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ON THE STRUCTURE OF (BCP)-OPERATORS AND RELATED ALGEBRAS. II

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1. INTRODUCTION

In this paper, we extend the methods and results of Part I [6] to a context first explored by B. Chevreau, C. Pearcy, and A. Shields in [3]. The results of Part I may be regarded as statements about the ultraweakly closed algebra generated by a (BCP)-operator (together with the identity operator). Here we study certain analogous algebras generated by several operators.

This paper proceeds from the general to the particular. In the next section we show how the techniques from Part I can be adapted to the study of a class of representations of $H^{\infty}(G)$, for G a bounded domain in the complex plane. Since the development proceeds very much as in Part I, most proofs are only sketched, or omitted. We also prove that the range of such a representation is a reflexive algebra, by adapting the argument given by H. Bercovici, C. Foiaş, J. Langsam, and C. Pearcy in [1] to establish the reflexivity of operators of class (BCP).

In Section 3, we specialize to the case of certain particular domains G which we call circular domains. For an operator suitably related to such a domain G, Chevreau, Pearcy, and Shields constructed in [3] an $H^{\infty}(G)$-functional calculus to which the analysis of Section 2 applies. Moreover, the range of this functional calculus is precisely the ultraweakly closed algebra generated by the operator together with several specific rational functions of it. We point out that two special cases of this situation are the ultraweakly closed algebras generated by certain polynomially bounded operators (in which case G is the open unit disc $\mathbf{D}$), and the ultraweakly closed algebras generated by T and T^{-1} for certain invertible operators T (in which case G is an annular region contained in $\mathbf{D}$). We give an example of a weighted bilateral shift operator of the latter type.

In the final Section 4, we return our attention to the class of (BCP)-operators, and make some concluding remarks which for reasons of space were not included in Part I.

2. REPRESENTATIONS OF $H^\infty(G)$

In general, we continue using the notation of Part I. Thus, $\mathscr{H}$ denotes an infinite-dimensional, separable, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ is the algebra of all (bounded, linear) operators on $\mathscr{H}$. For any bounded domain $G \neq \emptyset$ in $\mathbf{C}$, let $H^\infty(G)$ denote the Banach algebra of all bounded analytic functions on G, under the supremum norm. A subset Λ of G is said to be *dominating* for $H^\infty(G)$ if $\|f\| = \sup_{\lambda\in\Lambda} |f(\lambda)|$ for every $f \in H^\infty(G)$. The space $H^\infty(G)$ may be regarded as a subspace of $L^\infty(G)$, where G is endowed with Lebesgue area measure. The space $L^\infty(G)$ carries a weak* topology as the dual of $L^1(G)$, and $H^\infty(G)$ is in fact weak* closed in $L^\infty(G)$. A sequence in $H^\infty(G)$ converges weak* to zero if and only if it is bounded and converges pointwise to zero on G. Hence, for each $\lambda \in G$, the point evaluation $f \mapsto f(\lambda)$ is weak* continuous on $H^\infty(G)$ (note that $H^\infty(G)$ is the dual of a *separable* Banach space). For these facts, we refer to [7].

Throughout this section, let $G \neq \emptyset$ be a fixed bounded domain in $\mathbf{C}$, let $f_1 \in H^\infty(G)$ be the position function $f_1(\lambda) = \lambda$, and let $\Phi\colon H^\infty(G) \to \mathscr{L}(\mathscr{H})$ be a norm-continuous, unit-preserving algebra homomorphism. We set $T = \Phi(f_1)$. We make the following two standing assumptions:

(2.1) $\sigma_{\mathrm{le}}(T) \cap G$ is dominating for $H^\infty(G)$, and

(2.2) If a sequence $f_n \to 0$ weak* in $H^\infty(G)$, then $\Phi(f_n) \to 0$ in the strong (equivalently, ultrastrong) operator topology.

Set $\mathscr{A} = \Phi[H^\infty(G)]$. As shown in [3], $\mathscr{A}$ is ultraweakly closed, and Φ is a weak* homeomorphism of $H^\infty(G)$ onto $\mathscr{A}$. Let $Q = (\tau c)/{}^\perp\mathscr{A}$; then the bilinear form $\langle A, [K]\rangle = \mathrm{tr}(AK)$ on $\mathscr{A} \times Q$ allows us to identify $\mathscr{A}$ as the dual of Q. For any $\lambda \in G$, the map $A \mapsto [\Phi^{-1}(A)](\lambda)$ is a weak* continuous linear functional on $\mathscr{A}$, and hence corresponds to an element of Q, which we denote by $[C_\lambda]$.

Choose, once and for all, a countable dense subset $\Lambda = \{\lambda_k\}$ of $\sigma_{\mathrm{le}}(T) \cap G$. Note that Λ is obviously dominating for $H^\infty(G)$. The following is proved in the same way as [3, Lemma 3.7].

LEMMA 2.1. *The closed, absolutely convex hull* $\overline{\mathrm{aco}}\{[C_\lambda]: \lambda \in \Lambda\}$ *contains the closed ball in* Q, *centered at the origin, of radius* $\|\Phi\|^{-1}$.

The following lemma is a combination of Lemmas 3.4 and 3.5 of [3].

LEMMA 2.2. *Let* $\lambda \in \sigma_{\mathrm{le}}(T) \cap G$, *and let* $\{x_n\}$ *be an orthonormal sequence in* $\mathscr{H}$ *such that* $\|(T-\lambda)x_n\| \to 0$. *Then for any vector* $y \in \mathscr{H}$,

(A) $$\|[x_n \otimes y]\|_Q \to 0, \text{ and}$$

(B) $$\|[y \otimes x_n]\|_Q \to 0.$$

REMARK. In the present situation, as shown in [3], conclusion (B) holds for an arbitrary orthonormal sequence $\{x_n\}$.

The next result is part of [3, Lemma 3.3].

LEMMA 2.3. *Let $\lambda \in \sigma_{1e}(T) \cap G$ and let $\{x_n\}$ be an orthonormal sequence in $\mathscr{H}$ such that $\|(T-\lambda)x_n\| \to 0$. Then $\|[x_n \otimes x_n] - [C_\lambda]\|_Q \to 0$.*

As in Part I, we can construct a family of mutually orthogonal "drawers"

$$\mathscr{D}_{ij} := \{e_n^{k,i,j} : k, n \geqslant 1\} \quad (i,j \geqslant 1)$$

of orthonormal sets in $\mathscr{H}$ such that

$$(\mathscr{D}1) \qquad \|(T-\lambda_k)\, e_n^{k,i,j}\| \to 0 \quad \text{as } n \to \infty$$

for each $i, j, k \geqslant 1$, and

$$(\mathscr{D}2) \qquad \|[e_n^{k,i,j} \otimes e_n^{l,i,j}]\|_Q \to 0 \quad \text{as } n \to \infty$$

for each $i, j \geqslant 1$, whenever $k \neq l$.

Again as in Part I, we set $\mathscr{M}_i = \bigvee_j \mathscr{D}_{ij}$ and $\mathscr{M}^j = \bigvee_i \mathscr{D}_{ij}$.

We now proceed just as in Section 3 of Part I. One difference here is the appearance of the quantity $\|\Phi\|$ in Lemma 2.1. This necessitates very minor modifications, so that, via the analogues in the present context of Lemmas 3.9 and 3.10 of Part I, we obtain the following.

THEOREM 2.4. *Let $N > 0$ and let $u_i, v_j \in \mathscr{H}$, $[L_{ij}] \in Q$ $(1 \leqslant i, j \leqslant N)$. Let $\varepsilon > 0$ and let*

$$d_{ij} = \|[u_i \otimes v_j] - [L_{ij}]\|.$$

Then there exist $u_i', v_j' \in \mathscr{H}$ such that for all i and j,

$$[u_i' \otimes v_j'] = [L_{ij}],$$

$$\|u_i' - u_i\| < \|\Phi\|^{1/2} \sum_{j=1}^{N} d_{ij}^{1/2} + \varepsilon,$$

and

$$\|v_j' - v_j\| < \|\Phi\|^{1/2} \sum_{i=1}^{N} d_{ij}^{1/2} + \varepsilon.$$

Moreover, we can arrange it so that $u_i' - u_i \in \mathscr{M}_i$ and $v_j' - v_j \in \mathscr{M}^j$.

We now can apply the same inductive construction as in the proof of Theorem 3.12 of Part I to obtain the next result.

THEOREM 2.5. *Let* $[L_{ij}] \in Q$ $(i,j \geqslant 1)$ *and assume that*

$$\sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty \quad \textit{for each } i\,,$$

and

$$\sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty \quad \textit{for each } j\,.$$

Let $\varepsilon > 0$. *Then there exist* $u_i \in \mathscr{M}_i$ *and* $v_j \in \mathscr{M}^j$ *such that for each* $i,j \geqslant 1$, *we have*

$$[u_i \otimes v_j] = [L_{ij}]\,,$$

$$\|u_i\| < \|\Phi\|^{1/2} \sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \frac{\varepsilon}{2^{i-1}}\,,$$

and

$$\|v_j\| < \|\Phi\|^{1/2} \sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \frac{\varepsilon}{2^{j-1}}\,.$$

As in Part I, we obtain the following.

THEOREM 2.6. *Let* $[L_{ij}] \in Q$ $(i,j \geqslant 1)$. *Then there exist orthogonal sequences* $\{x_i\}$ *and* $\{y_j\}$ *in* $\mathscr{H}$ *such that* $[x_i \otimes y_j] = [L_{ij}]$ *for all* i *and* j.

We have now obtained analogues of the results of Section 3 of Part I. Henceforth, we abjure the use made previously of the symbols $\mathscr{M}_i$ and $\mathscr{M}^j$. The analogue of the main result of Section 4 of Part I goes as follows.

THEOREM 2.7. *Let* $\alpha_1, \ldots, \alpha_N$ *be distinct elements of* G. *Then there exist invariant subspaces* $\mathscr{M}$ *and* $\mathscr{N}$ *for the algebra* $\mathscr{A}$, *with* $\mathscr{M} \supset \mathscr{N}$, *such that for each* $f \in H^\infty(G)$, *the compression* $\Phi(f)_{\mathscr{M} \ominus \mathscr{N}}$ *is similar (via a fixed similarity) to* $f(\alpha_1) \oplus \ldots \oplus f(\alpha_N)$ *where each* $f(\alpha)$ *acts on an infinite-dimensional space.*

Proof. Let $\{\lambda_i\}$ be a sequence from $\{\alpha_1, \ldots, \alpha_N\}$ in which each α occurs infinitely often, and let $\{x_i\}$ and $\{y_j\}$ be sequences in $\mathscr{H}$ for which $[x_i \otimes y_j] = \delta_{ij}[C_{\lambda_i}]$, where δ_{ij} is the Kronecker delta. We set

$$\mathscr{M} = \bigvee \{Ax_i : A \in \mathscr{A},\ i > 0\},$$

$$\mathscr{M}_* = \bigvee \{A^* y_j : A \in \mathscr{A},\ j > 0\},$$

and

$$\mathscr{N} = \mathscr{M} \ominus \mathscr{M}_*\,.$$

The rest of the proof consists of verifications like those in the proof of the analogous earlier result. ▨

With one exception, the density theorems in Part I, Section 5 carry over with no additional hypothesis. The exception is that for the analogue of Part I, Corollary 5.5, we require that $0 \in G$. For the interpretation of these density results, we must bear in mind that the members of Q carry information about the entire algebra $\mathscr{A}$, which is likely to be much larger than just the ultraweak closure of the polynomials in T. For example, we have the following.

PROPOSITION 2.8. *The set of vectors $x \in \mathscr{H}$ for which $\mathscr{H} \ominus [\mathscr{A}x]^-$ is infinite-dimensional, is dense in $\mathscr{H}$.*

Also, the remark made at the end of Section 5, Part I concerning counterparts of the density theorems in which the roles of x and y are interchanged, is of special relevance here because the present situation (unlike that in Part I) is not symmetrical with respect to passing to adjoints.

The same techniques used in Section 6 of Part I yield the following result.

THEOREM 2.9. *There exist invariant subspaces $\mathscr{M}_i$ ($i \in \mathbf{Z}$) for the algebra $\mathscr{A}$, such that for each i, $\mathscr{M}_i \supset \mathscr{M}_{i+1}$ and $\sigma_{1e}(T_{\mathscr{M}_i \ominus \mathscr{M}_{i+1}}) \cap G$ is dominating for $H^\infty(G)$; $\bigcap\{\mathscr{M}_i : i \in \mathbf{Z}\} = \{0\}$, and $\bigvee\{\mathscr{M}_i : i \in \mathbf{Z}\} = \mathscr{H}$.*

REMARK. In Part I we obtained the analogue of the previous result by using Proposition 6.2 of that paper to obtain the $\mathscr{M}_i$ for $i \geqslant 0$, and we then obtained the $\mathscr{M}_i$ for $i < 0$ by applying the same proposition to $T^* \mid \mathscr{H} \ominus \mathscr{M}_0$. Since the present situation is not symmetrical with respect to adjoints, a word of explanation is required. The passage to $T^* \mid \mathscr{H} \ominus \mathscr{M}_0$ was merely a technical device to streamline the proof of the earlier result. It is a straightforward exercise to translate into the "language of T" the conclusion of Proposition 6.2 of Part I applied to T^*, and this assertion can then be proved (in the "language of T") using the same techniques as in the proof of the proposition. This approach carries over without difficulty to the present situation.

Notice that if for each i we define

$$\Phi_i(f) = \Phi(f)_{\mathscr{M}_i \ominus \mathscr{M}_{i+1}} \quad \text{for } f \in H^\infty(G),$$

then we obtain a representation of $H^\infty(G)$ on each $\mathscr{M}_i \ominus \mathscr{M}_{i+1}$ which has all the same properties we have assumed of Φ.

The proof of the following result is closely modelled on the proof [1] that (BCP)-operators are reflexive.

THEOREM 2.10. *The algebra $\mathscr{A}$ is reflexive.*

Proof. Recall that the assertion that $\mathscr{A}$ is reflexive means that $\mathrm{Alg}(\mathrm{Lat}\,\mathscr{A})) = \mathscr{A}$, where Alg (Lat ($\mathscr{A}$)) is the algebra of all operators on $\mathscr{H}$ which leave invariant each subspace in the lattice Lat($\mathscr{A}$) of invariant subspaces of the algebra $\mathscr{A}$. Let $A \in \mathrm{Alg}(\mathrm{Lat}(\mathscr{A}))$. As in [1], it suffices to show that whenever $n > 0$, $u_i, v_j \in \mathscr{H}$ $(1 \leqslant i, j \leqslant n)$ and $\sum_{i=1}^{n} [u_i \otimes v_i] = 0$, we have that $\sum_{i=1}^{n} (Au_i, v_i) = 0$. Clearly we may assume that $n > 1$, and it suffices to show that for any $\varepsilon > 0$, there exist u_i' and v_j' in $\mathscr{H}$ $(1 \leqslant i, j \leqslant n)$ such that $\|u_i' - u_i\| < \varepsilon$ and $\|v_j' - v_j\| < \varepsilon$ for all i and j, and $\sum_{i=1}^{n} (Au_i', v_i') = 0$. By Lemma 2.1, we can approximate each $[u_i \otimes v_j]$ arbitrarily well by a sum of the form $\sum_{k=1}^{N(i,j)} \alpha_k^{i,j} [C_{\lambda_k^{i,j}}]$, where the $\alpha_k^{i,j} \in \mathbf{C}$ and the $\lambda_k^{i,j} \in G$. We set $[L_{ij}] = \sum_{k=1}^{N(i,j)} \alpha_k^{i,j} [C_{\lambda_k^{i,j}}]$ for all pairs (i,j) other than $i = j = n$, and we set $[L_{nn}] = -\sum_{i=1}^{n-1} [L_{ii}]$. We then have that $\sum_{i=1}^{n} [L_{ii}] = 0$, and we can ensure that each $\|[u_i \otimes v_j] - [L_{ij}]\|_Q$ is as small as desired. Therefore, applying Theorem 2.4, we can obtain $u_i', v_j' \in \mathscr{H}$ such that $[u_i' \otimes v_j'] = [L_{ij}]$ and $\|u_i' - u_i\| < \varepsilon$, $\|v_j' - v_j\| < \varepsilon$ for all i and j. To complete the proof, we need to show that $\sum_{i=1}^{n} (Au_i', v_i') = 0$. Since $\sum_{i=1}^{n} [u_i' \otimes v_i'] = \sum_{i=1}^{n} [L_{ii}] = 0$, we have, in particular, that $\sum_{i=1}^{n} (q(T)u_i', v_i') = 0$ for all polynomials q. Hence it suffices to show that for some polynomial q, we have $(Au_i', v_j') = (q(T)u_i', v_j')$ for all i and j.

To this end, we set

$$\mathscr{M} = \bigvee \{Bu_i' : B \in \mathscr{A},\ i = 1, \ldots, n\},$$

$$\mathscr{M}_* = \bigvee \{B^* v_j' : B \in \mathscr{A},\ j = 1, \ldots, n\},$$

and

$$\mathscr{N} = \mathscr{M} \ominus \mathscr{M}_*.$$

Clearly, $\mathscr{M}$ and $\mathscr{N}$ are in Lat($\mathscr{A}$), and $\mathscr{M} \supset \mathscr{N}$. If p is the monic polynomial with simple zeros precisely at the $\lambda_k^{i,j}$, then a simple computation shows that $p(T_{\mathscr{M}\ominus\mathscr{N}}) = 0$. Therefore, by Part I, Proposition 4.1, $T_{\mathscr{M}\ominus\mathscr{N}}$ is similar to a diagonal operator with eigenvalues $\lambda_k^{i,j}$. It follows easily from this that $T_{\mathscr{M}\ominus\mathscr{N}}$ is reflexive. We claim now that the algebra

$$\mathscr{A}_{\mathscr{M}\ominus\mathscr{N}} = \{\Phi(f)_{\mathscr{M}\ominus\mathscr{N}} : f \in H^\infty(G)\}$$

consists of the polynomials in $T_{\mathcal{M}\ominus\mathcal{N}}$. This follows from the fact that if $x \in \mathcal{M}\ominus\mathcal{N}$, $T_{\mathcal{M}\ominus\mathcal{N}}x = \lambda x$ $(\lambda \in G)$, and if $f \in H^\infty(G)$, then $\Phi(f)_{\mathcal{M}\ominus\mathcal{N}}x = f(\lambda)x$. (To see this, write $f(\zeta) = f(\lambda) + (\zeta - \lambda)g(\zeta)$, where $g \in H^\infty(G)$, by virtue of [3, Proposition 2.1]. We then have $\Phi(f)_{\mathcal{M}\ominus\mathcal{N}}x = f(\lambda)x + \Phi(g)_{\mathcal{M}\ominus\mathcal{N}}(T_{\mathcal{M}\ominus\mathcal{N}} - \lambda)x = f(\lambda)x$.) The proof is now concluded thusly, as in [1]: since (the algebra of polynomials in) $T_{\mathcal{M}\ominus\mathcal{N}}$ is reflexive, we obtain (since $A_{\mathcal{M}\ominus\mathcal{N}} \in \mathrm{Alg}(\mathrm{Lat}(\mathcal{A}_{\mathcal{M}\ominus\mathcal{N}}))$) that $A_{\mathcal{M}\ominus\mathcal{N}}$ is a polynomial in $T_{\mathcal{M}\ominus\mathcal{N}}$, and from this the result follows. ▨

REMARK. One may wonder at the lack of symmetry in this section, with respect to adjoints. It is true (just as in Part I) that if $\lambda \in \sigma_{\mathrm{re}}(T) \cap G$ and if $\{x_n\}$ is an orthonormal sequence in $\mathcal{H}$ such that $\|(T^* - \bar{\lambda})x_n\| \to 0$, then $\|[x_n \otimes x_n] - [C_\lambda]\|_Q \to 0$. (The proof of this is, *mutatis mutandis*, the same as that of Part I, Lemma 3.6(B).) The difficulty here is to obtain conclusion (A) of Lemma 2.2. However, if we replace in condition (2.2) the strong operator topology by the strong* topology (in which adjunction is continuous), then by applying (the remark following) Lemma 2.2(B) both to Φ and to its obvious "conjugate" Φ_* (defined on $H^\infty(G_*)$, $G_* = \{\lambda : \bar{\lambda} \in G\}$, by $\Phi_*(f) = \Phi(f_*)^*$ where $f_*(\lambda) = \overline{f(\bar{\lambda})}$) we obtain that both of the conclusions of Lemma 2.2 hold for *any* orthonormal sequence $\{x_n\}$ in $\mathcal{H}$. Therefore, we may replace conditions (2.1) and (2.2) by

(2.1′) $\sigma_{\mathrm{e}}(T) \cap G$ is dominating for $H^\infty(G)$,

and

(2.2′) If a sequence $f_n \to 0$ in $H^\infty(G)$, then $\Phi(f_n) \to 0$ in the strong* (equivalently, ultrastrong*) topology,

and we may then develop all of the above theory in this context.

3. AN APPLICATION: THE CASE OF CIRCULAR DOMAINS

By a *circular domain*, we shall mean either the open unit disc $\mathbf{D}$, or a domain of the form $\mathbf{D}\setminus\bigcup_{i=1}^{n} D_i^-$, where $D_i^- = \{\xi \in \mathbf{C} : |\xi - \xi_i| \leqslant r_i\} \subset \mathbf{D}$ and the D_i^- are pairwise disjoint. Recall that a compact set $X \subset \mathbf{C}$ is called a *K-spectral set* for an operator $T \in \mathcal{L}(\mathcal{H})$ in case $\sigma(T) \subset X$ and for every rational function f with poles off X, $\|f(T)\| \leqslant K \sup_{\lambda \in X}|f(\lambda)|$. Let G be a circular domain, and let $T \in \mathcal{L}(\mathcal{H})$. Assume that

(3.1) G^- is a K-spectral set for T, for some $K > 0$,

(3.2) $\sigma_{\mathrm{le}}(T) \cap G$ is dominating for $H^\infty(G)$,

and

(3.3) The powers of T and (in case $G \neq \mathbf{D}$) each of $r_i(T - \xi_i)^{-1}$ tend strongly to 0.

Then Chevreau, Pearcy, and Shields constructed in [3] an $H^\infty(G)$ functional calculus $\Phi: H^\infty(G) \to \mathcal{L}(\mathcal{H})$ for T satisfying the conditions assumed in the previous section. Hence all of the analysis of that section applies to the algebra $\mathcal{A} = \Phi[H^\infty(G)]$. Moreover, $\mathcal{A}$ is precisely the ultraweakly closed algebra generated by I, T, and the $(T - \xi_i)^{-1}$.

In particular, if $G = \mathbf{D}$, then the assumptions are that T is a polynomially bounded operator whose powers tend strongly to 0, and for which $\sigma_{\mathrm{le}}(T) \cap \mathbf{D}$ is dominating for $H^\infty(\mathbf{D})$. Thus we obtain, for example, the following result, which was announced in [1].

THEOREM 3.1. *If T is a polynomially bounded operator whose powers tend strongly to 0, and for which $\sigma_{\mathrm{le}}(T) \cap \mathbf{D}$ is dominating for $H^\infty(\mathbf{D})$, then (the ultraweakly closed algebra containing I generated by) T is reflexive.*

Another special case of the present situation occurs in the following way. Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Recall that the *norm annulus* $\mathrm{N}(T)$ is defined by

$$\mathrm{N}(T) = \{\xi \in \mathbf{C} : \|T^{-1}\|^{-1} \leqslant |\xi| \leqslant \|T\|\}.$$

If the interior $\mathrm{N}(A)^0$ is nonempty, then, by [8, Proposition 23], $\mathrm{N}(T)$ is a K-spectral set for T, for some $K > 0$. We assume (without loss of generality) that $\|T\| = 1$. Hence, if T is invertible, $\mathrm{N}(T)^0 \neq \emptyset$, $\sigma_{\mathrm{le}}(T) \cap \mathrm{N}(T)^0$ is dominating for $H^\infty[\mathrm{N}(T)^0]$, and if the powers of each of T and $\|T^{-1}\|^{-1}T^{-1}$ tend strongly to 0, we obtain the following result, among others.

THEOREM 3.2. *Under the foregoing assumptions, the ultraweakly closed algebra generated by I, T, and T^{-1} is reflexive.*

EXAMPLE. We show here that the preceding theorem applies to certain weighted bilateral shifts. Let $\{e_n : n \in \mathbf{Z}\}$ be an orthonormal basis for $\mathcal{H}$, and let $\{w_n : n \in \mathbf{Z}\}$ be a bounded family of scalars. The operator T on $\mathcal{H}$ defined by requiring that $Te_n = w_n e_{n+1}$ (for all n) is the (weighted) bilateral shift with weight sequence $\{w_n\}$. Clearly, $\|T\| = \sup|w_n|$, and such an operator is invertible if and only if the weights are bounded away from zero, in which case $\|T^{-1}\| = \sup|w_n^{-1}|$.

Using the uniform boundedness principle, one readily sees that $T^n \to 0$ strongly if and only if

$$\prod_{n=k}^{k+N} w_n \to 0 \quad \text{as } N \to \infty \text{ for each } k, \tag{3.4}$$

and

$$\left\{\prod_{n=k}^{k+N} w_n : k \in \mathbf{Z},\ N \geqslant 0\right\} \text{ is bounded.} \tag{3.5}$$

Moreover, note that if $w_n = w_{n+1} = \ldots = w_{n+N} = a$ for some $a \in \mathbf{C}$, $n \in \mathbf{Z}$, and $N > 0$, and if

$$x = (N+1)^{-1/2} \sum_{k=n}^{n+N} e_k,$$

then $\|x\| = 1$ and $\|(T - a)x\| = \sqrt{2}(N+1)^{-1/2}|a|$.

Finally, recall that $\lambda \in \sigma_{le}(T)$ (for any $T \in \mathscr{L}(\mathscr{H})$) if and only if there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that $\|(T - \lambda)x_n\| \to 0$.

Let G be, for example, the annulus $\{\xi \in \mathbf{C} : 1/2 < |\xi| < 1\}$ and let $\{a_n\}$ be a dense sequence in G. Let $\{w_n : n \geqslant 0\}$ be a sequence in G in which each a_n occurs in arbitrarily long "strings" (of consecutive w's). Let $w_n = 1$ for $n < 0$. Then it is easy to check, using the remarks above, that $\mathrm{N}(T) = G^-$, and that assumptions (3.1)—(3.3) hold for this T and G. (We have that $\sigma_{le}(T) = G^-$.) Hence we have constructed an example satisfying the hypotheses of Theorem 3.2, and it is clear that this construction can be varied to obtain many such examples.

Remark. By the remark at the end of Section 2, and by Theorem 7.2 of [3], we may replace "σ_{le}" by "σ_e" in condition (3.2), provided that we also replace the strong operator topology by the strong* topology in condition (3.3).

4. SOME REMARKS CONCERNING (BCP)-OPERATORS

We conclude by making some observations relating to the following example. Let $\mathscr{F}$ be an infinite-dimensional Hilbert space, let $\mathscr{H} = \ell^2(\mathscr{F})$ be the space of all square-summable sequences (indexed by $\mathbf{Z}^+$) in $\mathscr{F}$, and let V be the (forward) unilateral shift in $\mathscr{H}$. Since each point of $\mathbf{D}$ is an eigenvalue of V^* of infinite multiplicity, and since $V^{*n} \to 0$ strongly, it follows that V and V^* are of class (BCP).

(1) Let $a \in \mathscr{F}$, $\|a\| = 1$, and let $y = (a, 0, 0, 0, \ldots) \in \mathscr{H}$ and $x_n = (0, \ldots, 0, a, 0, \ldots) \in \mathscr{H}$ (with a in the n-th position). Then

$$\|[y \otimes x_n]\|_Q \geqslant |(V^n y, x_n)| = \|a\|^2 = 1.$$

This shows, as remarked after the proof of Part I, Lemma 3.4, that the conclusion of that lemma need not hold for an arbitrary orthonormal sequence $\{x_n\}$.

(2) An infinite-dimensional subspace need not contain an invariant subspace for V. (Consider the subspace consisting of all vectors in $\mathscr{H}$ whose even-numbered components vanish.)

This observation forecloses one obvious strategy for attempting to produce two disjoint invariant subspaces for operators of class (BCP). One can also easily translate this problem into "$[x \otimes y]$ language", but that doesn't seem to be of much help, either.

One reason for interest in the question of the existence of two disjoint invariant subspaces stems from the following results. Foiaş, Pearcy, and Sz.-Nagy, in [5], showed that the existence of invariant subspaces for a broad class of operators would follow from an affirmative answer to the question: if T is an invertible (BCP)-operator, does there exist an invariant subspace which T maps *onto* itself? On the other hand, Bercovici, Foiaş, and Pearcy showed, in an early version of [2], that for any such operator T, either such an invariant subspace exists, or there is a pair of disjoint invariant subspaces. So one would like to know whether one or both of these alternatives always hold true.

(3) For a (BCP)-operator T, the lattice Lat(T) of invariant subspaces of T need not contain a pinch point.

R. Douglas and C. Pearcy, in [4], defined *a* pinch point of a lattice to be a member of the lattice which is comparable with every other member. They then showed that a pinch point of Lat(T) is hyperinvariant for T. Hence the existence of (nontrivial) pinch points would settle the question raised in the second remark.

However, a pinch point of Lat(T) would be *a fortiori* a pinch point of the lattice of all hyperinvariant subspaces for T. Douglas and Pearcy also showed in [4] that the lattice of all hyperinvariant subspaces of the unilateral shift (of any multiplicity) is isomorphic to the lattice of all inner functions in $H^{\infty}(D)$. This obviously has no pinch points.

Even more is true. Using the reflexivity theorem of Bercovici, Foiaş, Langsam, and Pearcy [1], one can show that Lat(T) *never* has a pinch point, for any (BCP)-operator T.

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