J. OPERATOR THEORY
12(1984), 221-233

© Copyright by INCREST, 1984

CONTRACTIONS WITH (σ, c) DEFECT OPERATORS

MITSURU UCHIYAMA

Let T be a contraction, that is $\|T\| \leqslant 1$, on a separable Hilbert space $\mathscr{H}$. Then $D_T = (I - T^*T)^{1/2}$ is well defined, which is called defect operator of T. In this case we have $\sigma(T) \subset \widetilde{\mathbf{D}}$, where $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ denote the open unit disc and its closure respectively. Contractions which have defect operators of finite ranks have been studied by many mathematicians. For investigations of contraction T with $D_T \in (\sigma, c)$, that is $I - T^*T \in (\tau, c)$, where (σ, c) and (τ, c) denote the Hilbert Schmidt class and the trace class respectively, some mathematicians added the condition $\sigma(T) \neq \widetilde{\mathbf{D}}$. Such a contraction T was called weak contraction by M. G. Kreĭn. The spectral decomposition for weak contractions T or accretive operators

$$(I + T)(I - T)^{-1}$$

were obtained by Sz.-Nagy and Foiaş, Brodskii and Ginzburg (cf. [7]).

Since T is a contraction, $\|T^n x\|$ is decreasing for each x. Sz.-Nagy and Foiaş defined contractions' classes as following:

$$C_{1\cdot} = \{T : \lim_{n\to\infty} \|T^n x\| > 0 \quad \text{for each } x \neq 0\},$$

$$C_{0\cdot} = \{T : \lim_{n\to\infty} \|T^n x\| = 0 \quad \text{for each } x\},$$

$$C_{\cdot 1} = \{T : T^* \in C_{1\cdot}\}, \quad C_{\cdot 0} = \{T : T^* \in C_{0\cdot}\},$$

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j} \quad (0 \leqslant i,j \leqslant 1).$$

These formal notations are playing important roles in the study of contractions. In particular they showed that every weak contraction in C_{00} belongs to C_0 (about this notation see [7]), and that every weak contraction has a C_0-part and a C_{11}-part. The Jordan models for weak contractions were constructed by P.Y.Wu [10].

In [9] the author applied the results of Bercovici and Voiculescu's paper [1] to investigate a contraction T satisfying $\sigma(T) = \widetilde{\mathbf{D}}$ and $D_T \in (\sigma, c)$, in particular, to show that T belongs to C_{10} iff there is a quasi-affinity X such that

$$XT = S_{\mathscr{E}} X,$$

where $\mathscr{E}$ is a Hilbert space with $\dim \mathscr{E} = -\text{index}\, T$ (this "index" is Fredholm index) and $S_{\mathscr{E}}$ is the unilateral shift on $\ell^2_+(\mathscr{E})$. From the results of [9], he conjectured that a contraction in C_{00} with (σ, c)-defect operator belongs to C_0. In [8] Takahashi and Uchiyama showed that this was true.

In this note we study the structure of a contraction T with D_T in (σ, c). In particular, setting

$$\alpha = \min\{\dim \mathrm{N}(T-\lambda) : \lambda \in \mathbf{D}\}, \quad \beta = \min\{\dim \mathrm{N}(T^* - \lambda) : \lambda \in \mathbf{D}\},$$

where $\mathrm{N}(T) = \{x : Tx = 0\}$, we will show that there are vector valued holomorphic functions $h_i(\lambda)$, $f_j(\lambda)$ $(1 \leqslant i \leqslant \alpha,\ 1 \leqslant j \leqslant \beta)$ defined on $\mathbf{D}$ satisfying

$$(T-\lambda)h_i(\lambda) \equiv 0, \quad (T^* - \lambda)f_j(\lambda) \equiv 0,$$

and that if $\alpha = \beta = 0$, then T is a weak contraction.

In Section 4, we will study the weighted shifts with finite matrices weights.

From now on, we use the symbol $D(T)$ instead of D_T for convenience.

1. UPPER TRIANGULATION

Let T be a contraction on $\mathscr{H}$ with $D(T) \in (\sigma, \mathrm{c})$. Then, since $\sum_i (1 - \|Te_i\|^2) < \infty$ for a C.O.N.B. $\{e_i\}$ of $\mathscr{H}$, we have $\dim \mathrm{N}(T) < \infty$. Let $T = V|T|$ be a polar decomposition for T with V chosen isometric or co-isometric, and $\dim \mathrm{N}(V) < \infty$. Therefore $V - \lambda$ is a semi-Fredholm operator and $\text{index}(V-\lambda)$ is constant in $\mathbf{D}$. Since $T - \lambda = V - \lambda - V(I - |T|)$, $T - \lambda$ is a semi-Fredholm operator, and $\text{index}(T - \lambda)$ is constant in $\mathbf{D}$ and less than ∞, because $I - |T| \in (\tau, \mathrm{c})$. Thus we have

$$\sigma(T) \cap \mathbf{D} = \{\sigma_{\mathrm{p}}(T) \cup \overline{\sigma_{\mathrm{p}}(T^*)}\} \cap \mathbf{D}, \tag{1.1}$$

where $\overline{\sigma_{\mathrm{p}}(T^*)}$ is the complex conjugate of $\sigma_{\mathrm{p}}(T^*)$.

Now we notice that if $\dim \mathrm{N}(T^*)$ is finite, then $T - \lambda$ is a Fredholm operator for each $\lambda \in \mathbf{D}$.

From the definition of $C_{1\cdot}$ it follows that

$$\sigma_{\mathrm{p}}(T) \cap \mathbf{D} = \emptyset \quad \text{for } T \in C_{1\cdot}. \tag{1.2}$$

In this section we obtain an upper triangulation of T whose diagonal elements were already studied.

The next lemma is trivial, but for the sake of completeness we prove it.

LEMMA 1.1. *Let Y be a bounded operator and F a Fredholm operator such that $FY \in (\tau, \mathrm{c})$. Then we have $Y \in (\tau, \mathrm{c})$.*

Proof. There are bounded operators F' and P such that

$$F'F = I - P, \quad \text{range}\, P = \mathrm{N}(F).$$

Thus $(I-P)Y = F'FY \in (\tau, c)$ implies $Y = (I-P)Y + PY \in (\tau, c)$. Q.E.D.

LEMMA 1.2. *Let T be a contraction with $D(T) \in (\sigma, c)$ and let*

$$T = \begin{bmatrix} T_{0\cdot} & B \\ 0 & T_{1\cdot} \end{bmatrix} \tag{1.3}$$

be the decomposition of T such that $T_{0\cdot} \in C_{0\cdot}$, $T_{1\cdot} \in C_{1\cdot}$ (see [7]*). Then $D(T_{0\cdot})$ and $D(T_{1\cdot})$ are in (σ, c) and B in (τ, c).*

Proof. Since $I - T^*T \in (\tau, c)$, $I - T_{0\cdot}^*T_{0\cdot}$, $B^*T_{0\cdot}$ and $I - (B^*B + T_{1\cdot}^*T_{1\cdot})$ belong to (τ, c), where I of "$I - T_{0\cdot}^*T_{0\cdot}$" is the identity on the space where $T_{0\cdot}$ is defined. From the next lemma, it follows that $T_{0\cdot}$ is a Fredholm operator. Thus, by Lemma 1.1, we have $B \in (\tau, c)$ and hence $I - T_{1\cdot}^*T_{1\cdot} \in (\tau, c)$. Q.E.D.

LEMMA 1.3. *Suppose $T_{0\cdot} \in C_{0\cdot}$ and $D(T_{0\cdot}) \in (\sigma, c)$. Then $T_{0\cdot}$ is a Fredholm operator.*

Proof. Let

$$T_{0\cdot} = \begin{bmatrix} T_{01} & A \\ 0 & T_0 \end{bmatrix} \tag{1.4}$$

be the decomposition of $T_{0\cdot}$ satisfying $T_{01} \in C_{01}$ and $T_0 \in C_{00}$ ([7]). Since $I - T_{0\cdot}^*T_{0\cdot} \in \in (\tau, c)$, $I - T_{01}^*T_{01}$, A^*T_{01} and $I - (A^*A + T_0^*T_0)$ are in (τ, c) too. From (1.2) we have $\sigma_p(T_{01}^*) \cap \mathbf{D} = \emptyset$, hence T_{01} is a Fredholm operator. Consequently, from Lemma 1.1, $A \in (\tau, c)$ and hence $I - T_0^*T_0 \in (\tau, c)$. Since $T_0 \in C_{00}$, we have $T_0 \in C_0$ [8], which implies $\dim \mathrm{N}(T_0) = \dim \mathrm{N}(T_0^*) < \infty$ [7]. Therefore T_0 is a Fredholm operator. Thus

$$T_{0\cdot} = \begin{bmatrix} T_{01} & 0 \\ 0 & T_0 \end{bmatrix} + \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

is a Fredholm operator. Q.E.D.

LEMMA 1.4. *Suppose $T_{1\cdot} \in C_{1\cdot}$ and $D(T_{1\cdot}) \in (\sigma, c)$ and let*

$$T_{1\cdot} = \begin{bmatrix} T_{11} & F \\ 0 & T_{\cdot 0} \end{bmatrix}$$

be a decomposition of $T_{1\cdot}$ such that $T_{11} \in C_{11}$, $T_{\cdot 0} \in C_{\cdot 0}$ ([7]*). Then $D(T_{11})$ and $D(T_{\cdot 0})$ are in (σ, c) and F in (τ, c), and $T_{\cdot 0} \in C_{10}$.*

Proof. $I - T_{11}^*T_{11}$, F^*T_{11} and $I-(F^*F + T_{\cdot 0}^* T_{\cdot 0})$ belong to (τ, c). From (1.2) we have

$$\sigma_p(T_{11}) \cap \mathbf{D} = \emptyset \quad \text{and} \quad \sigma_p(T_{11}^*) \cap \mathbf{D} = \emptyset,$$

and hence, by (1.1) we have

$$\sigma(T_{11}) \cap \mathbf{D} = \emptyset. \tag{1.5}$$

Thus $F \in (\tau, c)$ and hence $I - T_{\cdot 0}^* T_{\cdot 0} \in (\tau, c)$. To show $T_{\cdot 0} \in C_{10}$, decompose $T_{\cdot 0}$ as

$$T_{\cdot 0} = \begin{bmatrix} T_{00} & F_3 \\ 0 & T_{10} \end{bmatrix}, \tag{1.6}$$

where $T_{00} \in C_{00}$, $T_{10} \in C_{10}$. Then we have $I - T_{00}^* T_{00} \in (\tau, c)$ and hence $T_{00} \in C_0$, from which we get

$$\sigma(T_{00}) \cap \mathbf{D} \neq \mathbf{D}. \tag{1.7}$$

Denote the space on which $T_{1\cdot}$ is defined by $\mathscr{L}$, and let $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ be the decomposition of $\mathscr{L}$ corresponding to

$$T_{1\cdot} = \begin{bmatrix} T_{11} & F_1 & F_2 \\ 0 & T_{00} & F_3 \\ 0 & 0 & T_{10} \end{bmatrix},$$

where $[F_1, F_2] = F$. Set

$$T_2 = \begin{bmatrix} T_{11} & F_1 \\ 0 & T_{00} \end{bmatrix}. \tag{1.8}$$

Then, since $T_2 = T_{1\cdot} \mid \mathscr{L}_1 \oplus \mathscr{L}_2$, we have $T_2 \in C_{1\cdot}$ and $D(T_2) \in (\sigma, c)$. The above triangulation of T_2 implies that

$$\sigma(T_2) \subset \sigma(T_{11}) \cup \sigma(T_{00}).$$

From this relation and (1.5), (1.7), it follows that

$$\sigma(T_2) \cap \mathbf{D} \neq \mathbf{D}.$$

Therefore T_2 is a weak contraction. The $C_0 - C_{11}$ decomposition of T_2 ([7]) implies T_2 has no C_0-part, because $T_2 \in C_{1\cdot}$, and so $T_2 \in C_{11}$. From (1.8) we have $T_{00}^* = T_2^* \mid \mathscr{L}_2$, which belongs to $C_{0\cdot}$ and $C_{1\cdot}$; this is impossible. Thus $\mathscr{L}_2$ reduces to 0, so that from (1.6) we have $T_{\cdot 0} = T_{10} \in C_{10}$. Q.E.D.

THEOREM 1.5. *Let T be a contraction with $D(T)\in(\sigma, c)$. Then we have an upper triangulation:*

$$T = \begin{bmatrix} T_{01} & & & \\ 0 & T_0 & & * \\ 0 & 0 & T_{11} & \\ 0 & 0 & 0 & T_{10} \end{bmatrix},$$

where $D(T_{01})$, $D(T_0)$, $D(T_{11})$ and $D(T_{10})$ belong to (σ, c), and $T_{01}\in C_{01}$, $T_0\in C_0$, $T_{11}\in C_{11}$, $T_{10}\in C_{10}$, and $*$ belongs to (τ, c).

Proof. First decompose T as in Lemma 1.2, next decompose $T_{0\cdot}$ as in (1.4). In the proof of Lemma 1.3 we showed that T_{01} and T_0 satisfy the conditions of the theorem. At last decompose $T_{1\cdot}$ as in Lemma 1.4 and set $T_{10} = T_{\cdot 0}$. Q.E.D.

DEFINITION. The above upper triangulation is called the *canonical triangulation* for T with $D(T)\in(\sigma, c)$.

REMARK. We showed that T_{01} and T_0 are Fredholm operators and T_{11} is invertible. But $\dim N(T_{10}^*)$ may be infinite.

2. EIGENVECTORS

Let T be a contraction on $\mathscr{H}$ with $D(T)\in(\sigma, c)$. Set

$$\alpha = \min\{\dim N(T-\lambda) : \lambda\in \mathbf{D}\}, \quad \beta = \min\{\dim N(T^*-\lambda) : \lambda\in\mathbf{D}\},$$

$$\iota(\lambda) = \dim N(T-\lambda) - \alpha\ (<\infty), \quad \Lambda = \{\lambda\in\mathbf{D} : \iota(\lambda) > 0\}.$$

Now we note that if a bounded operator A is decomposed as $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3\end{bmatrix}$, where A_1 is a surjection, then $\dim N(A) = \dim N(A_1) + \dim N(A_3)$. In fact, we have

$$N(A) = (N(A_1)\oplus\{0\})\oplus\{(-B^{-1}A_2x, x) : x\in N(A_3)\},$$

where B is the restriction of A_1 to $N(A_1)^{\perp}$.

THEOREM 2.1. *Let T be a contraction with $D(T)\in(\sigma, c)$, and consider the canonical triangulation of T. Then*

$$\alpha = \dim N(T_{01}) \quad\text{and}\quad \beta = \dim N(T_{10}^*).$$

Proof. At first, we notice (1.3). Since $\sigma_p(T_{1\cdot})\cap\mathbf{D} = \emptyset$ it is not difficult to show $N(T-\lambda) = N(T_{0\cdot}-\lambda)$ for $\lambda\in\mathbf{D}$. Next we notice (1.4). Since $D(T_{01})\in\in(\sigma, c)$ and $\sigma_p(T_{01}^*)\cap\mathbf{D} = \emptyset$, $(T_{01}-\lambda)$ is a surjection for each $\lambda\in\mathbf{D}$. Thus we have

$$\dim N(T-\lambda) = \dim N(T_{0\cdot}-\lambda) = \dim N(T_{01}-\lambda) + \dim N(T_0-\lambda) = \tag{2.1}$$
$$= \operatorname{index}(T_{01}-\lambda) + \dim N(T_0-\lambda) = \operatorname{index} T_{01} + \dim N(T_0-\lambda).$$

$T_0 \in C_0$ implies that $\sigma(T_0) \cap \mathbf{D}$ is countable. Hence we have

$$\alpha = \operatorname{index} T_{01} = \dim N(T_{01}).$$

To show $\beta = \dim N(T_{10}^*)$, take the adjoint of (1.3), that is

$$T^* = \begin{bmatrix} T_{1\cdot}^* & B^* \\ 0 & T_{0\cdot}^* \end{bmatrix}.$$

Since $\sigma_p(T_{1\cdot}) \cap \mathbf{D} = \emptyset$ and $D(T_{1\cdot}) \in (\sigma, c)$, $(T_{1\cdot}^* - \lambda)$ is a surjection for each $\lambda \in \mathbf{D}$. Thus we have

$$\dim N(T^*-\lambda) = \dim N(T_{1\cdot}^*-\lambda) + \dim N(T_{0\cdot}^*-\lambda).$$

From (1.4), it follows that $N(T_{0\cdot}^*-\lambda) = N(T_0^*-\lambda)$ for $\lambda \in \mathbf{D}$, because $\sigma_p(T_{01}^*) \cap \cap \mathbf{D} = \emptyset$. Now we notice the decomposition of $T_{1\cdot}$ in Lemma 1.4 and remark that we set T_{10} instead of $T_{\cdot 0}$ in the canonical triangulation of T. Since $\sigma_p(T_{11}^*) \cap \mathbf{D} = \emptyset$, it is clear that $N(T_{1\cdot}^*-\lambda) = N(T_{10}^*-\lambda)$ for $\lambda \in \mathbf{D}$, so that

$$\dim N(T^*-\lambda) = \dim N(T_{10}^*-\lambda) + \dim N(T_0^*-\lambda).$$

Consequently we have $\beta = \dim N(T_{10}^*)$. Q.E.D.

COROLLARY 2.2. *Let T be a contraction with $D(T) \in (\sigma, c)$. Then*

$$\sum_{\lambda \in A} (1-|\lambda|)\cdot \iota(\lambda) < \infty.$$

Proof. From (2.1), we have $\iota(\lambda) = \dim N(T_0-\lambda)$. Thus, by [7] we can conclude the proof. Q.E.D.

THEOREM 2.3. *Let T be a contraction with $D(T) \in (\sigma, c)$. Then there are holomorphic vector valued functions $h_i(\lambda)$, $f_j(\lambda)$, $(1 \leq i \leq \alpha,\ 1 \leq j \leq \beta)$ defined on $\mathbf{D}$ such that*

$$(T-\lambda)h_i(\lambda) \equiv 0 \quad (T^*-\lambda)f_j(\lambda) \equiv 0,$$

and for each $\lambda \in \mathbf{D}$, $\{h_1(\lambda), \ldots, h_\alpha(\lambda)\}$ are linearly independent, and also $\{f_1(\lambda), \ldots, f_\beta(\lambda)\}$ are. In this case, setting $\mathcal{L}^\perp = \bigvee \{h_i(\lambda), f_j(\lambda): i, j, \lambda\}$, $P_{\mathcal{L}}T|\mathcal{L}$ is a weak contraction and satisfies $(P_{\mathcal{L}}T|\mathcal{L})^n = P_{\mathcal{L}}T^n|\mathcal{L}$.

Proof. We showed that T_{01} in the canonical triangulation of T is a Fredholm operator. Hence

$$T_{01}^*(I - T_{01}T_{01}^*) = (I - T_{01}^*T_{01})T_{01}^* \in (\tau, \mathrm{c})$$

implies, by Lemma 1.1, $D(T_{01}^*) \in (\sigma, \mathrm{c})$. Therefore there is a quasi-affinity X such that $XT_{01}^* = S_{\mathscr{E}}X$, where $\dim \mathscr{E} = -\operatorname{index} T_{01}^* = \dim \mathrm{N}(T_{01}) = \alpha < \infty$ [9]. Let $\{e_1, \ldots, e_\alpha\}$ be a C.O.N.B. of $\mathscr{E}$. Then $g_i(\lambda) = \{e_i, \lambda e_i, \lambda^2 e_i, \ldots\}$ $(1 \leqslant i \leqslant \alpha)$ is a holomorphic function defined on $\mathbf{D}$ with value in $\ell_+^2(\mathscr{E})$. And for each $\lambda \in \mathbf{D}$, $\{g_1(\lambda), \ldots, g_\alpha(\lambda)\}$ are orthogonal to each other. It is trivial to show that

$$(S_{\mathscr{E}}^* - \lambda)g_i(\lambda) \equiv 0, \quad \bigvee_{i,\lambda} g_i(\lambda) = \ell_+^2(\mathscr{E}).$$

Since $T_{01}X^* = X^*S_{\mathscr{E}}^*$,

$$h_i(\lambda) = \begin{bmatrix} X^*g_i(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1 \leqslant i \leqslant \alpha)$$

satisfy the conditions given in the theorem. Since $T_{10} \in C_{10}$ and $D(T_{10}) \in (\sigma, \mathrm{c})$, there is a quasi-affinity Y such that

$$YT_{10} = S_{\mathscr{F}}Y, \quad \text{where } \dim \mathscr{F} = \beta \leqslant \infty.$$

We can show the existence of $f_j(\lambda)$ in the same way as above; hence we omit it. We must show the latter half of assertions. To this end, notice that $\{h_i(\lambda) : 1 \leqslant i \leqslant \alpha, \lambda \in \mathbf{D}\}$ and $\{f_j(\lambda) : 1 \leqslant j \leqslant \beta, \lambda \in \mathbf{D}\}$ span the spaces on which T_{01} and T_{10}, respectively, are defined. Thus, by Theorem 1.5 we have

$$P_{\mathscr{L}}T|\mathscr{L} = \begin{bmatrix} T_0 & * \\ 0 & T_{11} \end{bmatrix}. \tag{2.2}$$

In this case $*$ clearly belongs to (τ, c). Now we set $T_{\mathscr{L}} = P_{\mathscr{L}}T|\mathscr{L}$. From (2.2), $D(T_0) \in (\sigma, \mathrm{c})$ and $D(T_{11}) \in (\sigma, \mathrm{c})$ imply that $D(T_{\mathscr{L}}) \in (\sigma, \mathrm{c})$. Since T_{11} is invertible, we have

$$\sigma_{\mathrm{p}}(T_{\mathscr{L}}) = \sigma_{\mathrm{p}}(T_0) \quad \sigma_{\mathrm{p}}(T_{\mathscr{L}}^*) = \boldsymbol{\sigma}_{\mathbf{p}}(\boldsymbol{T}_{\mathbf{0}}^*).$$

$T_0 \in C_0$ implies that $\sigma_{\mathrm{p}}(T_0^*) = \overline{\sigma_{\mathrm{p}}(T_0)} \neq \mathbf{D}$ [7]. Thus by (1.1) we have $\sigma(T_{\mathscr{L}}) \cap \mathbf{D} = \sigma_{\mathrm{p}}(T_0) = \Lambda \neq \mathbf{D}$. Thus $T_{\mathscr{L}}$ is a weak contraction. $(P_{\mathscr{L}}T|\mathscr{L})^n = P_{\mathscr{L}}T^n|\mathscr{L}$ is obvious. Q.E.D.

THEOREM 2.4. *Let T be a contraction with $D(T) \in (\sigma, \mathrm{c})$; then the following conditions are equivalent:*

(a) $\alpha = \beta = 0$;

(b) *T is a weak contraction;*

(c) *T is decomposable (about definition see* [2]*).*

Proof. (a)$\Rightarrow$(b): From Theorem 2.1, $N(T_{01})=0$, which implies T_{01} is a weak contraction. Therefore there is a C_0-C_{11} decomposition of T_{01}, but this is impossible because $T_{01}\in C_{01}$. Thus the space on which T_{01} is defined reduces to 0. Similarly the space on which T_{10} is defined reduces to 0. Thus $\mathscr{L}$ in Theorem 2.3 is $\mathscr{H}$. Therefore T is a weak contraction.

(b) $\Rightarrow$ (c): This was shown by Jafarian [5].

(c) $\Rightarrow$ (a): Since decomposable T has the single valued extension property, $\alpha=0$ follows. Thus for $\lambda\notin\Lambda$, $(T-\lambda)$ is an injective semi-Fredholm operator. Hence $\sigma_p(T)\cap\mathbf{D}\subset\Lambda$. Thus we have $\sigma(T)\cap\mathbf{D}\subset\Lambda$ (see p. 30 of [2]). Consequently $\beta=0$. Q.E.D.

PROPOSITION 2.5. *Let T be a contraction on $\mathscr{H}$ with $D(T)\in(\sigma,\mathrm{c})$. Then $T\in C_{10}$ if and only if there are vector valued holomorphic functions $h_i(\lambda)$ such that*

$$(T^*-\lambda)h_i(\lambda)\equiv 0,\qquad \bigvee_{i,\lambda} h_i(\lambda)=\mathscr{H}.$$

Proof. "Only if" part follows from Theorem 2.3 and its proof. We must show "if" part. Since

$$T^{*n}h_i(\lambda)=\lambda^n h_i(\lambda)\to 0\quad\text{as } n\to\infty,$$

T^{*n} strongly converges to 0 on linear span of $\{h_i(\lambda): i,\lambda\}$. Suppose

$$T^{*n}x_i\to 0\quad(n\to\infty)\quad\text{and}\quad x_i\to x\quad(i\to\infty).$$

Since

$$\|T^{*n}x\|\leqslant\|T^{*n}x_i\|+\|T^{*n}(x-x_i)\|\leqslant\|T^{*n}x_i\|+\|x-x_i\|,$$

we have $\varlimsup_{n\to\infty}\|T^{*n}x\|\leqslant\|x-x_i\|$. Since we can make the right side arbitrary small, $T^{*n}x\to 0$ $(n\to\infty)$. Thus T belongs to $C_{\cdot 0}$, therefore the canonical triangulation of T becomes

$$T=\begin{bmatrix} T_0 & * \\ 0 & T_{10}\end{bmatrix}.$$

Let P be the orthogonal projection to the space which T_0 is defined on. Then we have

$$0=P(T^*-\lambda)h_i(\lambda)=P(T^*-\lambda)Ph_i(\lambda)=(T_0^*-\lambda)Ph_i(\lambda).$$

Since $\sigma_p(T_0^*)$ are countable, $Ph_i(\lambda)\equiv 0$. Consequently $P\mathscr{H}=0$ and hence $T=T_{10}$. Q.E.D.

Alternately we have:

PROPOSITION 2.6. *Let T be a contraction on $\mathscr{H}$ with $D(T)\in(\sigma,\mathrm{c})$. Then $T\in C_{01}$ iff there are vector valued holomorphic functions $f_j(\lambda)$ defined on $\mathbf{D}$ such that*

$$(T-\lambda)f_j(\lambda)\equiv 0\quad \bigvee_{j,\lambda} f_j(\lambda)=\mathscr{H}.$$

3. m-ACCRETIVE OPERATORS

Let A be an m-accretive operator densely defined in $\mathscr{H}$ (about the definition see [6]). Then

$$T = (A - I)(A + I)^{-1} \tag{3.1}$$

is a contraction defined on $\mathscr{H}$ and

$$\sigma_p(T) \not\ni 1 \quad \text{and} \quad T^* = (A^* - I)(A^* + I)^{-1}$$

(see Chapter IV of [7]). It is trivial to show that

$$((I - T^*T)h, h) = 4\mathrm{Re}(A(A + I)^{-1}h, (A + I)^{-1}h) \quad \text{for } h \in \mathscr{H}.$$

Since $A(A + I)^{-1}$ and $(A + I)^{-1}$ are bounded, we have the relation:

$$I - T^*T \in (\tau, c) \Leftrightarrow u(A) \in (\tau, c),$$

where $u(A) = \mathrm{Re}((A^* + I)^{-1}A(A + I)^{-1})$. In this section we denote the open right half plane by Ω. The mapping

$$\psi : \mu \to \frac{\mu - 1}{\mu + 1}$$

transforms Ω onto $\mathbf{D}$. It is clear that

$$(A - \mu)x = 0 \Leftrightarrow (T - \psi(\mu))(A + I)x = 0. \tag{3.2}$$

Set

$$\alpha = \min\{\dim \mathrm{N}(A - \mu) : \mu \in \Omega\}, \quad \beta = \min\{\dim \mathrm{N}(A^* - \mu) : \mu \in \Omega\},$$

$$\imath(\mu) = \dim \mathrm{N}(A - \mu) - \alpha, \quad \Gamma = \{\mu : \imath(\mu) > 0\}.$$

PROPOSITION 3.1. *Let A be an m-accretive operator densely defined in $\mathscr{H}$. If $u(A) \in (\tau, c)$, then it follows that*

$$\sum_{\mu \in \Gamma} \left(\frac{\mathrm{Re}\,\mu}{1 + |\mu|^2}\right) \cdot \imath(\mu) < \infty.$$

Proof. Since the range of $(A + I)$ is $\mathscr{H}$, by (3.2), we have

$$\dim \mathrm{N}(A - \mu) = \dim \mathrm{N}(T - \psi(\mu)), \quad \alpha = \min\{\dim \mathrm{N}(T - \lambda) : \lambda \in \mathbf{D}\},$$

$$\dim \mathrm{N}(T - \lambda) - \alpha = \dim \mathrm{N}(A - \psi^{-1}(\lambda)) - \alpha = \imath(\psi^{-1}(\lambda)),$$

$$\{\lambda : \imath(\psi^{-1}(\lambda)) > 0\} = \psi(\Gamma).$$

Thus from Corollary 2.2, it follows that

$$\sum_{\lambda\in\psi(\Gamma)} (1 - |\lambda|)\cdot\iota(\psi^{-1}(\lambda)) < \infty$$

so that

$$\sum_{\mu\in\Gamma} (1 - |\psi(\mu)|)\cdot\iota(\mu) < \infty.$$

Therefore we have

$$\sum_{\mu\in\Gamma} \frac{\mathrm{Re}\,\mu}{1 + |\mu|^2}\cdot\iota(\mu) < \infty \qquad \text{(cf. p. 132 of [4]).}$$

THEOREM 3.2. *Let A be an m-accretive operator densely defined in $\mathscr{H}$. If $u(A) \in \in (\tau, c)$, then there are vector valued holomorphic functions $x_i(\mu), y_j(\mu)$, $(1 \leqslant i \leqslant \alpha$, $1 \leqslant j \leqslant \beta)$ defined on Ω such that*

$$(A - \mu)x_i(\mu) \equiv 0 \quad \textit{and} \quad (A^* - \mu)y_j(\mu) \equiv 0 .$$

Proof. From Theorem 2.3, for T defined by (3.1) there are holomorphic functions $h_i(\lambda)$ $(1 \leqslant i \leqslant \alpha)$ such that

$$(T - \lambda)h_i(\lambda) \equiv 0 .$$

Then

$$x_i(\mu) = (A + I)^{-1}h_i(\psi(\mu))$$

is a holomorphic function defined on Ω, and for each $\mu \in \Omega$, $x_i(\mu)$ belongs to the domain of A. From (3.2), we have

$$(A - \mu)x_i(\mu) \equiv 0.$$

We can similarly show the existence of $y_j(\mu)$ from the alternate relation of (3.2), that is

$$(A^* - \mu)x = 0 \Leftrightarrow (T^* - \psi(\mu))(A^* + I)x = 0.$$

Q.E.D.

4. WEIGHTED UNILATERAL SHIFTS

In this section we study weighted unilateral shifts with $(\sigma$, c)-defect operators. Let $\mathscr{E}$ be an N-dimensional finite Hilbert space, and A_n $(n = 0, 1, 2, \ldots)$ invertible contractions on $\mathscr{E}$. Let T be a weighted unilateral shift on $\ell^2_+(\mathscr{E})$ defined by

$$T\{x_0, x_1, \ldots\} = \{0, A_0x_0, A_1x_1, \ldots\}$$

LEMMA 4.1. *Let B be an invertible operator on $\mathscr{E}$. Then we have*

$$\|B^{-1}\| \leqslant \frac{\|B\|^{N-1}}{|\det B|}, \qquad \frac{1}{|\det B|} \leqslant \|B^{-1}\|^N.$$

Proof. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_N > 0$ be eigenvalues of B^*B. Then we have

$$\|B^{-1}\|^2 = \|(B^*B)^{-1}\| = \frac{1}{\lambda_N} \leqslant \frac{\lambda_1^{N-1}}{\lambda_1 \ldots \lambda_N} = \frac{\|B^*B\|^{N-1}}{\det(B^*B)}.$$

Thus we have

$$\|B^{-1}\| \leqslant \frac{\|B\|^{N-1}}{|\det B|}.$$

The second inequality follows similarly (cf. p. 200 of [3]). Q.E.D.

Now we remember the following fact: for scalars a_n such that $0 < |a_n| < 1$, $\prod_{n=0}^{\infty} |a_n|$ converges iff $\sum_{n=0}^{\infty} (1 - |a_n|) < \infty$.

THEOREM 4.2. *Let T be the contractive weighted shift defined above. Then the following conditions are equivalent:*

(a) $T \in C_{10}$;

(b) $D(T) \in (\sigma, c)$;

(c) *T is similar with simple shift $S_{\mathscr{E}}$;*

(d) *there is a $\delta > 0$ such that*

$$\|A_n \ldots A_0 x\| \geqslant \delta\|x\| \quad \textit{for every } x \in \mathscr{E} \textit{ and every } n.$$

Proof. (d) $\Rightarrow$ (c): For each m we have

$$\|A_{m+n} \ldots A_m x\| = \|A_{m+n} \ldots A_m A_{m-1} \ldots A_0 (A_{m-1} \ldots A_0)^{-1} x\| \geqslant$$

$$\geqslant \delta\|(A_{m-1} \ldots A_0)^{-1} x\| \geqslant \delta \frac{1}{\|A_{m-1} \ldots A_0\|} \|x\| \geqslant \delta\|x\|,$$

because each A_i is a contraction. Thus for each $f \in \ell^2_+(\mathscr{E})$, we have

$$\|T^n f\| \geqslant \delta\|f\| \quad \text{for every } n.$$

By the well known Sz.-Nagy's theorem, T is similar with an isometry V. Since T belongs to $C_{\cdot 0}$ so does V, hence V is a unilateral shift. Since

$$\dim \mathrm{N}(V^*) = \dim \mathrm{N}(T^*) = \dim \mathscr{E} = N$$

the dimension of the wandering space for V is N. Thus V is unitarily equivalent with $S_{\mathscr{E}}$.

(c) $\Rightarrow$ (a): This is obvious.

(a) $\Rightarrow$ (d): Set

$$l(x) = \lim_{n \to \infty} \|T^n\{x, 0, 0, \ldots\}\| \quad \text{for } x \in \mathscr{E}.$$

Since l is continuous and $l(x) \neq 0$ for $x \neq 0$, there is a $\delta > 0$ such that

$$l(x) \geqslant \delta \quad \text{for } x \text{ in the unit surface of } \mathcal{E}.$$

Since $l(\alpha x) = |\alpha| l(x)$, we have

$$\lim_{n\to\infty} \|A_n \ldots A_0 x\| = l(x) \geqslant \delta\|x\| \quad \text{for } x \in \mathcal{E}.$$

(b) $\Rightarrow$ (d): From

$$\infty > \|I - T^*T\|_1 = \sum_{n=0}^{\infty} \|I - A_n^* A_n\|_1 \geqslant \sum_{n=0}^{\infty} \|I - A_n^* A_n\|,$$

it follows that

$$\prod_{n=0}^{\infty} (1 - \|I - A_n^* A_n\|)$$

converges and we denote its limit by δ^2. In view of

$$\|A_i^{-1}\|^2 = \|(A_i^* A_i)^{-1}\| = \|(I - (I - A_i^* A_i))^{-1}\| \leqslant \frac{1}{1 - \|I - A_i^* A_i\|},$$

we have

$$\|A_n \ldots A_0 x\|^2 \geqslant \frac{\|x\|^2}{\|(A_n \ldots A_0)^{-1}\|^2} \geqslant \frac{\|x\|^2}{\|A_n^{-1}\|^2 \ldots \|A_0^{-1}\|^2} \geqslant$$

$$\geqslant \prod_{i=0}^{n} (1 - \|I - A_i^* A_i\|)\|x\|^2 \geqslant \delta^2 \|x\|^2 \quad \text{for every } n.$$

(d) $\Rightarrow$ (b): Since each A_n is an invertible contractive matrix, we have

$$\|1 - A_n^* A_n\| = 1 - \min\{\lambda : \lambda \in \sigma_p(A_n^* A_n)\} =$$

$$= 1 - \frac{1}{\|(A_n^* A_n)^{-1}\|} = 1 - \frac{1}{\|A_n^{-1}\|^2} \leqslant 2\left(1 - \frac{1}{\|A_n^{-1}\|}\right) \leqslant \qquad \text{from Lemma 4.1,}$$

$$\leqslant 2\left(1 - \frac{|\det A_n|}{\|A_n\|^{N-1}}\right) \leqslant 2(1 - |\det A_n|).$$

From (d) and Lemma 4.1, we have

$$|\det A_n| \ldots |\det A_0| = |\det(A_n \ldots A_0)| \geqslant$$

$$\geqslant \|(A_n \ldots A_0)^{-1}\|^{-N} \geqslant \delta^N,$$

which implies that $\prod_{n=0}^{\infty} |\det A_n|$ converges, and hence

$$\sum_{n=0}^{\infty} \|I - A_n^* A_n\| \leqslant 2 \sum_{n=0}^{\infty} (1 - |\det A_n|) < \infty.$$

Q.E.D.

Acknowledgement. The author thanks the referee for several suggestions.

REFERENCES

1. BERCOVICI, H.; VOICULESCU, D., Tensor operations on characteristic functions of C_0 contractions, *Acta Sci. Math. (Szeged)*, **39**(1977), 205–231.
2. COLOJOARĂ, I.; FOIAŞ, C., *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
3. FUHRMANN, P., *Linear systems and operators in Hilbert space*, McGraw-Hill, 1981.
4. HOFFMAN, K., *Banach spaces of analytic functions*, Englewood Cliffs, N. J., 1962.
5. JAFARIAN, A., Weak contractions of Sz.-Nagy and Foiaş are decomposable, *Rev. Roumaine Math. Pures Appl.*, **22**(1977), 489–497.
6. KATO, T., *Perturbation theory for linear operators*, Springer, 1966.
7. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North-Holland, Akademiai Kiado, 1970.
8. TAKAHASHI, K.; UCHIYAMA, M., Every C_{00} contraction with Hilbert-Schmith defect operator is of class C_0, *J. Operator Theory*, **10**(1983), 331–335.
9. UCHIYAMA, M., Contractions and unilateral shifts, *Acta Sci. Math. (Szeged)*, **46**(1983), 345–356.
10. WU, P. Y., Jordan model for weak contractions, *Acta Sci. Math. (Szeged)*, **40**(1978), 189–196.

MITSURU UCHIYAMA
Department of Mathematics,
Fukuoka University of Education,
Munakata, Fukuoka, 811–41,
Japan.

Received November 4, 1982; revised May 7, 1983.