(4.1)\quad q_{j,\varepsilon}(f, g) = \lambda_j(\varepsilon) \int dx\, \varepsilon^{-2} V_j((x - x_j)/\varepsilon)\, \bar{f}(x)\, g(x), \quad \mathscr{D}(q_{j,\varepsilon}) = H^{1,2}(\mathbf{R}), \varepsilon > 0,$$

and

$$(4.2)\qquad q_{\alpha_j}(f, g) = \alpha_j \bar{f}(x_j)\, g(x_j), \quad \mathscr{D}(q_{\alpha_j}) = H^{1,2}(\mathbf{R})$$

with α_j real-valued

$$(4.3)\qquad |\alpha_j| \leqslant C_0 < \infty, \quad j \in \mathbf{Z}$$

and $\lambda_j(\varepsilon)$ real-valued for ε real and analytic near $\varepsilon = 0$, $\lambda_j(0) = 0$, $j \in \mathbf{Z}$.

In order to define Hamiltonians whose interactions are formally given by $\sum_{j\in\mathbf{Z}} q_{j,\varepsilon}$ resp. $\sum_{j\in\mathbf{Z}} q_{\alpha_j}$ we follow the lines of [26] and use a variant of a powerful technique developed by Morgan [39].

A quadratic form q is called summably form-bounded with respect to H_0 if it is form-bounded with respect to H_0 and there are constants $a_j \geqslant 0$, $b_j \geqslant 0$ such that

$$|q(\varphi, \varphi)| \leqslant a_j\|\varphi'\|^2 + b_j\|\varphi\|^2 \tag{4.4}$$

for all $\varphi \in H^{1,2}(\mathbf{R})$ with $\operatorname{supp}\varphi \subset [j - 3/2, j + 3/2)$ with

$$\sum_{j\in\mathbf{Z}} a_j < 1 \quad \text{and} \quad \sum_{j\in\mathbf{Z}} b_j < \infty . \tag{4.5}$$

Let

$$f_{x_j}(x) = f(x + x_j)$$

and define

$$q_{x_j}(f, g) = q(f_{x_j}, g_{x_j})$$

then we have ([26]):

LEMMA 4.1. *If q is summably form-bounded with respect to H_0 (with constants a_j, b_j, $j \in \mathbf{Z}$) and satisfies*

$$q(\chi f, g) = q(f, \chi g) \tag{4.6}$$

for all $f, g \in H^{1,2}(\mathbf{R})$ and all $\chi \in C_0^\infty(\mathbf{R})$ then

$$q_{\{\alpha_j\}}(f, g) = \sum_{j\in\mathbf{Z}} q_{x_j}(f, g) \tag{4.7}$$

is a well defined quadratic form on $H^{1,2}(\mathbf{R})$ satisfying

$$|q_{\{\alpha_j\}}(f,f)| \leqslant C(\sum_{j\in\mathbf{Z}} a_j)\, \|f'\|^2 + [D \sum_{j\in\mathbf{Z}} a_j + \sum_{j\in\mathbf{Z}} b_j]\, \|f\|^2, \quad f \in H^{1,2}(\mathbf{R}) \tag{4.8}$$

where C and D only depend on the set $\{x_i\}$.

Proof. Follows from the proof of Satz 4 in [26]. ▨

Lemma 4.1 immediately implies that

$$Q_\varepsilon(f, g) = (f', g') + \sum_{j\in\mathbf{Z}} q_{j,\varepsilon}(f, g), \quad \mathscr{D}(Q_\varepsilon) = H^{1,2}(\mathbf{R}), \quad \varepsilon > 0 \tag{4.9}$$

and

$$Q_{\{\alpha_j\}}(f, g) = (f', g') + \sum_{j\in\mathbf{Z}} q_{\alpha_j}(f, g), \quad \mathscr{D}(Q_{\{\alpha_j\}}) = H^{1,2}(\mathbf{R}) \tag{4.10}$$

are closed forms bounded from below. The corresponding self-adjoint Hamiltonians are denoted by H_ε resp. $H_{\{\alpha_j\}}$, and by definition $H_{\{\alpha_j\}}$ describes point interactions of strength $\alpha_j \int dx V_j(x)$ centered at x_j, $j \in \mathbf{Z}$.

The main result of this section then reads:

THEOREM 4.1. *Let* $H_\varepsilon = H_0 \dot{+} \varepsilon^{-2} \sum_{j\in\mathbf{Z}} \lambda_j(\varepsilon) V_j\left(\frac{1}{\varepsilon}(\cdot - x_j)\right)$, $\varepsilon > 0$, *and* $H_{\{\alpha_j\}}$ *be the Hamiltonians associated with* Q_ε *and* $Q_{\{\alpha_j\}}$. *Then, as* $\varepsilon \to 0$, H_ε *converges to* $H_{\{\alpha_j\}}$ *in norm resolvent sense*

$$\text{n-}\lim_{\varepsilon\to 0}(H_\varepsilon - k^2)^{-1} = (H_{\{\alpha_j\}} - k^2)^{-1}, \quad k^2 \notin \sigma(H_\varepsilon) \cap \sigma(H_{\{\alpha_j\}}) \tag{4.11}$$

with

$$\alpha_j = \lambda_j'(0)\int dx\, V_j(x). \tag{4.12}$$

Proof. To keep the proof as short as possible we assume without loss of generality that $\lambda_j(\varepsilon) = \alpha_j \varepsilon$, $j \in \mathbf{Z}$ and that $\int dx V_j(x) = 1$. Define

$$q_{\varepsilon,j}(f, g) = \alpha_j \int dx\, \varepsilon^{-1} V_j(x/\varepsilon)\, \bar{f}(x) g(x), \quad \mathscr{D}(q_\varepsilon) = H^{1,2}(\mathbf{R}), \quad \varepsilon > 0, \tag{4.13}$$

$$q_{\alpha_j}(f, g) = \alpha_j \bar{f}(0) g(0), \quad \mathscr{D}(q_\alpha) = H^{1,2}(\mathbf{R}). \tag{4.14}$$

Assume $f, g \in C_0^\infty(\mathbf{R})$, $0 \leqslant \eta(x) \leqslant 1$, $\eta \in C_0^\infty(\mathbf{R})$ and

$$\eta(x) = \begin{cases} 1 & |x| \leqslant \delta \\ 0 & |x| \geqslant 2\delta, \text{ for some } \delta > 0. \end{cases}$$

Then

$$|q_{\varepsilon,j}(f,g) - q_{\alpha_j}(f,g)| \leqslant |\alpha_j| \int_{-\delta}^{\delta} dx\, \varepsilon^{-1} |V_j(x/\varepsilon)|\, |\bar{f}(x) g(x) - \bar{f}(0) g(0)| +$$

$$+ 2|\alpha_j|\, \|f\|_\infty \|g\|_\infty \int_{(-\infty, -\delta] \cup [\delta, \infty)} dx\, \varepsilon^{-1} |V_j(x/\varepsilon)| \leqslant \tag{4.15}$$

$$\leqslant |\alpha_j| \sup_{|x|\leqslant\delta} |\bar{f}(x) g(x) - \bar{f}(0) g(0)| + 2|\alpha_j|\, \|f\|_\infty \|g\|_\infty \int_{(-\infty, -\delta/\varepsilon] \cup [\delta/\varepsilon, \infty)} dx\, W(x) \leqslant$$

$$\leqslant |\alpha_j| \int_{-2\delta}^{2\delta} dx\, \eta(x)[|f'(x)||g(x)| + |f(x)||g'(x)|] + 2|\alpha_j|\, \|f\|_\infty \|g\|_\infty \int_{(-\infty, -\delta/\varepsilon] \cup [\delta/\varepsilon, \infty)} dx\, W(x) \leqslant$$

$$\leqslant |\alpha_j| \left(\int_{-2\delta}^{2\delta} dx |\eta(x)|^2 \right)^{1/2} [\|f'\|\, \|g\|_\infty + \|f\|_\infty \|g'\|] + 2|\alpha_j|\, \|f\|_\infty \|g\|_\infty \int_{(-\infty, -\delta/\varepsilon] \cup [\delta/\varepsilon, \infty)} dx\, W(x) \leqslant$$

$$\leqslant C_0 c \left\{ \left(\int_{-2\delta}^{2\delta} dx |\eta(x)|^2 \right)^{1/2} + 2c \int_{(-\infty, -\delta/\varepsilon] \cup [\delta/\varepsilon, \infty)} dx\, W(x) \right\} \|f\|_{+1} \|g\|_{+1}$$

where

$$\|h\|_{+1}^2 = \|h'\|^2 + \|h\|^2$$

and

$$\|h\|_\infty \leqslant c\|h\|_{+1}, \quad h \in H^{1,2}(\mathbf{R}).$$

Since $C_0(\mathbf{R})$ is a form core of H_0, (4.15) extends to all $f, g \in H^{1,2}(\mathbf{R})$. Next assume φ_j, $\psi_j \in H^{1,2}(\mathbf{R})$ and $\operatorname{supp}\varphi_j \subset [j - 3/2, j + 3/2)$ and $\operatorname{supp}\psi_j \subset [j - 3/2, j + 3/2)$. Then for $j = 0, \pm 1$ the above method shows that for any $\nu > 0$ there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$|q_\varepsilon(\varphi_j, \psi_j) - q_\alpha(\varphi_j, \psi_j)| \leqslant \nu\|\varphi_j\|_{+1}\|\psi_j\|_{+1}, \quad j = 0, \pm 1. \tag{4.16}$$

If $j \in \mathbf{Z}\setminus\{0, \pm 1\}$ we get from $\varphi_j(0) = \psi_j(0) = 0$

$$\begin{aligned} |q_{\varepsilon,j}(\varphi_j, \psi_j) - q_{\alpha_j}(\varphi_j, \psi_j)| &= |q_{\varepsilon,j}(\varphi_j, \psi_j)| \leqslant \\ &\leqslant |\alpha_j|\,\|\varphi_j\|_\infty\|\psi_j\|_\infty \int_{j/\varepsilon - 3/2\varepsilon}^{j/\varepsilon + 3/2\varepsilon} \mathrm{d}x\, W(x) \leqslant \\ &\leqslant C_0 c^2 \int_{j/\varepsilon - 3/2\varepsilon}^{j/\varepsilon + 3/2\varepsilon} \mathrm{d}x\, W(x)\, \|\varphi_j\|_{+1}\|\psi_j\|_{+1}, \quad j \in \mathbf{Z}\setminus\{0, \pm 1\}. \end{aligned} \tag{4.17}$$

From (4.16) and

$$\sum_{j \in \mathbf{Z}\setminus\{0,\pm 1\}} \int_{j/\varepsilon - 3/2\varepsilon}^{j/\varepsilon + 3/2\varepsilon} \mathrm{d}x\, W(x) \leqslant 3 \int_{-\infty}^{-1/2\varepsilon} \mathrm{d}x\, W(x) + 3 \int_{1/2\varepsilon}^{\infty} \mathrm{d}x\, W(x) \tag{4.18}$$

we actually infer that for any $\nu > 0$ there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$|q_{\varepsilon,j}(\varphi, \psi) - q_{\alpha_j}(\varphi, \psi)| \leqslant a_j\|\varphi\|_{+1}\|\psi\|_{+1} \tag{4.19}$$

for all $\varphi, \psi \in H^{1,2}(\mathbf{R})$, $\operatorname{supp}\varphi \subset [j - 3/2, j + 3/2)$, $\operatorname{supp}\psi \subset [j - 3/2, j + 3/2)$ with

$$\sum_{j \in \mathbf{Z}} a_j < \nu. \tag{4.20}$$

Consequently Lemma 4.1 applies and gives for all $\nu > 0$

$$|Q_\varepsilon(f, f) - Q_{\{\alpha_j\}}(f, f)| \leqslant \nu\|f\|_{+1}^2, \quad f \in H^{1,2}(\mathbf{R})$$

which implies

$$|Q_\varepsilon(f,g) - Q_{\{\alpha_j\}}(f,g)| \leqslant \nu \|f\|_{+1}\|g\|_{+1}, \quad f, g \in H^{1,2}(\mathbf{R}). \tag{4.21}$$

But (4.21) implies norm resolvent convergence of H_ε to $H_{\{\alpha_j\}}$ by Theorem VIII. 25c) of [46]. ▨

REMARK 4.1. Again if $\lambda'_{j_0}(0)\int dx\, V_{j_0}(x) = 0$ for some j_0 the point interaction at x_{j_0} disappears in $H_{\{\alpha_j\}}$. In correspondence to the finite center case, H_ε converges in norm resolvent sense to H_0 if and only if $\lambda'_j(0)\int dx\, V_j(x) = 0$ for all $j \in \mathbf{Z}$.

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APPENDIX

We collect some results on Hamiltonians defined as quadratic forms which are not only useful for the present paper but are certainly of interest for quite general systems in quantum mechanics. The basic ideas of our proofs of Lemmas A1–A3 are taken from Simon's monograph [52].

Let $H_0 \geqslant 0$ be self-adjoint in some Hilbert space $\mathscr{H}$ and let E_j, F_l, $j, l = 1, \ldots, N$ be closed operators in $\mathscr{H}$ which are infinitesimally bounded with respect to $H_0^{1/2}$. Define

$$H = H_0 \dotplus \sum_{j=1}^{N} E_j^* F_j \tag{A.1}$$

by the method of forms [25, 52]. Let $\mathscr{H}^N = \bigoplus_{j=1}^{N} \mathscr{H}$ and introduce the family of bounded operators $K(k)$

$$K(k) : \mathscr{H}^N \to \mathscr{H}^N, \quad (K(k)(f_1, \ldots, f_N))_j = \sum_{l=1}^{N} K_{jl}(k) f_l \tag{A.2}$$

where [1)]

$$\text{(A.3)} \qquad K_{jl}(k) = F_j(H_0 - k^2)^{-1}E_l^*, \quad \operatorname{Im} k > 0.$$

We first state:

LEMMA A.1. *Under the above hypotheses*

$$\text{(A.4)} \; (H-k^2)^{-1}=(H_0-k^2)^{-1}-\sum_{j=1}^{N}(H_0-k^2)^{-1}E_j^*F_j(H-k^2)^{-1}, \quad \operatorname{Im} k>0,\ k^2 \notin \sigma(H).$$

If in addition [2)] $K_{jl}(k) \in \mathscr{B}_\infty(\mathscr{H})$, $j, l = 1, \ldots, N$ *for all* $\operatorname{Im} k > 0$ *then*

$$\text{(A.5)} \quad (H-k^2)^{-1}=(H_0-k^2)^{-1}-\sum_{j,l=1}^{N}(H_0-k^2)^{-1}E_j^*(1+K(k))_{jl}^{-1}F_l(H_0-k^2)^{-1}$$

holds for $\operatorname{Im} k > 0$, $k^2 \notin \sigma(H)$ *with the possible exception of a discrete set. In particular for* $\operatorname{Im} k > 0$ *large enough and* $k^2 \notin \sigma(H)$

$$\text{(A.6)} \; (H-k^2)^{-1}=(H_0-k^2)^{-1}-\sum_{m=0}^{\infty}\sum_{j,l=1}^{N}(-1)^m(H_0-k^2)^{-1}E_j^*(K(k)^m)_{jl}F_l(H_0-k^2)^{-1}$$

with convergence in the norm sense.

Proof. Let

$$Q_{H_0}(f, g) = (H_0^{1/2}f, H_0^{1/2}g), \quad \mathscr{D}(Q_{H_0}) = \mathscr{D}(H_0^{1/2}),$$

$$Q_H(f, g) = (H_0^{1/2}f, H_0^{1/2}g) + \sum_{j=1}^{N}(E_jf, F_jg), \quad \mathscr{D}(Q_H) = \mathscr{D}(H_0^{1/2}).$$

Then for $f, g \in \mathscr{D}(H_0^{1/2})$

$$Q_H((H_0-\bar{k}^2)^{-1}f,\ (H-k^2)^{-1}g) - Q_{H_0}((H_0-\bar{k}^2)^{-1}f,\ (H-k^2)^{-1}g) =$$

$$= \sum_{j=1}^{N}(E_j(H_0-\bar{k}^2)^{-1}f,\ F_j(H-k^2)^{-1}g) = ((H_0-\bar{k}^2)^{-1}f, g) - (f, (H-k^2)^{-1}g)$$

[1)] If no confusions arise we always use a simplified notation identifying operators of the type $(H_0 - k^2)^{-1}E_l^*$ and $(E_l(H_0 - \bar{k}^2)^{-1})^*$ etc.

[2)] We use the notation $\mathscr{B}(\mathscr{H})$, $\mathscr{B}_\infty(\mathscr{H})$ for bounded resp. compact operators on $\mathscr{H}$. We also use $\mathscr{B}_p(\mathscr{H})$, $p \in N$ (cf. e.g. [25])

by a simple computation. This proves (A.4). From the infinitesimal boundedness of $F_j(E_j)$ with respect to $H_0^{1/2}$ we obtain e.g.

$$\lim_{E \to +\infty} \|F_j(H_0 + E)^{-1/2}\| = 0$$

and thus $\|K_{jl}(\mathrm{i}\sqrt{E})\| < 1$ for $E > 0$ large enough. Therefore we can iterate (A.4) in order to obtain the norm convergent expansion (A.6). Since $K_{jl}(k)$, $j, l = 1, \ldots, N$ are compact and analytic in k^2 in the region $\operatorname{Im} k > 0$ the same is true for $K(k)$. Moreover since $(1 + K(\mathrm{i}\sqrt{E}))^{-1} \in \mathscr{B}(\mathscr{H}^N)$ for $E > 0$ large enough, $(1 + K(k))^{-1}$ exists and is analytic for $\operatorname{Im} k > 0$ except possibly for a set of discrete points, by the analytic Fredholm theorem [48].

Thus (A.5) holds by analytic continuation of both sides. ▨

Next we give

LEMMA A.2. *Assume in addition that* $F_j(H_0 + 1)^{-1/2} \in \mathscr{B}_\infty(\mathscr{H})$ *or* $E_l(H_0 + 1)^{-1/2} \in \mathscr{B}_\infty(\mathscr{H})$ *for all* $j, l = 1, \ldots, N$. *Then*

$$\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0). \tag{A.7}$$

Proof. By the hypothesis and (A.5) $(H + E)^{-1} - (H_0 + E)^{-1}$, $E > 0$ large enough, is compact and we only need to apply Weyl's theorem [48]. ▨

Regarding bound states of H we have

LEMMA A.3. *Assume* $K_{jl}(k) \in \mathscr{B}_\infty(\mathscr{H})$, $j, l = 1, \ldots, N$ *for all* $\operatorname{Im} k > 0$. *Let* $E_0 = k_0^2$, $\operatorname{Im} k_0 > 0$, *then* H *has the eigenvalue* E_0 *with geometric multiplicity* n_0 *if and only if* $K(k_0)$ *has the eigenvalue* -1 *with the same geometric multiplicity* n_0. *In particular if*

$$K(k_0)\varphi_0 = -\varphi_0, \quad \varphi_0 = (\varphi_{01}, \ldots, \varphi_{0N}), \ \varphi_{0j} \in \mathscr{H}, \ j = 1, \ldots, N$$

then

$$\psi_0 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \varphi_{0l} \tag{A.8}$$

fulfills

$$H\psi_0 = E_0\psi_0, \quad \psi_0 \in \mathscr{D}(H). \tag{A.9}$$

Conversely if

$$H\tilde{\psi}_0 = E_0\tilde{\psi}_0, \quad \tilde{\psi}_0 \in \mathscr{D}(H)$$

then

$$\tilde{\varphi}_0 = (\tilde{\varphi}_{01}, \ldots, \tilde{\varphi}_{0N}), \quad \tilde{\varphi}_{0j} = -F_j\tilde{\psi}_0, \quad j = 1, \ldots, N \tag{A.10}$$

fulfills

(A.11) $$K(k_0)\tilde{\varphi}_0 = -\tilde{\varphi}_0$$

and

(A.12) $$\tilde{\psi}_0 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \varphi_{0l}.$$

Proof. a) Assume $K(k_0)\varphi_0 = -\varphi_0$ and define

$$\psi_0 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \varphi_{0l}.$$

Then obviously

$$\psi_0 \in \mathscr{D}(H_0^{1/2}) \subset \mathscr{D}(F_j), \quad j = 1, \ldots, N,$$

$$-F_j\psi_0 = -\sum_{l=1}^{N} F_j(H_0 - k_0^2)^{-1} E_l^* \varphi_{0l} = -\varphi_{0j}, \quad 1 \leqslant j \leqslant N,$$

and

(A.13) $$\psi_0 = -\sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* F_l \psi_0.$$

Noting

(A.14) $$f + \sum_{l=1}^{N} (H_0 - k^2)^{-1} E_l^* F_l f = (H_0 - k^2)^{-1}(H + E_1)^{1/2}(H + E_1)^{1/2} f - (E_1 + k^2)(H_0 - k^2)^{-1} f, \quad f \in \mathscr{D}(H_0^{1/2})$$

for $E_1 > 0$ large enough and $k^2 \in \mathbf{C}\setminus[0, \infty)$ ((A.14) is proved similar to (A.4)) we get, after applying (A.14) to (A.13) with $f = \psi_0$, $k = k_0$,

$$0 = \psi_0 + \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* F_l \psi_0 =$$

$$= (H_0 - k_0^2)^{-1}(H + E_1)^{1/2}(H + E_1)^{1/2}\psi_0 - (E_1 + k_0^2)(H_0 - k_0^2)^{-1}\psi_0$$

and thus

$$Q_{H-k_0^2}((H_0 - \bar{k}_0^2)^{-1} g, \psi_0) = 0, \quad \text{for all } g \in \mathscr{H},$$

where $Q_{H-k_0^2}$ denotes the form associated with $H - k_0^2$. Since $\mathscr{D}(H_0)$ is a core for $Q_{H-k_0^2}$ we obtain

$$\psi_0 \in \mathscr{D}(H) \quad \text{and} \quad (H - k_0^2)\psi_0 = 0.$$

Next assume that

$$K(k_0)\varphi_1 = -\varphi_1, \quad \varphi_1 \in \mathscr{H}^N$$

where φ_1 is linearly independent of φ_0. Define

$$\psi_1 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \varphi_{1l} \tag{A.15}$$

then also

$$-F_j \psi_1 = \varphi_{1j}, \quad 1 \leqslant j \leqslant N$$

holds. Suppose that $\psi_1 = \text{const} \cdot \psi_0$ then obviously $\varphi_{1j} = \text{const} \cdot \varphi_{0j}$, $1 \leqslant j \leqslant N$ yields a contradiction.

b) Assume now

$$H\tilde{\psi}_0 = E_0 \tilde{\psi}_0, \quad \tilde{\psi}_0 \in \mathscr{D}(H).$$

If $-k^2 > 0$ is large enough such that $(1 + K(k))^{-1} \in \mathscr{B}(\mathscr{H}^N)$ then (A.5) implies

$$F_j(H - k^2)^{-1} = \sum_{l=1}^{N} (1 + K(k))_{jl}^{-1} F_l (H_0 - k^2)^{-1}$$

and thus

$$\sum_{l=1}^{N} (1 + K(k))_{jl} F_l (H - k^2)^{-1} = F_j (H_0 - k^2)^{-1}. \tag{A.16}$$

Applying (A.16) to vectors of the type $(H - k^2)\psi$, $\psi \in \mathscr{D}(H)$ one obtains after analytic continuation of both sides

$$F_j \psi + \sum_{l=1}^{N} K_{jl}(k) F_l \psi = F_j (H_0 - k^2)^{-1} (H - k^2)\psi, \; k^2 \in \mathbb{C} \setminus [0, \infty). \tag{A.17}$$

Taking $\psi = \tilde{\psi}_0$ and $k = k_0$ we get

$$F_j \tilde{\psi}_0 = -\sum_{l=1}^{N} F_j (H_0 - k_0^2)^{-1} E_l^* F_l \tilde{\psi}_0, \quad 1 \leqslant j \leqslant N. \tag{A.18}$$

Let

$$\tilde{\varphi}_{0j} = -F_j \tilde{\psi}_0, \quad 1 \leqslant j \leqslant N$$

and assume

$$\tilde{\varphi}_{0j} = 0 \quad \text{for all } 1 \leqslant j \leqslant N.$$

Then $\tilde{\psi}_0 \in \mathrm{Ker}(F_j)$, $1 \leqslant j \leqslant N$ yields a contradiction since

$$Q_H(\tilde{\psi}_0, \tilde{\psi}_0) = \|H_0^{1/2} \tilde{\psi}_0\|^2 = k_0^2(\tilde{\psi}_0, \tilde{\psi}_0), \quad \mathrm{Im}\, k_0 > 0 \tag{A.19}$$

is clearly impossible. Consequently (A.18) implies $-1 \in \sigma_p(K(k_0))$, $K(k_0)\tilde{\varphi}_0 = -\tilde{\varphi}_0$.

Next let

$$\tilde{\Psi}_0 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \tilde{\varphi}_{0l}, \tag{A.20}$$

then Part a) of the proof shows $\tilde{\Psi}_0 \in \mathscr{D}(H)$ and $(H - k_0^2)\tilde{\Psi}_0 = 0$.

Thus

$$H(\tilde{\Psi}_0 - \tilde{\psi}_0) = E_0(\tilde{\Psi}_0 - \tilde{\psi}_0)$$

and by (A.18)

$$F_j(\tilde{\Psi}_0 - \tilde{\psi}_0) = 0, \quad 1 \leqslant j \leqslant N.$$

As in (A.19) this yields a contradiction unless $\tilde{\Psi}_0 = \tilde{\psi}_0$ i.e.

$$\tilde{\psi}_0 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \tilde{\varphi}_{0l}. \tag{A.21}$$

Finally let $\tilde{\psi}_1 \in \mathscr{D}(H)$ be another eigenfunction linearly independent from $\tilde{\psi}_0$ such that $H\tilde{\psi}_1 = E_0\tilde{\psi}_1$. Define $\tilde{\varphi}_{1j} = -F_j\tilde{\psi}_1$, then also

$$\tilde{\psi}_1 = \sum_{l=1}^{N} (H_0 - k_0^2)^{-1} E_l^* \tilde{\varphi}_{1l}$$

which proves that $\tilde{\varphi}_1$ is linearly independent of $\tilde{\varphi}_0$. ▨

REMARK A.1. For $N = 1$ the above results are well known and widely used for quantum Hamiltonians in $L^2(\mathbf{R}^n)$, $n \geqslant 1$ (see e.g. [31, 33, 47, 48, 52, 53] and the references therein). In the case $N = 2$ they first appeared in [31, 32] and for general N in [19, 21] in the context of the multiple well problem.

Finally we give a discussion on eigenvalues and resonances in the case where H is self-adjoint.

Assume $K_{jl}(k) \in \mathscr{B}_\infty(\mathscr{H})$, $j, l = 1, \ldots, N$ for all $\mathrm{Im}\, k > 0$. Then, by Lemma A.3 there is a one to one correspondence between negative eigenvalues $E_0 = k_0^2 < 0$ of H and the eigenvalue -1 of $K(k_0)$, $k_0 = \mathrm{i}\sqrt{-E_0}$. If $K_{jl}(k)$, $j, l = 1, \ldots, N$ are analytic with respect to k in $\mathrm{Im}\, k > 0$ and may be analytically continued in the region $\mathrm{Im}\, k > -a$ for some $a > 0$ such that $K_{jl}(k)$, $j, l = 1, \ldots, N$ are compact for all $\mathrm{Im}\, k > -a$, we call k_1, $-a < \mathrm{Im}\, k_1 < 0$ a resonance of H if $K(k_1)$ has an eigenvalue -1 ([4]).

Concerning multiplicities of eigenvalues and resonances of H we state:

LEMMA A.4. *Assume H to be self-adjoint.*

a) *Suppose $K_{jl}(k)$, $j, l = 1, \ldots, N$ are analytic with respect to k in* $\operatorname{Im} k > 0$. *Let for some $p \in \mathbf{N}$, $K_{jl}(k) \in \mathscr{B}_p(\mathscr{H})$, $j, l = 1, \ldots, N$ for all* $\operatorname{Im} k > 0$ *and* $E_0 = k_0^2 < 0$, $k_0 = i\sqrt{-E_0}$, *be an eigenvalue of H. Then the multiplicity of E_0 coincides with the multiplicity of the zero of the (modified) Fredholm determinant* $\det_p(1 + K(k))$ at $k = k_0$.

b) *Suppose that $K_{jl}(k)$, $j, l = 1, \ldots, N$ are analytic with respect to k in* $\operatorname{Im} k > -a$ *for some $a > 0$. Assume in addition that for some $p \in \mathbf{N}$, $K_{jl}(k) \in \mathscr{B}_p(\mathscr{H})$, $j, l = 1, \ldots, N$ for all* $\operatorname{Im} k > -a$. *Then, if H has a resonance at k_1,* $-a < \operatorname{Im} k_1 < 0$, $(1 + K(k))^{-1}$ *has a norm convergent Laurent expansion around* $k = k_1$

$$(1 + K(k))^{-1} = \sum_{m=-M}^{\infty} K_m (k - k_1)^m$$

where each $K_m \in \mathscr{B}(\mathscr{H}^N)$ and for $-M \leqslant m \leqslant -1$, K_m is of finite rank. Moreover the multiplicity of this resonance k_1 defined as the zero of the (modified) Fredholm determinant $\det_p(1 + K(k))$ *at $k = k_1$ coincides with the geometric multiplicity of the eigenvalue -1 of $K(k_1)$ if and only if $M = 1$.*

For the proof of Lemma A.4 one needs the following result of Howland [23].

LEMMA A.5. a) *Let $\Omega \subset \mathbf{C}$ be open and connected, let $\mathscr{K}$ be some Hilbert space and assume $L(z)\colon \Omega \to \mathscr{B}_\infty(\mathscr{K})$ to be analytic. Suppose furthermore $z_0 \in \Omega$, $-1 \in \sigma(L(z_0))$, and $(1 + L(z_1))^{-1} \in \mathscr{B}(\mathscr{K})$ for some $z_1 \in \Omega$. Then, for z near z_0*

$$(1 + L(z))^{-1} = \sum_{m=-M}^{\infty} L_m (z - z_0)^m, \tag{A.22}$$

with convergence in the norm sense, where each $L_m \in \mathscr{B}(\mathscr{K})$ and, for $-M \leqslant m \leqslant -1$, L_m are of finite rank.

With respect to the decomposition

$$\mathscr{K} = P(z_0)\mathscr{K} + (1 - P(z_0))\mathscr{K}$$

we get in obvious matrix notation

$$1 + U(z)L(z)U(z)^{-1} = (1 + U(z)L(z)U(z)^{-1})P(z_0) + (1 + U(z)L(z)U(z)^{-1})(1 - P(z_0)) =$$

(A.23)

$$= \begin{pmatrix} 1 + A(z) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 + B(z) \end{pmatrix} = \begin{pmatrix} 1 + A(z) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 + B(z) \end{pmatrix}$$

where

$$U(z) = P(z_0)P(z)+(1-P(z_0))(1-P(z)), \tag{A.24}$$

$$P(z) = (2\pi i)^{-1} \oint_{|\zeta+1|=\varepsilon} d\zeta(\zeta - L(z))^{-1}. \tag{A.25}$$

b) *Assume in addition that for some* $p \in \mathbf{N}$, $L(z) \in \mathscr{B}_p(\mathscr{K})$ *for all* $z \in \Omega$. *Then*

$$\det{}_p(1 + L(z)) = \det\left(1 + \begin{pmatrix} A(z) & 0 \\ 0 & 0 \end{pmatrix}\right) \exp\left\{-\operatorname{Tr}\left[\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \begin{pmatrix} A(z) & 0 \\ 0 & 0 \end{pmatrix}^j\right]\right\} \cdot \det{}_p\left(1 + \begin{pmatrix} 0 & 0 \\ 0 & B(z) \end{pmatrix}\right). \tag{A.26}$$

Let ν *be the order of the zero of* $\det_p(1 + L(z))$ *at* $z = z_0$ *(*ν *coincides with the order of the zero of* $\det\left(1 + \begin{pmatrix} A(z) & 0 \\ 0 & 0 \end{pmatrix}\right)$ *at* $z = z_0$*) and* $\mu = \dim \operatorname{Ker}(1 + L(z_0))$. *Then* $\nu = \mu$ *if and only if* $M = +1$.

Proof. See Howland [23].

From Lemma A.5 we immediately get:

Proof of Lemma A.4. a) Identifying $z = k^2$, $\Omega = \{z \mid \operatorname{Im} z > 0\}$, $L(z) = K(k)$, and $\mathscr{K} = \mathscr{H}^N$ we only have to prove that $K(k)$ has a simple pole in k^2 at k_0^2. But this follows from (A.5) since $(H - k^2)^{-1}$, as the resolvent of a self-adjoint operator, has a simple pole at k_0^2.

b) is obvious from Lemma A.5. ▨

REMARK A.2. a) For general discussions on resonances in quantum mechanical systems cf. e.g. [4, 24, 44, 51] and the literature cited therein. For the fact that resonance poles need not necessarily be simple see [43]. Note that $(1 + L(z))^{-1}$ has a simple pole at $z = z_0$ if and only if $(1 + L(z_0)+(z - z_0)L^1(z_0))^{-1}$ has a simple pole at $z = z_0$ [23].

b) Using different arguments the result of Lemma A.4 a) has been derived previously by Newton [41] for a special class of potentials in $\mathbf{R}^3$. Modified Fredholm determinants are discussed e.g. in [55, 56, 40].

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