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LIMITS OF SPECTRA OF STRONGLY CONVERGING COMPRESSIONS

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0. PRELIMINARIES

Let H denote a complex separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $\mathscr{L}(H)$ ($\mathscr{LC}(H)$) stands for the set of bounded (compact) linear operators acting in H. For $A \in \mathscr{L}(H)$, $\sigma(A)$ denotes the spectrum of A. By a projection we shall mean an operator $P = P^* = P^2 \in \mathscr{L}(H)$. $\mathscr{P}_f(H)$ denotes the set of finite-dimensional projections. For a projection P and an operator $A \in \mathscr{L}(H)$, $A_P=(PA)|PH$— the restriction of PA to the range of P is called the compression of A to PH.

For two compact nonempty subsets M, N of the plane $\mathbf{C}$ of complex numbers the Hausdorff distance is defined as follows: $\mathrm{dist}(M, N) = \inf\{\varepsilon > 0$ such that $M \subset N + \varepsilon\mathbf{B}$, $N \subset M + \varepsilon\mathbf{B}\}$, where $\mathbf{B}$ denotes the closed unit disc in $\mathbf{C}$.

We are interested in answering the following:

QUESTION. Given a sequence $\{P_n\}_1^\infty$ of projections in H converging strongly to the identity operator in H ($P_n \xrightarrow{s} I_H$), what is the asymptotical behaviour of $\sigma(A_{P_n})$ for $A \in \mathscr{L}(H)$?

This problem is equivalent to the characterization of the family of sets $S(A)$ defined below.

DEFINITION. For $A \in \mathscr{L}(H)$, $S(A)$ denotes the family of all compact nonvoid subsets of $\mathbf{C}$ such that for every $\Omega \in S(A)$ there is a sequence $\{P_n\}_1^\infty$ of projections in H such that $P_n \xrightarrow{s} I_H$ and $\mathrm{dist}(\Omega, \sigma(A_{P_n})) \to 0$.

We shall show that $S(A)$ can be characterized with the use of $\sigma(A)$ and $W_e(A)$ — the essential numerical range of A, i.e. $W_e(A) = \bigcap_{P \in \mathscr{P}_f(H)} \overline{W(A_{(1-P)})}$, where $W(A) = \{\langle Ax, x\rangle ; x \in H, \|x\| = 1\}$ is the numerical range of A. Note that $W_e(A) = \{0\}$ if and only if $A \in \mathscr{LC}(H)$ (cf. [7]). It follows from [6], Theorem 2 that each $\lambda \in \sigma(A) \setminus W_e(A)$ is an isolated point of $\sigma(A)$, and λ is an eigenvalue of a finite multiplicity, i.e. the spectral projection $E(\lambda, A)$ is finite-dimensional.

It was shown in [3], Theorem 1 that for each $A \in \mathscr{L}(H)$, $\Omega \in S(A)$ we have

(1) $$\sigma(A) \setminus W_e(A) \subset \Omega \subset \sigma(A) \cup W_e(A) \quad \text{and} \quad \Omega \cap W_e(A) \neq \emptyset.$$

The set $W_e(A)$ is always convex and closed. It was shown in [4] that if $W_e(A)$ contains an interior point then a closed set Ω belongs to $S(A)$ if and only if (1) holds. If $W_e(A)$ has an empty interior it is either a single point set or a nondegenerate interval. In the first case it follows from (1) that $S(A) = \{\sigma(A)\}$. If $W_e(A)$ is a nondegenerate interval then $A = \alpha\tilde{A} + \beta I$, where $W_e(\tilde{A}) = \mathfrak{I} = [-1, 1], \alpha, \beta \in \mathbf{C}$. Then $S(A) = \{\alpha\Omega + \beta\,; \Omega \in S(\tilde{A})\}$, since for each projection P, $\sigma(A_P) = \alpha\sigma(\tilde{A}_P) + \beta$. Therefore we shall consider only the operators A such that $W_e(A) = \mathfrak{I}$; the imaginary parts of these operators are compact.

In the case when $\operatorname{Im} A = (A - A^*)/(2i) \in \mathfrak{S}_\omega$ (the Macaev ideal of compact operators) it was shown in [5] that $\Omega = \bar{\Omega} \in S(A)$ if and only if $\sigma(A) \subset \Omega \subset$ $\subset \sigma(A) \cup W_e(A)$. Thus in order to give a complete answer for the stated question it suffices to consider the case when $W_e(A) = \mathfrak{I}$ and $\operatorname{Im} A \in \mathscr{L}\mathscr{C}(H) \setminus \mathfrak{S}_\omega$. Corollary 2.7 gives the desired characterization of $S(A)$ in this up to now unsolved case.

We say (cf. [1], III, § 1.3) that a compact operator $K \in \mathfrak{S}_\omega$ if $\|K\|_\omega = \sum_1^\infty s_j(2j-1)^{-1} < \infty$, where $s_1 \geq s_2 \geq \ldots$ are all positive eigenvalues of $\sqrt{KK^*}$ repeated according to their multiplicities; if only n positive eigenvalues exist we set $s_m = 0$ for $m > n$.

1. COMPRESSIONS OF A DIRECT SUM OF OPERATORS

We shall not distinguish between $x \in H_1$ and $(x, 0) \in H_1 \oplus H_2$ (the orthogonal direct sum of H_1 and H_2), consequently we refer to H_1 as a subspace of $H_1 \oplus H_2$; and alike an operator $C \in \mathscr{L}(H_1)$ is identified with its trivial extension $C \oplus 0_{H_2} \in \mathscr{L}(H_1 \oplus H_2)$.

PROPOSITION 1.1. *Suppose that $V_j \in \mathscr{L}(H_j)$ and $\tilde{V}_j \in \mathscr{L}(H)$ are unitarily equivalent operators ($V_j \overset{u}{\approx} \tilde{V}_j$), $\alpha_j \in [0, 1]$, ($j = 1, 2, \ldots, n$), $\sum_1^n \alpha_j = 1$. Then there exists a projection $P \in \mathscr{L}\left(\bigoplus_1^n H_j\right)$ such that*

$$\left(\bigoplus_1^n V_j\right)_P \overset{u}{\approx} \sum_1^n \alpha_j \tilde{V}_j$$

and

$$\|(1-P)x\| \leq \sqrt{1-\alpha_1}\,\|x\| \quad \textit{for all } x \in H_1.$$

Proof. We proceed by induction with respect to n; since the induction step is nearly the same as the proof for $n = 2$, this last case is given only.

Let $U_j \in \mathscr{L}(H, H_j)$ be unitary operators such that $U_j V_j U_j^* = V_j$, $j = 1, 2$. Then $U =: U_1 U_2^{-1} \in \mathscr{L}(H_2, H_1)$ is unitary too. Note that

$$\tilde{U} = \begin{bmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} U \\ \sqrt{\alpha_2} U^{-1} & -\sqrt{\alpha_1} \end{bmatrix} \in \mathscr{L}(H_1 \oplus H_2)$$

is also unitary and $\tilde{U} = \tilde{U}^* = \tilde{U}^{-1}$. An easy calculation gives

$$\tilde{U} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \tilde{U} = \begin{bmatrix} \alpha_1 V_1 + \alpha_2 U V_2 U^{-1} & * \\ * & * \end{bmatrix};$$

thus setting

$$\tilde{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \tilde{U}\tilde{P}\tilde{U} = \begin{bmatrix} \alpha_1 & \sqrt{\alpha_1 \alpha_2} U \\ \sqrt{\alpha_1 \alpha_2} U^{-1} & \alpha_2 \end{bmatrix}$$

we see that

$$(V_1 \oplus V_2)_P \overset{u}{\approx} (\tilde{U}(V_1 \oplus V_2)\tilde{U})_{\tilde{P}} = \alpha_1 U_1 \tilde{V}_1 U_1^{-1} + \alpha_2 U_1 U_2^{-1} (U_2 \tilde{V}_2 U_2^{-1}) U_2 U_1^{-1} =$$

$$= U_1(\alpha_1 \tilde{V}_1 + \alpha_2 \tilde{V}_2) U_1^{-1}.$$

For $x \in H_1$, $Px = (\alpha_1 x, \sqrt{\alpha_1 \alpha_2} U^{-1} x)$, hence $\|(1 - P)x\|^2 = (1 - \alpha_1)^2 \|x\|^2 + \alpha_1 \alpha_2 \|x\|^2 = (1 - \alpha_1)\|x\|^2$, for all $x \in H_1$, and this ends the proof.

For any Hilbert space H we shall write λ instead of λI_H, and since $\mathbf{C}$ is itself a Hilbert space isomorphic with $\mathscr{L}(\mathbf{C})$, we shall write $[\lambda]$ to point out that $\lambda \in \mathscr{L}(\mathbf{C})$.

Suppose that $\lambda = \eta + i\xi$, $\eta = \operatorname{Re} \lambda \in \mathfrak{I}$, $\mu \in [(-1 + \eta)/2, (1 + \eta)/2]$. Let $\tilde{K} = \mu + i \sum_{1}^{n} s_j \langle \cdot, f_j \rangle f_j \in \mathscr{L}(H_n)$, where $\{f_j\}_1^n$ is an orthonormal basis of a Hilbert space H_n, $s_1 \geqslant s_2 \geqslant \ldots s_n > 0$.

PROPOSITION 1.2. *There exist a selfadjoint contraction $\tilde{C} \in \mathscr{L}(\mathscr{H})$ and a projection $P \in \mathscr{L}(\mathbf{C} \oplus \mathscr{H} \oplus H_n)$ such that*

i) $\sigma(([\lambda] \oplus \tilde{C} \oplus \tilde{K})_P) = \{\mu\}$,

ii) $\|(1 - P)z\|^2 \leqslant 2\|\operatorname{Im} \tilde{K}\|_\infty^{-1} + \dfrac{2|\xi|}{\pi(1 - |\mu|)} \|z\|^2$ *for* $z \in \mathbf{C}$,

iii) $\tilde{C} - \mu \in \mathscr{L}\mathscr{C}(\mathscr{H})$.

Proof. Let $\psi_j(t) = \exp((2j-1)\pi i t)$, $\varphi_j(t) = \exp(-2j\pi i t)$, $\{\psi_j\}_{-\infty}^{\infty}$, $\{\varphi_j\}_{-\infty}^{\infty}$ are two orthonormal sequences in $L^2(0, 1)$. Set $\mathscr{K}(t) = \sum_{j=1}^{n} s_j \psi_j(t)$. The formulae (cf.[1], Chapter III, § 10)

$$(Wf)(t) = 2i\int_t^1 \mathscr{K}(t-s)f(s)\,ds, \quad (Vf)(t) = 2i\int_t^1 f(s)\,ds$$

define two quasinilpotent operators W and $V \in \mathscr{L}\mathscr{C}(L^2(0, 1))$. The eigenvalue expansions of the real and imaginary parts of the operators V and W are

$$F = \operatorname{Im} V = \langle \cdot, \varphi_0\rangle\varphi_0, \quad L = \operatorname{Re} V = -\frac{2}{\pi}\sum_{-\infty}^{\infty}(2j-1)^{-1}\langle \cdot, \psi_j\rangle \psi_j,$$

$$K = \operatorname{Im} W = \sum_{j=1}^{n} s_j\langle \cdot, \psi_j\rangle\psi_j, \quad G = \operatorname{Re} W = \sum_{-\infty}^{\infty} \eta_j\langle \cdot, \varphi_j\rangle\varphi_j,$$

where

$$\eta_k = -\frac{2}{\pi}\sum_{j=1}^{n}\frac{s_j}{2j+2k-1}.$$

Note that

$$|\eta_k| \leqslant \eta_0 = \frac{2}{\pi}\|K\|_\omega = \|G\|, \quad k = \pm 1, \pm 2, \ldots$$

and

(2) $$\sigma(\alpha W + \beta V) = \{0\} \quad \text{for any } \alpha, \beta \in \mathbf{C}.$$

Now we set

$$\tilde{G} = \left(\mu + \frac{\eta-\mu}{\eta_0}G\right)_{(1-F)}, \quad \tilde{L} = \mu + \frac{1-|\mu|}{2}\pi\hat{L},$$

where

$$\hat{L} = \begin{cases}\operatorname{sign}(\xi(\eta-\mu))L & \text{if } \eta \neq \mu,\\ -\operatorname{sign}\xi L & \text{if } \eta = \mu.\end{cases}$$

Note that $\|\tilde{G}\| \leqslant |\mu| + |\eta-\mu|\,\|G\|/\eta_0 = |\mu| + |\eta-\mu| \leqslant 1$ and $\|\tilde{L}\| \leqslant |\mu| + (1-|\mu|)\pi\|L\|/2 = 1$. Therefore the operator $\tilde{C} = \tilde{G} \oplus \tilde{L} \oplus \mu I_H$, where H is some infinite-dimensional Hilbert space, is a selfadjoint contraction and $\tilde{C} - \mu$ is compact.

Note also that

$$[\lambda] \oplus \tilde{G} \overset{u}{\approx} \lambda F + (1 - F)\left(\mu + \frac{\eta - \mu}{\eta_0} G\right) = \mu + \mathrm{i}\xi F + (\eta - \mu)F +$$

$$+ \frac{\eta - \mu}{\eta_0}(1 - F)G = \mu + \mathrm{i}\xi F + \frac{\eta - \mu}{\eta_0}(\eta_0 F + (1 - F)G) = \mu + \mathrm{i}\xi F + \frac{\eta - \mu}{\eta_0} G$$

and $(\mu I_H \oplus \tilde{K}) \overset{u}{\approx} \mu + \mathrm{i}K$. Hence

$$(3) \qquad \begin{aligned} &[\lambda] \oplus \tilde{C} \oplus \tilde{K} = ([\lambda] \oplus \tilde{G}) \oplus \tilde{L} \oplus (\mu I_H \oplus \tilde{K}) \overset{u}{\approx} \\ &\overset{u}{\approx} \mu + \left(\left(\mathrm{i}\xi F + \frac{\eta - \mu}{\eta_0} G\right) \oplus \frac{1 - |\mu|}{2} \pi \hat{L} \oplus (\mathrm{i}K)\right). \end{aligned}$$

Let α, β, γ satisfy the equations:

$$\alpha|\eta - \mu| = \eta_0 \gamma, \quad 2\alpha|\xi| = \pi(1 - |\mu|)\beta, \quad \alpha + \beta + \gamma = 1,$$

then

$$(4) \quad 1 - \alpha = \beta + \gamma = \alpha\left(\frac{|\eta - \mu|}{\eta_0} + \frac{2}{\pi} \frac{|\xi|}{1 - |\mu|}\right) \leqslant \frac{2}{\|\mathrm{Im}\,\tilde{K}\|_\omega} + \frac{2}{\pi} \frac{|\xi|}{1 - |\mu|}.$$

It follows from Proposition 1.1 and (3) that there is a projection $P \in \mathscr{L}((\mathbf{C} \oplus \oplus \operatorname{ran}(1 - F)) \oplus L^2(0, 1) \oplus (H \oplus H_n)) = \mathscr{L}(\mathbf{C} \oplus \mathscr{H} \oplus H_n)$ such that

$$(5) \qquad \|(1 - P)x\| \leqslant \sqrt{1 - \alpha}\, \|x\| \quad \text{for } x \in \mathbf{C}$$

and

$$([\lambda] \oplus \tilde{C} \oplus \tilde{K})_P \overset{u}{\approx} \mu + \alpha\left(\mathrm{i}\xi F + \frac{\eta - \mu}{\eta_0} G\right) + \beta \frac{1 - |\mu|}{2} \pi \hat{L} + \mathrm{i}\gamma K =$$

$$= \mu + \gamma(\operatorname{sign}(\eta - \mu)G + \mathrm{i}K) + \alpha|\xi|(\mathrm{i} \operatorname{sign} \xi F + \hat{L}) =$$

$$= \begin{cases} \mu + \gamma W + \alpha\xi V & \text{if } \eta \geqslant \mu \\ \mu - (\gamma W^* + \alpha\xi V^*) & \text{if } \eta < \mu . \end{cases}$$

Thus in both cases i) is satisfied, since by (2) $\sigma(\gamma W^* + \alpha\xi V^*) = \sigma(\gamma W + \alpha\xi V) = \{0\}$. The remaining condition ii) follows from (5) and (4).

In the rest of this section we shall study the compressions of direct sums of operators with an operator $K = -K^* \in \mathscr{LC}(H) \setminus \mathfrak{S}_\omega$. We shall assume that $K = \mathrm{i} \sum_{j=1}^{\infty} s_j \langle \cdot, f_j \rangle f_j$ is the eigenvalue expansion of K and that $s_j \geqslant s_{j+1} > 0$, $j = 1, 2, \ldots$.

We put $H_n = \bigvee_1^n f_j$ (the subspace spanned by $f_1, \ldots, f_n$).

PROPOSITION 1.3. *Assume that* $\lambda \in (1+\beta)\mathfrak{I} + i\beta\mathfrak{I}$ *and* $\mu \in [(-1+\eta)/2, (1+\eta)/2]$, *where* $\beta > 0$, $\eta = \min\{1, |\mathrm{Re}\,\lambda|\}\,\mathrm{sign}\,\mathrm{Re}\,\lambda$. *Then for any* $\varepsilon > 0$ *there exist: a natural* n, *a selfadjoint contraction* $C \in \mathscr{L}(\mathscr{H})$ *and* $P \in \mathscr{P}_f(\mathbf{C} \oplus H_n \oplus \mathscr{H})$ *such that* $\sigma(([\lambda] \oplus K|H_n \oplus C)_P) \subset \mu + \varepsilon\mathbf{B}$ *and*

$$\|(1-P)z\| \leqslant 2\sqrt{\frac{\beta}{1-|\mu|}}\,\|z\| \quad \textit{for all } z \in \mathbf{C}.$$

Proof. Let $\mathscr{H}_0$ be a one-dimensional space, $0 = C_0 \in \mathscr{L}(\mathscr{H}_0)$ and $\alpha = 1 - (\max\{1, |\mathrm{Re}\,\lambda|\})^{-1}$. Then $0 \leqslant \alpha < 1 - (1+\beta)^{-1} < \beta$, and $\mathrm{Re}(1-\alpha)\lambda = \eta$. It follows from Proposition 1.1 that there is a one-dimensional projection $Q \in \mathscr{L}(\mathbf{C} \oplus \mathscr{H}_0)$ such that

$$F = ([\lambda] \oplus C_0)_Q \overset{u}{\approx} [(1-\alpha)\lambda] \quad \text{and} \quad \|(1-Q)z\| \leqslant \sqrt{\alpha}\,\|z\| \leqslant \sqrt{\beta}\,\|z\| \quad \text{for} \quad z \in \mathbf{C}.$$

Since $\|K|H_n\|_\omega \to \infty$, we may take n such that $\|K|H_n\|_\omega > 6\beta^{-1}$. Let $\mathscr{H}_1$ be an n-dimensional Hilbert space, and put $C_1 = \mathrm{sign}\,\mu \in \mathscr{L}(\mathscr{H}_1)$. By Proposition 1.1 there is a projection $R \in \mathscr{L}(H_n \oplus \mathscr{H}_1)$ such that $\tilde{K} = (K|H_n \oplus C_1)_R \overset{u}{\approx} (\mu + (1-|\mu|)K)|H_n$, consequently $\|\mathrm{Im}\,\tilde{K}\|_\omega = (1-|\mu|)\|K|H_n\|_\omega > 6(1-|\mu|)\beta^{-1}$. Using Proposition 1.2 we can find a selfadjoint contraction $C_2 \in \mathscr{L}(\mathscr{H}_2)$ and a projection $S \in \mathscr{L}(\mathrm{ran}\,Q \oplus \mathrm{ran}\,R \oplus \mathscr{H}_2)$ such that

$$(C_2 - \mu) \in \mathscr{L}\mathscr{C}(\mathscr{H}_2), \quad \sigma((F \oplus \tilde{K} \oplus C_2)_S) = \{\mu\}$$

and

$$\|(1-S)Q\|^2 \leqslant \frac{2}{\|\mathrm{Im}\,\tilde{K}\|_\omega} + \frac{2}{\pi}\,\frac{|\mathrm{Im}(1-\alpha)\lambda|}{1-|\mu|} \leqslant$$

$$\leqslant \frac{\beta}{1-|\mu|}\left(\frac{2}{6} + \frac{2}{\pi}\right) \leqslant \beta/(1-|\mu|).$$

Now we define: the Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$, the selfadjoint contraction $C = C_0 \oplus C_1 \oplus C_2$ and the projection $\tilde{P} = S(Q \oplus R \oplus I_{\mathscr{H}_2})$. One can easily verify that $\sigma(([\lambda] \oplus (K|H_n) \oplus C)_{\tilde{P}}) = \{\mu\}$ and

$$\|(1-\tilde{P})z\| \leqslant \|(1-\tilde{P})Qz\| + \|(1-\tilde{P})(1-Q)z\| \leqslant$$

$$\leqslant \sqrt{\frac{\beta}{1-|\mu|}}\,\|Qz\| + \sqrt{\beta}\,\|z\| \leqslant 2\sqrt{\frac{\beta}{1-|\mu|}}\,\|z\| \quad \text{for } z \in \mathbf{C}.$$

Since the operator $([\lambda] \oplus (K|H) \oplus C)_{\tilde{P}} - \mu$ is compact and quasinilpotent, there exists $P \in \mathscr{P}_f(\mathbf{C} \oplus H_n \oplus \mathscr{H})$ such that $P < \tilde{P}$ (that is $\operatorname{ran} P \subset \operatorname{ran} \tilde{P}$), $\tilde{P}\mathbf{C} \subset \operatorname{ran} P$ and $\sigma(([\lambda] \oplus (K|H_n) \oplus C)_P) \subset \mu + \varepsilon\mathbf{B}$. Thus P satisfies the thesis, since

$$\|(1-P)z\| = \|(1-\tilde{P})z\| \leqslant 2\sqrt{\frac{\beta}{1-|\mu|}}\,\|z\|, \quad \text{for } z \in \mathbf{C}.$$

We shall generalize Proposition 1.3 for finite-dimensional operators. Suppose that $\mathfrak{H}$ is an m-dimensional Hilbert space, $A \in \mathscr{L}(\mathfrak{H})$, $\sigma(A) \subset [a, b] + i\beta\mathfrak{I}$, where $\beta > 0$, $[a, b] \subset (1+\beta)\mathfrak{I}$, and let $a' = \max\{a, -1\}, b' = \min\{b, 1\}$.

PROPOSITION 1.4. *Suppose that* $\mu \in [(-1+b')/2,\ (1+a')/2]$ *and* $\varepsilon > 0$. *Then we can find a natural number* n, *a selfadjoint contraction* $C \in \mathscr{L}(\mathscr{H})$ *and a projection* $P \in \mathscr{P}_f(\mathfrak{H} \oplus H_n \oplus \mathscr{H})$ *such that*

$$\sigma((A \oplus (K|H_n) \oplus C)_P) \subset \mu + \varepsilon\mathbf{B}$$

and

$$\|(1-P)x\| \leqslant 2\sqrt{\frac{\beta}{1-|\mu|}}\,\|x\| \quad \textit{for } x \in \mathfrak{H}.$$

Proof. By the theorem on the triangular matrix form there exists an orthonormal basis $\{e_j\}_1^m$ of $\mathfrak{H}$ such that setting $E_j = \langle\cdot, e_j\rangle e_j$ we get $E_k A E_j = 0$ for $j < k$, $E_j A E_j = \lambda_j E_j$, $j = 1, 2, \ldots, m$; obviously $\lambda_j \in \sigma(A)$. Put $\eta_j = \min\{1, |\operatorname{Re}\lambda_j|\}\operatorname{sign}\operatorname{Re}\lambda_j$. Then $a' \leqslant \eta_j \leqslant b'$ and $\mu \in \left[\frac{-1+b'}{2}, \frac{1+a'}{2}\right] \subset \left[\frac{-1+\eta_j}{2}, \frac{1+\eta_j}{2}\right]$. Note that for each natural n, $\operatorname{Im}(K|H_n^\perp) \geqslant 0$ and $K|H_n^\perp \notin \mathfrak{S}_\omega$, therefore, by Proposition 1.3, we can find one by one natural numbers $0 = k_0 < k_1 < \ldots < k_m = n$, selfadjoint contractions $C_j \in \mathscr{L}(\mathscr{H}_j)$, and projections $P_j \in \mathscr{P}_f(\operatorname{ran} E_j \oplus \hat{H}_j \oplus \mathscr{H}_j)$, where $\hat{H}_j = H_{k_{j-1}}^\perp \cap H_{k_j}$, such that

$$\sigma((A_{E_j} \oplus (K|\hat{H}_j) \oplus C_j)_{P_j}) \subset \mu + \varepsilon\mathbf{B},$$

and

$$\|(1-P_j)e_j\| \leqslant 2\sqrt{\beta/(1-|\mu|)}, \quad j = 1, 2, \ldots, m.$$

Let $\mathscr{H} = \bigoplus_1^m \mathscr{H}_j$, $\tilde{\mathscr{H}} = \mathfrak{H} \oplus H_n \oplus \mathscr{H}$, $C = \bigoplus_1^m C_j$, $P = \bigoplus_1^m P_j$, $B = A \oplus (K|H_n) \oplus C$ and $Q_j \in \mathscr{L}(\tilde{\mathscr{H}})$ be the projection onto $\hat{H}_j \oplus \mathscr{H}_j$. Identifying the operators with their trivial extensions in $\tilde{\mathscr{H}}$ we can write

$$(E_k + Q_k)B(E_j + Q_j) = E_k A E_j = 0 \quad \text{for } j < k,$$

and since $P_j < E_j + Q_j$ $(j = 1, 2, \ldots, m)$, we get $P_k B P_j = 0$ for $j < k$. Hence

$$B_P = \begin{bmatrix} B_{P_1} & & & * \\ & B_{P_2} & & \\ & & \ddots & \\ 0 & & & B_{P_m} \end{bmatrix},$$

therefore $\sigma(B_P) = \bigcup_1^m \sigma(B_{P_j}) = \bigcup_1^m \sigma((A_{E_j} \oplus (K|\hat{H}_j) \oplus C_j)_{P_j}) \subset \mu + \varepsilon\mathbf{B}$. Since $\sum_1^m (E_j + Q_j) = I_{\tilde{\mathscr{H}}}$ and $P(E_j + Q_j) = P_j(E_j + Q_j)$, we have

$$(1 - P)x = \sum_j (1 - P)E_j x = \sum (1 - P)(E_j + Q_j)E_j x =$$

$$= \sum (E_j + Q_j)(1 - P_j)x, \quad \text{for each } x \in \mathfrak{H}$$

and this gives

$$\|(1 - P)x\|^2 = \sum \|(1 - P_j)E_j x\|^2 \leqslant 4\frac{\beta}{1 - |\mu|} \sum \|E_j x\|^2 = 4\frac{\beta}{1 - |\mu|}\|x\|^2,$$

which ends the proof.

Lemma 1.5. *Suppose that* $K = i\operatorname{Im} K \in \mathscr{L}\mathscr{C}(H) \setminus \mathfrak{S}_\omega$ *and* $\operatorname{Im} K \geqslant 0$, $A \in \mathscr{L}(\mathfrak{H})$, $\dim \mathfrak{H} < \infty$, $\sigma(A) \subset \mathfrak{I} + \beta\mathbf{B}$, $\beta > 0$, $\varepsilon > 0$, $\mu \in (-1, 1)$.

Then there exist a selfadjoint contraction $C \in \mathscr{L}(\mathscr{H})$ *and two projections* $R \in \mathscr{P}_f(H)$, $P \in \mathscr{P}_f(\mathfrak{H} \oplus \operatorname{ran} R \oplus \mathscr{H})$ *such that*

$$\sigma((A \oplus K_R \oplus C)_P) \subset \mu + \varepsilon\mathbf{B},$$

$$\|(1 - P)x\| \leqslant 3\sqrt{\beta}\|x\| \quad \textit{for } x \in \mathfrak{H}.$$

Proof. We put $t = |\mu|/(1 + |\mu|) < 1/2$ and define the sequence : $\mu_0 = 0$, $\mu_k = \sum_{j=1}^k t^j \operatorname{sign} \mu$, which converges monotonically to μ. We shall show by induction on k that for any $\varepsilon > 0$ there exist a natural number n, a selfadjoint contraction $C \in \mathscr{L}(\mathscr{H})$ and a projection $P \in \mathscr{P}_f(\mathfrak{H} \oplus H_n \oplus \mathscr{H})$ such that

$$\sigma(A \oplus (K|H_n) \oplus C) \subset \mu_k + \varepsilon\mathbf{B} \tag{6}$$

and

$$\|(1 - P)x\| \leqslant (3 - 2^{-k})\sqrt{\beta}\|x\| \quad \text{for all } x \in \mathfrak{H}. \tag{7}$$

This thesis for $k = 0$ is a simple application of Proposition 1.4; suppose that it holds for some $k \geqslant 0$. In particular for $\varepsilon = \min\{1 + |\mu_k| - 2|\mu_{k+1}|, \beta(1 - |\mu_{k+1}|)^{4-k-2}\}$ we can find n, C and P satisfying (7). We put $B = (A \oplus (K|H_n) \oplus C)_P$, then

$$\sigma(B) \subset \mu_k + \varepsilon\mathbf{B} \subset [\mu_k - \varepsilon, \mu_k + \varepsilon] + i\varepsilon\,\mathfrak{I} \subset \mathfrak{I} + i\varepsilon\,\mathfrak{I}$$

and

$$\mu_{k+1} \in [(-1 + \mu_k + \varepsilon)/2, (1 + \mu_k - \varepsilon)/2].$$

The operators B and $K|H_n^\perp$ satisfy the assumptions of Proposition 1.4; applying this proposition we can find (for a given $\tilde{\varepsilon} > 0$) $\hat{C} = \hat{C}^* \in \mathscr{L}(\hat{\mathscr{H}})$, $\tilde{n} > n$ and $\hat{P} \in \mathscr{P}_f(\operatorname{ran} P \oplus (H_n^\perp \cap H_{\tilde{n}}) \oplus \hat{\mathscr{H}})$ such that $\|\hat{C}\| \leqslant 1$,

$$\sigma((B \oplus (K|H_n^\perp \cap H_{\tilde{n}}) \oplus \hat{C})_{\hat{P}}) \subset \mu_{k+1} + \tilde{\varepsilon}\mathbf{B} \tag{8}$$

and

$$\|(1 - \hat{P})P\| \leqslant 2\sqrt{\frac{\varepsilon}{1 - |\mu_{k+1}|}} \leqslant 2^{-k-1}\sqrt{\beta}. \tag{9}$$

Now we set $\tilde{\mathscr{H}} = \mathscr{H} \oplus \hat{\mathscr{H}}$, $\tilde{C} = C \oplus \hat{C} \in \mathscr{L}(\tilde{\mathscr{H}})$ and $\tilde{P} = \hat{P}(P \oplus I_{H_n^\perp \cap H_{\tilde{n}}} \oplus I_{\hat{\mathscr{H}}}) \in \mathscr{P}_f(\mathfrak{H} \oplus H_{\tilde{n}} \oplus \tilde{\mathscr{H}})$. Then it follows from (8) that

$$\sigma((A \oplus (K|H_{\tilde{n}}) \oplus \tilde{C})_{\tilde{P}}) \subset \mu_{k+1} + \tilde{\varepsilon}\mathbf{B}$$

and by (9) and (7) we have for any $x \in \mathfrak{H}$

$$\|(1 - \tilde{P})x\| \leqslant \|(1 - \tilde{P})Px\| + \|(1 - \tilde{P})(1 - P)x\| \leqslant \|(1 - \hat{P})Px\| +$$

$$+ \|(1 - P)x\| \leqslant \sqrt{\beta}(2^{-k-1} + 3 - 2^{-k})\|x\| = (3 - 2^{-k-1})\sqrt{\beta}\|x\|.$$

Thus we have shown that (6)—(7) hold for $k + 1$ and therefore for all $k \geqslant 0$.

If $|\mu_k - \mu| < \varepsilon/2$, then the operators C, P satisfying (6)—(7) with $\varepsilon/2$ instead of ε and the projection $R \in \mathscr{P}_f(H)$ on H_n satisfy also the thesis of the lemma.

2. COMPRESSIONS OF AN OPERATOR WITH AN IMAGINARY PART IN $\mathscr{LC}(H) \setminus \mathfrak{S}_\omega$

In this section we assume that $A \in \mathscr{L}(H)$, $W_e(A) = \mathfrak{I}$, $(\operatorname{Im} A)_+ = \sum_1^\infty s_j\langle\cdot, f_j\rangle f_j \notin \mathfrak{S}_\omega$, where $s_1 \geqslant s_2 \geqslant \ldots$ are all positive eigenvalues of $\operatorname{Im} A$ corresponding to the eigenvectors f_j.

PROPOSITION 2.1. *Suppose that B is a selfadjoint contraction acting in a complex separable Hilbert space $\mathscr{H}$, $P \in \mathscr{P}_f(H)$, $\varepsilon > 0$. Then there exist a projection $Q \in \mathscr{L}(H)$ and a unitary operator $U \in \mathscr{L}(QH, \mathscr{H})$ such that $QP = QAP = QA^*P = 0$ and $\|B - U(A_Q)U^*\| < \varepsilon$.*

Proof. By the Weyl-von Neuman theorem ([2]) there exists $C = C^* \in \mathscr{L}(\mathscr{H})$ with a pure point spectrum such that $\|B - C\| < \varepsilon/2$. Let $C = \sum_1^\infty \lambda_j \langle \cdot, \varphi_j \rangle \varphi_j$, where $\lambda_i \in \mathfrak{J}$, be the eigenvalue expansion of C.

We put $H_0 = \operatorname{ran} P \vee \operatorname{ran} AP \vee \operatorname{ran} A^*P$ (the smallest closed subspace containing the respective subsets). Then we can find one by one unit vectors e_j and subspaces H_j $(j = 1, 2, \ldots)$ satisfying $e_j \perp H_{j-1}$, $|\langle Ae_j, e_j\rangle - \lambda_j| < \varepsilon/2$, $H_j = H_{j-1} \vee \{e_j, Ae_j, A^*e_j\}$.

Then the operators $Q = \sum_{j=1}^\infty \langle \cdot, e_j\rangle e_j$ and $U = \sum_{j=1}^\infty \langle \cdot, e_j \rangle \varphi_j$ satisfy the thesis.

PROPOSITION 2.2. *If $Q \in \mathscr{P}_f(H)$ then there exists a projection $P \in \mathscr{L}(H)$ such that $PQ = PAQ = PA^*Q = 0$, $\operatorname{Re} A_P = 0$, $\operatorname{Im} A_P \geqslant 0$ and $A_P \notin \mathfrak{S}_\omega$.*

Proof. It suffices to define an orthonormal sequence $\{e_j\}_1^\infty$ and a sequence of subspaces $\{H_j\}_0^\infty$ such that

i) $H_0 = QH \vee AQH \vee A^*QH$,

ii) $H_j = H_{j-1} \vee e_j \vee Ae_j \vee A^*e_j$, $\quad \dim H_j \leqslant 3(m + j)$,

iii) $e_j \perp H_{j-1}$,

iv) $\operatorname{Re}\langle Ae_j, e_j\rangle = 0$, $\quad \operatorname{Im}\langle Ae_j, e_j\rangle \geqslant \dfrac{s_{3(m+j)}}{8\|A\|}$,

where $m = \dim QH$. When the sequence $\{e_j\}_1^\infty$ is defined, the projection $P = \sum_1^\infty \langle \cdot, e_j\rangle e_j$ satisfies the thesis.

Suppose that $\{e_1, \ldots, e_n\}$, H_n are already defined in such a way that i) — iv) hold. Then we find a unit vector $e \in H_n^\perp \vee \bigvee_{j=1}^{3(m+n+1)} f_j$ and we have

$$\operatorname{Im}\langle Ae, e\rangle = \langle (\operatorname{Im} A)e, e\rangle = \sum_1^{3(m+n+1)} s_j |\langle e, f_j\rangle|^2 \geqslant s_{3(m+n+1)}.$$

It follows from the definition of $W_e(A)$ that there is a unit vector $x \in (H_n \vee e \vee Ae \vee A^*e)^\perp$ such that

$$|\operatorname{Im}\langle Ax, x\rangle| \leqslant \frac{s_{3(m+n+1)}}{8\|A\|}, \quad \operatorname{Re}\langle Ax, x\rangle \begin{cases} > 1/2 & \text{if } \operatorname{Re}\langle Ae, e\rangle < 0 \\ < -1/2 & \text{if } \operatorname{Re}\langle Ae, e\rangle \geqslant 0, \end{cases}$$

then

$$1 \geqslant \operatorname{Re}\langle Ax, x\rangle(\operatorname{Re}(\langle Ax, x\rangle - \langle Ae, e\rangle))^{-1} =$$

$$= \alpha \geqslant |\operatorname{Re}\langle Ax, x\rangle|(|\langle Ax, x\rangle| + |\langle Ae, e\rangle|)^{-1} \geqslant 1/(4\|A\|).$$

Thus setting $e_{n+1} = \sqrt{\alpha}e + \sqrt{1-\alpha}x$ we have obviously $\|e_{n+1}\| = 1$, $e_{n+1} \perp H_n$, and since $\langle Ae_{n+1}, e_{n+1}\rangle = \alpha\langle Ae, e\rangle + (1-\alpha)\langle Ax, x\rangle$, we get $\operatorname{Re}\langle Ae_{n+1}, e_{n+1}\rangle = 0$ and

$$\operatorname{Im}\langle Ae_{n+1}, e_{n+1}\rangle = \alpha\operatorname{Im}\langle Ae, e\rangle + (1-\alpha)\operatorname{Im}\langle Ax, x\rangle \geqslant s_{3(m+n+1)}/(4\|A\|) -$$

$$- s_{3(m+n+1)}/(8\|A\|) = s_{3(m+n+1)}/(8\|A\|).$$

Thus we have shown that the sequences $\{H_j\}$ and $\{e_j\}$ may be defined recursively, and this ends the proof.

LEMMA 2.3. *Suppose that* $P \in \mathscr{P}_f(H)$, $\mu \in \mathfrak{I}$, $0 < \varepsilon < \beta$. *Then there exists* $Q \in \mathscr{P}_f(H)$ *such that*

$$\sigma(A_P) \setminus (\mathfrak{I} + \beta\mathbf{B}) = \sigma(A_Q) \setminus (\mathfrak{I} + \beta\mathbf{B}),$$

$$\sigma(A_Q) \cap (\mathfrak{I} + \beta\mathbf{B}) \subset \mu + \varepsilon\mathbf{B}$$

and

$$\|(1-Q)P\| \leqslant 3\sqrt{\beta}.$$

Proof. Let P_0 be the projection on $\bigvee_{\lambda \in \sigma(A_P)\setminus(\mathfrak{I}+\beta\mathbf{B})} \operatorname{ran} E(\lambda, A_P) = \operatorname{ran} E(\mathbf{C}\setminus(\mathfrak{I} + \beta\mathbf{B}), A_P)$ and $P_1 = P - P_0$. Then

$$A_P = \begin{bmatrix} A_{P_0} & * \\ 0 & A_{P_1} \end{bmatrix}$$

and consequently $\sigma(A_{P_0}) = \sigma(A_P) \setminus (\mathfrak{I} + \beta\mathbf{B})$, $\sigma(A_{P_1}) = \sigma(A_P) \cap (\mathfrak{I} + \beta\mathbf{B})$. It follows from Proposition 2.2 that there is a projection $S \in \mathscr{L}(H)$ such that $SP = SAP = SA^*P = 0$, $\operatorname{Re} A_S = 0$, $\operatorname{Im} A_S \geqslant 0$ and $A_S \notin \mathfrak{S}_\omega$. Lemma 1.5 implies now that there exist a projection $S_0 \in \mathscr{P}_f(H)$, $(S_0 < S)$, a selfadjoint contraction $C \in \mathscr{L}(\mathscr{H})$ and $\tilde{Q} \in \mathscr{P}_f(\operatorname{ran} P_1 \oplus \operatorname{ran} S_0 \oplus \mathscr{H})$ such that

$$\sigma((A_{P_1} \oplus A_{S_0} \oplus C)_{\tilde{Q}}) \subset \mu + \frac{\varepsilon}{2}\mathbf{B}$$

and

$$\|(1-\tilde{Q})P_1\| \leqslant 3\sqrt{\beta}.$$

The operator $(A_{P_1} \oplus A_{S_0} \oplus C)_{\tilde{Q}}$ is finite-dimensional, therefore there is $\delta > 0$ such that if only $\|\tilde{C} - C\| < \delta$ then $\sigma((A_{P_1} \oplus A_{S_0} \oplus \tilde{C})_{\tilde{Q}}) \subset \mu + \varepsilon\mathbf{B}$. It follows from Proposition 2.1 that one can find a projection $R \in \mathscr{L}(H)$ and a unitary operator $U \in \mathscr{L}(\operatorname{ran} R, \mathscr{H})$ such that $R(P + S_0) = RA^*(P + S_0) = RA(P + S_0) = 0$ and $\|UA_R U^* - C\| < \delta$. Then $\hat{Q} = (I_{(P_1+S_0)} \oplus U^*)\tilde{Q}(I_{(P_1+S_0)} \oplus U) \in \mathscr{P}_f(\operatorname{ran}(P_1 + S_0 + R))$, $\sigma((A_{P_1} \oplus A_{S_0} \oplus A_R)_{\hat{Q}}) \subset \mu + \varepsilon\mathbf{B}$ and $\|(1 - \hat{Q})P_1\| \leqslant 3\sqrt{\beta}$. Hence setting $Q = P_0 + \hat{Q}(P_1 + S_0 + R)$ it follows from the properties of S, R and $\hat{Q}$ that $Q \in \mathscr{P}_f(H)$, $A_Q = \begin{bmatrix} A_{P_0} & * \\ 0 & A_{\hat{Q}} \end{bmatrix}$ and $\|(1 - Q)P\| = \|(1 - \hat{Q})P_1\| \leqslant 3\sqrt{\beta}$. Since

$$\sigma(A_Q) = \sigma(A_{P_0}) \cup \sigma(A_{\hat{Q}}) \subset (\sigma(A) \setminus (\mathfrak{I} + \beta\mathbf{B})) \cup (\mu + \varepsilon\mathbf{B})$$

the lemma is proved.

PROPOSITION 2.4. *Let* $H_d = \bigvee_{\lambda \notin W_e(A)} \operatorname{ran} E(\lambda, A)$ *and* P_e *be the projection on* $H_e = H_d^{\perp}$, *then* $\sigma(A_{P_e}) \subset W_e(A)$.

The proposition follows from the fact that for each $\lambda \in \sigma(A) \setminus W_e(A)$ we have $\operatorname{ran} E(\bar{\lambda}, A^*) \cap H_e = \{0\}$.

PROPOSITION 2.5. *Suppose that* $0 < \beta_n \to 0$, *then there exists a sequence* $\{R_n\}_1^\infty \subset \mathscr{P}_f(H)$ *such that* $R_n \xrightarrow{s} 1$ *and* $\sigma(A_{R_n}) \setminus (\mathfrak{I} + \beta_n\mathbf{B}) \subset \sigma(A)$.

Proof. Let Q_n be the orthogonal projection on $\operatorname{ran} E(\mathbf{C} \setminus (\mathfrak{I} + \beta_n\mathbf{B}), A)$. Then $Q_n \xrightarrow{s} P_d$, where P_d stands for the projection on H_d (defined in Proposition 2.4) and $\sigma(A_{Q_n}) = \sigma(A) \setminus (\mathfrak{I} + \beta_n\mathbf{B})$.

Let $\{\tilde{Q}_n\}_1^\infty \subset \mathscr{P}_f(H)$ be a sequence such that $\tilde{Q}_n < P_e$ and $\tilde{Q}_n \xrightarrow{s} P_e$. Since $A_{\tilde{Q}_n} = (A_{P_e})_{\tilde{Q}_n}$ it follows from Proposition 2.4 and (1) that passing to the subsequence if needed, we may assume also that $\sigma(A_{\tilde{Q}_n}) \subset \mathfrak{I} + \beta_n\mathbf{B}$. Setting $R_n = Q_n + \tilde{Q}_n$ we see that $R_n \in \mathscr{P}_f(H)$, $R_n \xrightarrow{s} P_e + P_d = I$ and $\tilde{Q}_n A Q_n = \tilde{Q}_n P_e(P_d A Q_n) = 0$. This last relation implies that

$$A_{R_n} = \begin{bmatrix} A_{Q_n} & * \\ 0 & A_{\tilde{Q}_n} \end{bmatrix}$$

and consequently

$$\sigma(A_{R_n}) = \sigma(A_{Q_n}) \cup \sigma(A_{\tilde{Q}_n}) \subset \sigma(A) \cup (\mathfrak{I} + \beta_n\mathbf{B}),$$

which ends the proof.

THEOREM 2.6. *Suppose that* $\emptyset \neq \Omega = \overline{\Omega} \subset \mathfrak{I}$ *and that* $0 < \beta_n \to 0$. *Then there exists a sequence* $\{P_n\}_1^\infty \subset \mathscr{P}_f(H)$ *such that* $P_n \xrightarrow{s} 1$,

$$\sigma(A_{P_n}) \setminus (\mathfrak{I} + \beta_n \mathbf{B}) = \sigma(A) \setminus (\mathfrak{I} + \beta_n \mathbf{B})$$

and

$$\operatorname{dist}(\Omega, \sigma(A_{P_n}) \cap (\mathfrak{I} + \beta_n \mathbf{B})) \to 0.$$

Proof. Let $\mu \in \Omega$ and $\{R_n\}_1^\infty \subset \mathscr{P}_f(H)$ be a sequence satisfying the thesis of Proposition 2.5; then, by Lemma 2.3, there is a sequence $\{Q_n\}_1^\infty \subset \mathscr{P}_f(H)$ such that

$$\sigma(A_{Q_n}) \setminus (\mathfrak{I} + \beta_n \mathbf{B}) = \sigma(A_{R_n}) \setminus (\mathfrak{I} + \beta_n \mathbf{B}) = \sigma(A) \setminus (\mathfrak{I} + \beta_n \mathbf{B}),$$

(10)

$$\sigma(A_{Q_n}) \cap (\mathfrak{I} + \beta_n \mathbf{B}) \subset \mu + \beta_n \mathbf{B}$$

and

(11) $$\|(1 - Q_n) R_n\| \leqslant 3\sqrt{\beta_n}.$$

Let $\Omega_n \subset \Omega$ be a finite set such that $\operatorname{dist}(\Omega_n, \Omega) < \beta_n/2$. Then it follows from the proof of Proposition 2.1 that there is a sequence $\{S_n\}_1^\infty \subset \mathscr{P}_f(H)$ such that $S_n Q_n = S_n A Q_n = S_n A^* Q_n = 0$ and

(12) $$\operatorname{dist}(\Omega_n, \sigma(A_{S_n})) \leqslant \frac{\beta_n}{2}.$$

Then $P_n = S_n + Q_n \in \mathscr{P}_f(H)$ and since $A_{P_n} = A_{S_n} \oplus A_{Q_n}$ we have $\sigma(A_{P_n}) = \sigma(A_{S_n}) \cup \sigma(A_{Q_n})$. This with (10) and (12) shows that the spectra of the compressions A_{P_n} have the desired properties.

Making use of the relations $1 - P_n \leqslant 1 - Q_n$, $R_n \xrightarrow{s} 1$ and (11) we have for any $x \in H$

$$\|(1 - P_n)x\| \leqslant \|(1 - P_n) R_n x\| + \|(1 - P_n)(1 - R_n)x\| \leqslant$$

$$\leqslant \|(1 - Q_n) R_n x\| + \|(1 - R_n)x\| \leqslant 3\sqrt{\beta_n}\|x\| + \|(1 - R_n)x\| \to 0.$$

Thus $P_n \xrightarrow{s} 1$ and this ends the proof.

The theorem above and (1) imply immediately the following corollary.

COROLLARY 2.7. *If* $\operatorname{Im} A \in \mathscr{LC}(H) \setminus \mathfrak{S}_\omega$ *then a closed set* Ω *belongs to* $S(A)$ *if and only if*

$$\sigma(A) \setminus W_e(A) \subset \Omega \subset \sigma(A) \cup W_e(A) \quad \textit{and} \quad \Omega \cap W_e(A) \neq \emptyset.$$

REMARK 2.8. It follows from Theorem 1 of [4], Theorem 2 of [5] and Theorem 2.6 that for any $A \in \mathscr{L}(H)$ and $\Omega \in S(A)$ there exists a sequence $\{P_n\}_1^\infty \subset \mathscr{P}_f(H)$ such that $P_n \xrightarrow{s} 1$ and $\operatorname{dist}(\sigma(A_{P_n}), \Omega) \to 0$. Using the spectral perturbation theorem and the proposition below one may show that the projections $\{P_n\}$ may be also assumed to be ordered.

PROPOSITION 2.9. *Suppose that* $\{P_n\}_1^\infty \subset \mathscr{P}_f(H)$, $P_n \xrightarrow{s} 1$ *and* ε_n *is a nonincreasing sequence of positive numbers,* $\varepsilon_n \to 0$. *Then there exists an increasing sequence* $\{k_n\}_1^\infty$ *of natural numbers and a sequence* $\{Q_n\}_1^\infty \subset \mathscr{P}_f(H)$ *such that* $Q_n \leqslant Q_{n+1}$, $Q_n \xrightarrow{s} 1$ *and* $\|P_{k_n} - Q_n\| \leqslant \varepsilon_{k_n}$.

Sketch of the proof. We choose an increasing sequence $\{k_n\}_1^\infty$ of natural numbers in such a way that $\|P_{k_n} - P_{k_{n+1}}\| \leqslant 2^{-n}\varepsilon_{k_n}$. Then setting $Q_{n,0} = P_{k_n}$ we can construct sequences $\{Q_{n,j}\}_{n=1}^\infty \subset \mathscr{P}_f(H)$ $(j = 1, 2, \ldots)$ such that $Q_{n,j+1} \leqslant Q_{n+1,j}$ and $\|Q_{n,j+1} - Q_{n,j}\| \leqslant 2^{-n-j}\varepsilon_{k_{n+j}}$. The required projections Q_n are defined by $Q_n = \lim_{j\to\infty} Q_{n,j}$.

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