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GLOBAL CROSS SECTIONS OF UNITARY AND SIMILARITY ORBITS OF HILBERT SPACE OPERATORS

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INTRODUCTION

In [2] D. Deckard and L. A. Fialkow characterized the operators on a separable Hilbert space which have local unitary cross sections. These are the operators of the form $A \oplus B \otimes 1$, where A and B are operators on finite dimensional spaces and 1 is the identity operator on a complex separable infinite dimensional Hilbert space $\mathscr{H}$.

The purpose of this article is to characterize the operators which have global unitary cross sections or global similarity cross sections. To make these statements more precise, it is necessary to introduce some standard notation, which will be used throughout the paper: $\mathscr{L}(\mathscr{H})$ denotes the algebra of all bounded linear operators on $\mathscr{H}$, $\mathscr{U}(\mathscr{H})$ denotes the group of unitary operators in $\mathscr{L}(\mathscr{H})$, $\mathscr{U}(T) = \{U^*TU : U \in \mathscr{U}(\mathscr{H})\}$ is the *unitary orbit* of an operator T in $\mathscr{L}(\mathscr{H})$ and $\tau : \mathscr{U}(\mathscr{H}) \to \mathscr{U}(T)$ is the (norm continuous) function defined by $\tau(U) = U^*TU$. A local unitary cross section for τ is a pair $(\varphi, \mathscr{B})$ such that $\mathscr{B}$ is a relatively open subset of $\mathscr{U}(T)$ which contains T and $\varphi : \mathscr{B} \to \mathscr{U}(\mathscr{H})$ is a norm continuous function such that $\tau \circ \varphi = 1_{\mathscr{B}}$ and $\varphi(T) = 1$; φ is a *global unitary cross section* when $\mathscr{B} = \mathscr{U}(T)$.

In Section 3 it is shown that τ has a global unitary cross section if and only if T has the form $B \otimes 1$, where B is an operator on a finite dimensional space.

Let $\tilde{T}$ denote the canonical image of an operator T in the Calkin algebra. If $T = A \oplus B \otimes 1$ (A, B acting on finite dimensional spaces), then $\tilde{T}$ admits a global unitary cross section only in the trivial case when $\tilde{T}$ is a multiple of the identity. As a corollary, it is shown that if $T = B \otimes 1$ has a global unitary cross section φ, but T is not a multiple of the identity, then it is impossible to construct φ so that $\varphi(T_1) - \varphi(T_2)$ is a compact operator whenever $T_1, T_2 \in \mathscr{U}(T)$ and $T_1 - T_2$ is compact.

The second part of the paper is devoted to the study of similarity cross sections. Let $\mathcal{G}(\mathcal{H})$ denote the group of invertible operators in $\mathcal{L}(\mathcal{H})$ and let $\mathcal{S}(T) = \{W^{-1}TW : W \in \mathcal{G}(\mathcal{H})\}$ be the similarity orbit of an operator T in $\mathcal{L}(\mathcal{H})$. The (norm continuous) function $s : \mathcal{G}(\mathcal{H}) \to \mathcal{S}(T)$ is defined by $s(W) = W^{-1}TW$. L. A. Fialkow and D. A. Herrero ([5], [6]) proved that s has a local cross section if and only if T is similar to a nice Jordan operator, that is, an operator of the form

$$\bigoplus_{i=1}^{n} \left(\lambda_i 1_{\mathcal{H}_i} + \bigoplus_{j=1}^{m_i} q_{k_{ij}}^{(\alpha_{ij})}\right),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct complex numbers, q_k is the $k \times k$ nilpotent Jordan cell, and for each i, $k_{i1} < k_{i2} < \ldots < k_{im_i}$ and $\alpha_{ij} = \infty$ for at most one value of j.

In Theorem 17 it is shown that s has a global cross section if and only if T is similar to a very nice Jordan operator, that is, a nice Jordan operator of the form

$$\bigoplus_{i=1}^{n} \left(\lambda_i 1_{\mathcal{H}_i} + q_{k_i}^{(\infty)}\right).$$

For operators on finite dimensional spaces and for the Calkin algebra, it is shown that there are no global cross sections except in the trivial case when the operator T, or respectively its image $\tilde{T}$ in the Calkin algebra, is a multiple of the identity. As in the case of the unitary orbits, it follows from the Calkin algebra result that if T has a global similarity cross section ζ, then ζ cannot be constructed so that $\zeta(T_1) - \zeta(T_2)$ is compact whenever $T_1, T_2 \in \mathcal{S}(T)$ and $T_1 - T_2$ is compact, except, of course, in the trivial case when T is a multiple of the identity operator.

[14] and [15] are used as standard references for Algebraic Topology throughout the paper.

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PART I

1. A CHARACTERIZATION OF THE UNITARY ORBIT AS A HOMOGENEOUS SPACE

Let T be an operator in $\mathcal{L}(\mathcal{H})$ and let $\mathcal{A}'(T) = \{A \in \mathcal{L}(\mathcal{H}) : AT = TA\}$ be the commutant of T.

PROPOSITION 1. *If $T = A \oplus B \otimes 1$, where A and B are operators on finite dimensional spaces, then the unitary orbit $\mathcal{U}(T)$ (norm topology) of T is homeomorphic with the homogeneous space of right cosets $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ (quotient topology).*

Proof. Let $p : \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ be the canonical projection. Since $\tau^{-1}(\{T\}) = \mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$, there is a bijective continuous map $\varphi : \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \to \mathcal{U}(T)$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\;p\;} & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \\ {\scriptstyle\tau}\big\downarrow & \swarrow{\scriptstyle\varphi} & \\ \mathcal{U}(T) & & \end{array} \tag{1.1}$$

If T satisfies the hypothesis, then τ is an open mapping [2], [4], and it follows easily that φ is indeed a homeomorphism. ▨

REMARK. Since $\mathcal{U}(T)$ is pathwise connected, $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ has the same property.

Consider the commutative diagram (1.1). In [2] it was shown that τ has a local cross section; this implies that p also has a local cross section via the homeomorphism φ. Since $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ is a closed subgroup of $\mathcal{U}(\mathcal{H})$, it follows [15, p. 57] that

$$\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \xrightarrow{\;i\;} \mathcal{U}(\mathcal{H}) \xrightarrow{\;p\;} \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$$

is a fibre bundle ($i : \mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \to \mathcal{U}(\mathcal{H})$ is the inclusion map.) Via φ, the existence of a global (norm continuous) cross section for τ is equivalent to the existence of a global cross section for p.

From now on we will only consider the latter question, that is, we will work with the above mentioned fibre bundle.

First of all we need to determine the homotopy type of $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$. This is done in the next section.

2. CHARACTERIZATION OF $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$

PROPOSITION 2. *If $A = N^{(m)} \simeq N \otimes 1_m$ for some irreducible operator N in $\mathcal{L}(\mathbf{C}^n)$, then the unitary operators which commute with A are the operators of the form $1_n \otimes U$ with $U \in \mathcal{U}(\mathbf{C}^m)$.*

Let $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$ and $B = B_1 \oplus B_2 \oplus \dots \oplus B_m$, where B_i and A_j are irreducible operators for each i, j, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. If $B_i \not\simeq A_j$ for every i, j, we say that A and B are disjoint. In this case, it follows from [3, p. 8] and a simple calculation that if $U(A \oplus B) = (A \oplus B)U$ for some unitary operator U, then $U = U_1 \oplus U_2$, where U_1 commutes with A and U_2 commutes with B.

Let $T = A \oplus B \otimes 1$ with $A = A_1^{(s_1)} \oplus A_2^{(s_2)} \oplus \dots \oplus A_n^{(s_n)} \in \mathcal{L}(\mathbf{C}^s)$, $B = B_1^{(t_1)} \oplus B_2^{(t_2)} \oplus \dots \oplus B_m^{(t_m)} \in \mathcal{L}(\mathbf{C}^t)$, where $A_i \in \mathcal{L}(\mathbf{C}^{k_i})$ and $B_j \in \mathcal{L}(\mathbf{C}^{r_j})$ are irreducible operators, $s = \sum s_i k_i$ and $t = \sum t_j r_j$.

We can assume that A and B are disjoint: if $A_i \simeq B_j$ for some pair i, j, then we can write

$$T \simeq A_1^{(s_1)} \oplus \ldots \oplus A_{i-1}^{(s_{i-1})} \oplus A_{i+1}^{(s_{i+1})} \oplus \ldots \oplus A_n^{(s_n)} \oplus B_1^{(t_1)} \otimes 1 \oplus \ldots \oplus B_{j-1}^{(t_{j-1})} \otimes 1 \oplus$$

$$\oplus \{A_i^{(s_i)} \oplus B_j^{(t_j)} \otimes 1\} \oplus B_{j+1}^{(t_{j+1})} \otimes 1 \oplus \ldots \oplus B_m^{(t_m)} \otimes 1 \simeq$$

$$\simeq A_1^{(s_1)} \oplus \ldots \oplus A_{i-1}^{(s_{i-1})} \oplus A_{i+1}^{(s_{i+1})} \oplus \ldots \oplus A_n^{(s_n)} \oplus B \otimes 1.$$

(That is, all the common irreducible terms can be "absorbed" in B). Also notice that $B \otimes 1 \simeq B_1 \otimes 1 \oplus \ldots \oplus B_m \otimes 1$, because $B_j^{(t_j)} \otimes 1 \simeq B_j \otimes 1_{t_j} \otimes 1 \simeq \simeq B_j \otimes 1$. Hence, we can directly assume that $t_j = 1$ for all j, $1 \leqslant j \leqslant m$.

From Proposition 2, the above remarks and the fact that $\pi_1(\mathcal{U}(\mathbf{C}^k)) = \mathbf{Z}$ for all $k \geqslant 1$, and $\mathcal{U}(\mathcal{H})$ is contractible [11], we obtain:

PROPOSITION 3. *For an operator $T = A \oplus B \otimes 1$ of the above form,*

$$\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) = \{1_{k_1} \otimes U_1 \oplus \ldots \oplus 1_{k_n} \otimes U_n \oplus 1_{r_1} \otimes V_1 \oplus \ldots \oplus 1_{r_m} \otimes V_m :$$

$$: U_i \in \mathcal{U}(\mathbf{C}^{s_i}),\ \ V_j \in \mathcal{U}(\mathcal{H}_j)\}$$

and

$$\pi_1(\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) = \pi_1(\mathcal{U}(\mathbf{C}^{s_1})) \oplus \ldots \oplus \pi_1(\mathcal{U}(\mathbf{C}^{s_n})) \oplus$$

$$\oplus \pi_1(\mathcal{U}(\mathcal{H}_1)) \oplus \ldots \oplus \pi_1(\mathcal{U}(\mathcal{H}_m)) = \mathbf{Z}^{(n)}.$$

3. THE EXISTENCE OF GLOBAL CROSS SECTIONS

THEOREM 4. *Let $T = B \otimes 1$, where B is an operator on a finite dimensional space. Then the fibre bundle*

$$\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \xrightarrow{\ i\ } \mathcal{U}(\mathcal{H}) \xrightarrow{\ p\ } \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) \tag{3.1}$$

has a global (continuous) cross section.

Proof. By Proposition 3, $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T) = \left\{ \bigoplus_{j=1}^{m} I_{r_j} \otimes V_j : V_j \in \mathcal{U}(\mathcal{H}_j), \right.$ $\left. j = 1, 2, \ldots, m \right\}$. It follows that $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ is isomorphic with a (finite) product of copies of $\mathcal{U}(\mathcal{H})$ and therefore it is contractible [11].

We will prove that the base space $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ is also contractible. Since a fibre bundle with contractible base is equivalent to a product fibre bundle (which obviously has global cross sections), it will follow that p has a global cross section.

Consider the exact homotopy sequence

$$\text{(3.2)}\quad \begin{aligned} \ldots &\xrightarrow{\partial} \pi_n(\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) \xrightarrow{i_*} \pi_n(\mathscr{U}(\mathscr{H})) \xrightarrow{p_*} \pi_n(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) \xrightarrow{\partial} \\ &\xrightarrow{\partial} \pi_{n-1}(\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) \xrightarrow{i_*} \ldots \xrightarrow{\partial} \pi_0(\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) \end{aligned}$$

corresponding to the fibre bundle (3.1). Since $\pi_n(\mathscr{U}(\mathscr{H})) = 0$ for all n and $\pi_n(\mathscr{U}(\mathscr{H}) \cap \cap \mathscr{A}'(T)) = 0$ for all n, we obtain $\pi_n(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) = 0$ for all $n \geqslant 1$. Since $\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T) \simeq \mathscr{U}(T)$ is pathwise connected, we also have $\pi_0(\mathscr{U}(\mathscr{H})/ /\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) = 0$.

This means that $\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)$ is weakly homotopy equivalent [14] to a contractible space. To prove that it is actually contractible, it will be enough to show that it is homotopy equivalent to a CW-complex [14]. This is done in the following auxiliary result.

If $\mathscr{R}$ is a subset of $\mathscr{L}(\mathscr{H})$, then we define $\mathscr{R}^* = \{R^* : R \in \mathscr{R}\}$.

LEMMA 5. *$(\mathscr{G}(\mathscr{H}), \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T))$, $(\mathscr{G}(\mathscr{H}), [\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)] \cap [\mathscr{G}(\mathscr{H}) \cap \cap \mathscr{A}'(T)]^*)$ and $(\mathscr{U}(\mathscr{H}), \mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T))$ are homotopy equivalent relative to* CW-*complexes.*

Proof. Since $\mathscr{G}(\mathscr{H})$ is paracompact and $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$ is a closed subset of $\mathscr{G}(\mathscr{H})$, according to [13] it is enough to prove that the following property is satisfied: there is a neighborhood Ω of the diagonal in $\mathscr{G}(\mathscr{H}) \times \mathscr{G}(\mathscr{H})$, a (continuous) function $\lambda : \Omega \times [0, 1] \to \mathscr{G}(\mathscr{H})$ and an open covering $\{\Phi_V\}$ of $\mathscr{G}(\mathscr{H})$ such that the following properties are satisfied:

$\lambda(V, W, 0) = W$ for every (V, W) in Ω,

$\lambda(V, W, 1) = V$ for every (V, W) in Ω,

$\lambda(V, V, t) = V$ for every V in $\mathscr{G}(\mathscr{H})$ $(0 \leqslant t \leqslant 1)$,

$\lambda(V, W, t) \in \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$ if (V, W) is in $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T) \times \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$ $(0 \leqslant t \leqslant 1)$,

$\Phi_V \times \Phi_V \subset \Omega$

and

$\lambda(\Phi_V \times \Phi_V \times [0, 1]) = \Phi_V$ for every Φ_V in the covering.

Let $\Omega = \{(V, W) \in \mathscr{G}(\mathscr{H}) \times \mathscr{G}(\mathscr{H}) : \|V - W\| < \min\{\|V^{-1}\|^{-1}, \|W^{-1}\|^{-1}\}\}$, let $\Phi_V = \{W \in \mathscr{G}(\mathscr{H}) : \|V - W\| < 1/(3\|V^{-1}\|)\}$ for each V in $\mathscr{G}(\mathscr{H})$ and let $\lambda : \Omega \times \times [0, 1] \to \mathscr{G}(\mathscr{H})$ be defined by $\lambda(V, W, t) = tV + (1 - t)W$.

It is easy to see that all the required conditions are satisfied. This shows that $(\mathscr{G}(\mathscr{H}), \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T))$ is a relative CW-complex.

To prove the second statement of the lemma, we notice that $\lambda(V, W, t)$ belongs to $[\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*$ if V and W are in $[\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*$, so that the same proof holds in this case.

For the last statement, consider the map $r : \mathcal{G}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ defined by

$$r(V) = V(V^*V)^{-1/2} \quad \text{for } V \text{ in } \mathcal{G}(\mathcal{H}).$$

If $i : \mathcal{U}(\mathcal{H}) \to \mathcal{G}(\mathcal{H})$ denotes the inclusion map, then $r \circ i = 1_{\mathcal{U}(\mathcal{H})}$. Thus r is a retraction.

If $V \in [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*$, then both V and V^* commute with T and, a fortiori, the whole W^*-algebra $W^*(V)$ generated by V is contained in $\mathcal{A}'(T)$. In particular, $(V^*V)^{-1/2} \in \mathcal{A}'(T)$. Since V is invertible, so is $(V^*V)^{-1/2}$, and therefore $r(V) = V(V^*V)^{-1/2} \in \mathcal{U}(\mathcal{H}) \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^* = = \mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$. Hence,

$$r \,\big|\, [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^* : [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^* \to \mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T).$$

Therefore r induces a map of pairs

$$r : (\mathcal{G}(\mathcal{H}), [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*) \to (\mathcal{U}(\mathcal{H}), \mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)),$$

where $[\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*$ and $\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ are closed subsets of $\mathcal{G}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$, respectively. Since a retract of a relative CW-pair is also a relative CW-pair [12, p. 127], [14], the proof is complete. ▨

The proof of Theorem 4 is now complete.

By using Lemma 5 and [14, p. 402] we obtain the following:

COROLLARY 6. *$\mathcal{G}(\mathcal{H})/\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)$, $\mathcal{G}(\mathcal{H})/[\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)] \cap [\mathcal{G}(\mathcal{H}) \cap \mathcal{A}'(T)]^*$ and $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)$ are homotopy equivalent to* CW-*complexes.*

THEOREM 7. *Let $T = A \oplus B \otimes 1$, where A and B are disjoint operators on finite dimensional spaces, and $A \neq 0$. Then the fibre bundle* (3.1) *does not have a global cross section.*

Proof. In the exact homotopy sequence (3.2) we have $\pi_n(\mathcal{U}(\mathcal{H})) = 0$ for all n and $\pi_1(\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) = \mathbf{Z}^{(m)}$ (Proposition 3) for some $m \geqslant 1$. Therefore $\pi_{n+1}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) \simeq \pi_n(\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T))$ and, in particular,

$$\pi_2(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) \simeq \pi_1(\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) \simeq \mathbf{Z}^{(m)}.$$

If there exists a global cross section for p, then the exact homotopy sequence

$$\ldots \xrightarrow{\partial} \pi_2(\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) \xrightarrow{i_*} \pi_2(\mathcal{U}(\mathcal{H})) \xrightarrow{p_*} \pi_2(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cap \mathcal{A}'(T)) \xrightarrow{\partial} \ldots$$

splits [14, p. 418], that is, $\pi_2(\mathscr{U}(\mathscr{H})) \simeq \pi_2(\mathscr{U}(\mathscr{H}) \cap \mathscr{A}'(T)) \oplus \pi_2(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap \cap \mathscr{A}'(T))$. But $\mathbf{Z}^{(m)}$ cannot be a direct summand of $\pi_2(\mathscr{U}(\mathscr{H})) = 0$; therefore p does not have a global cross section. ▨

4. THE FINITE DIMENSIONAL CASE

In what follows, the notation $U(n) = \mathscr{U}(\mathbf{C}^n)$ will be used. Let A be an operator on $\mathbf{C}^n$. Proceeding exactly as in Proposition 1, we see that $\mathscr{U}(A)$ is homeomorphic with $U(n)/U(n) \cap \mathscr{A}'(A)$ (right cosets) and that

$$U(n) \cap \mathscr{A}'(A) \xrightarrow{i} U(n) \xrightarrow{p} U(n)/U(n) \cap \mathscr{A}'(A) \tag{4.1}$$

is a fibre bundle. (Indeed, this can also be proved by a direct argument.)

LEMMA 8. *Let $A = N^{(m)} \simeq N \otimes 1_m$, where N is an irreducible operator in $\mathscr{L}(\mathbf{C}^q)$, $n = qm$. Then the fibre bundle* (4.1) *does not have a global cross section unless $q = 1$, that is, unless A is a multiple of the identity.*

Proof. By Proposition 2, $U(n) \cap \mathscr{A}'(A) = \{1_q \otimes U : U \in U(m)\}$. Hence, $\pi_1(U(n) \cap \mathscr{A}'(A)) \simeq \pi_1(U(m)) \simeq \mathbf{Z}$.

The last terms of the exact homotopy sequence of the bundle we are considering are

$$\mathbf{Z} \xrightarrow{i_*} \mathbf{Z} \xrightarrow{p_*} \pi_1(U(n)/U(n) \cap \mathscr{A}'(A)) \longrightarrow 0.$$

We will show that $i_*(z) = qz$ for every $z \in \mathbf{Z}$. The generator of $\pi_1(U(n))$ is the homotopy class of the map $\gamma_n : [0, 1] \to U(n)$, where $\gamma_n(t)$ is the diagonal matrix $\mathrm{diag}(e^{2\pi i t}, 1, 1, \ldots, 1)$. Since $i \circ \gamma_m(t) = \mathrm{diag}(e^{2\pi i t}, 1, \ldots, 1, e^{2\pi i t}, 1, \ldots, 1, \ldots, e^{2\pi i t}, 1, \ldots, 1)$, where $e^{2\pi i t}$ appears q times on the diagonal and the remaining elements on the diagonal are equal to 1, it is easy to show that $i \circ \gamma_m$ and $q\gamma_n$ are homotopic. Therefore, $i_*[\gamma_m] = q[\gamma_n]$.

Since $\pi_1(U(n)) = \mathbf{Z}$ is commutative, if there is a global cross section, then the exact homotopy sequence will also split at π_1. But

$$\pi_1(U(n)/U(n) \cap \mathscr{A}'(A)) \simeq \pi_1(U(n))/\ker p_* \simeq \pi_1(U(n))/\mathrm{im}\, i_* \simeq \mathbf{Z}/q\mathbf{Z} = \mathbf{Z}_q.$$

Since $\mathbf{Z}_q$ cannot be a direct summand of $\mathbf{Z}$, there is no global cross section unless $q = 1$.

LEMMA 9. *Let $A = A_1^{(s_1)} \oplus A_2^{(s_2)} \oplus \ldots \oplus A_n^{(s_n)}$, $n \geqslant 2$, where the operators $A_i \in \mathscr{L}(\mathbf{C}^{k_i})$ are irreducible and disjoint, and $\sum s_i k_i = s$. Then the fibre bundle*

$$U(s) \cap \mathscr{A}'(A) \xrightarrow{i} U(s) \xrightarrow{p} U(s)/U(s) \cap \mathscr{A}'(A) \tag{4.2}$$

does not have a global cross section.

Proof. $\pi_1(U(s) \cap \mathscr{A}'(A)) = \mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} = \mathbf{Z}^{(n)}$ (Proposition 3) and $\pi_1(U(s)) = \mathbf{Z}$.

The last terms of the exact homotopy sequence of the fibre bundle (4.2) are

$$\ldots \to \mathbf{Z}^{(n)} \to \mathbf{Z} \to \pi_1(U(s)/U(s) \cap \mathscr{A}'(A)) \to 0.$$

Since $n \geqslant 2$, $\mathbf{Z}^{(n)}$ cannot be a direct summand of $\mathbf{Z}$; therefore the exact homotopy sequence cannot split. This implies that the fibre bundle has no global cross section. ▨

Combining Lemmas 8 and 9, we obtain the following result.

THEOREM 10. *An operator acting on a finite dimensional space has a global unitary cross section if and only if it is a multiple of the identity.*

5. UNITARY CROSS SECTIONS IN THE CALKIN ALGEBRA

We need to introduce the following notation: $\mathscr{K}$ denotes the ideal of compact operators in $\mathscr{L}(\mathscr{H})$, $\mathscr{A} = \mathscr{L}(\mathscr{H})/\mathscr{K}$ is the *Calkin algebra*, $\mathscr{U}(\mathscr{A})$ denotes the unitary group in $\mathscr{A}$ and $\mathscr{U}_0(\mathscr{A})$ is the pathwise connected component of $\tilde{1}$ in $\mathscr{U}(\mathscr{A})$, where $\tilde{T}$ denotes the image of $T \in \mathscr{L}(\mathscr{H})$ under the canonical projection onto $\mathscr{A}$. Let $\mathscr{U}(\tilde{T}) := \{\tilde{V}^*\tilde{T}\tilde{V} : \tilde{V} \text{ in } \mathscr{U}(\mathscr{A})\}$. In this section we will only consider operators of the form $T := A \oplus B \otimes 1$, $A \in \mathscr{L}(\mathbf{C}^s)$, $B \in \mathscr{L}(\mathbf{C}^n)$, for which $\tau \colon \mathscr{U}(\mathscr{A}) \to \mathscr{U}(\tilde{T})$ is known to have a local unitary cross section [2]. Also, notice that $\tilde{T}$ is unitarily equivalent to $(B \otimes 1)^{\sim}$.

As before, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{U}(\mathscr{A}) & \xrightarrow{p} & \mathscr{U}(\mathscr{A})/\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \\ \tau\downarrow & \swarrow \varphi & \\ \mathscr{U}(\tilde{T}) & & \end{array}$$

where the map φ is continuous and bijective. We will prove that if $\tilde{T}$ is not a multiple of the identity, then p does not have a global cross section, in which case τ cannot have one either: if $\xi \colon \mathscr{U}(\tilde{T}) \to \mathscr{U}(\mathscr{A})$ is a (norm continuous) mapping such that $\tau \circ \xi = 1_{\mathscr{U}(\tilde{T})}$, $\xi(\tilde{T}) = 1$, then $\varphi \circ p \circ \xi \circ \varphi = \tau \circ \xi \circ \varphi = \varphi$. Since φ is injective, this implies that $p \circ \xi \circ \varphi = 1$. Therefore $\xi \circ \varphi$ is a cross section for p.

THEOREM 11. *Let* $T = B \otimes 1$, *where* $B \in \mathscr{L}(\mathbf{C}^n)$. *The fibre bundle*

$$\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \xrightarrow{\iota} \mathscr{U}(\mathscr{A}) \xrightarrow{p} \mathscr{U}(\mathscr{A})/\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$$

does not have a global cross section unless $\tilde{T}$ *is a multiple of the identity.*

Proof. Consider the following morphism of bundles:

$$
\begin{array}{ccc}
\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) & \xrightarrow{\ i'\ } & \mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \\
\Big\downarrow{\scriptstyle \iota_0} & & \Big\downarrow{\scriptstyle \iota} \\
\mathscr{U}_0(\mathscr{A}) & \xrightarrow{\ i\ } & \mathscr{U}(\mathscr{A}) \\
\Big\downarrow{\scriptstyle p_0} & & \Big\downarrow{\scriptstyle p} \\
\mathscr{U}_0(\mathscr{A})/\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) & \xrightarrow{\ i''\ } & \mathscr{U}(\mathscr{A})/\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})
\end{array}
\tag{5.1}
$$

where ι, ι_0, i and i_0 are inclusion mappings, p and p_0 are the canonical projections onto the classes of right cosets and i'' is induced by the inclusions i and i' so that the diagram is commutative. It is easy to see that i'' is well defined and injective. Since $\mathscr{U}_0(\mathscr{A})$ is the (pathwise connected) component of $\tilde{1}$ in $\mathscr{U}(\mathscr{A})$, we have

$$i_* : \pi_n(\mathscr{U}_0(\mathscr{A}),\ \tilde{1}) \simeq \pi_n(\mathscr{U}(\mathscr{A}),\ \tilde{1}) \quad \text{for } n \geqslant 1.$$

Observe that the path component of $\tilde{1}$ in $\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$ is equal to the path component of $\tilde{1}$ in $\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$, whence

$$i'_* : \pi_n(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}),\ \tilde{1}) \simeq \pi_n(\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}),\ \tilde{1}) \quad \text{for } n \geqslant 1.$$

Now consider the exact homotopy sequences of the bundles in (5.1) and the induced homomorphism i_*, i'_* and i''_*. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
\ldots \xrightarrow{\partial} & \pi_n(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) & \xrightarrow{\iota_{0*}} & \pi_n(\mathscr{U}_0(\mathscr{A}), \tilde{1}) & \xrightarrow{p_{0*}} & \pi_n(\mathscr{U}_0(\mathscr{A})/\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p_0(\tilde{1})) & \xrightarrow{\partial} & \\
& \Big\downarrow{\scriptstyle i'_*} & & \Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle i''_*} & & \Big\downarrow{\scriptstyle i'_*} \\
\ldots \xrightarrow{\partial} & \pi_n(\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) & \xrightarrow{\iota_*} & \pi_n(\mathscr{U}(\mathscr{A}), \tilde{1}) & \xrightarrow{p_*} & \pi_n(\mathscr{U}(\mathscr{A})/\mathscr{U}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p(\tilde{1})) & \xrightarrow{\partial} & .
\end{array}
\tag{5.2}
$$

Since i_* and i'_* are isomorphisms, the 5-lemma implies that i''_* is also an isomorphism. Therefore if one of the exact sequences splits, the same holds for the other one. We will show that the upper sequence in (5.2) does not split unless $\tilde{T}$ is a multiple of $\tilde{1}$. This implies that the exact homotopy sequence of $p : \mathscr{U}(\mathscr{A}) \to \mathscr{U}(\mathscr{A})/\mathscr{U}(\mathscr{A}) \cap$ $\cap\, \mathscr{A}'(\tilde{T})$ does not split either, whence p cannot have a global cross section. To this end, first we compute $\pi_n(\mathscr{U}_0(\mathscr{A}), \tilde{1})$. We use the fact that $\mathscr{U}_0(\mathscr{A})$ is isomorphic with $\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})$, which is a deformation retract of $\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$, where $1 + \mathscr{K} = \{1 + K : K \in \mathscr{K}\}$.

Let $r : \mathscr{G}(\mathscr{H}) \to \mathscr{U}(\mathscr{H})$ and $h : \mathscr{G}(\mathscr{H}) \times [0, 1] \to \mathscr{G}(\mathscr{H})$ be defined by

$$r(V) = V(V^*V)^{-1/2} \quad \text{for } V \text{ in } \mathscr{G}(\mathscr{H})$$

and

$$h(V, t) = V[(1 - t)(V^*V)^{-1/2} + t1] \quad \text{for } V \text{ in } \mathscr{G}(\mathscr{H}),\ 0 \leqslant t \leqslant 1,$$

respectively. It is easy to see (by using the spectral theorem for compact hermitian operators) that $h(\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K}) \times [0, 1]) \subset \mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$ and $r(\mathscr{G}(\mathscr{H}) \cap \cap (1 + \mathscr{K})) \subset \mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})$. Therefore r and h give maps of pairs

$$r : (\mathscr{G}(\mathscr{H}), \mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})) \to (\mathscr{U}(\mathscr{H}), \mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K}))$$

and

$$h : (\mathscr{G}(\mathscr{H}) \times [0, 1], \mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K}) \times [0, 1]) \to (\mathscr{G}(\mathscr{H}), \mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K}))$$

with $h(V, 0) = r(V)$ and $h(V, 1) = V$. Thus, $\mathscr{U}(\mathscr{H})$ is a deformation retract of $\mathscr{G}(\mathscr{H})$, $\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})$ is a deformation retract of $\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$ and $\mathscr{U}(\mathscr{H})/ /\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})$ is a deformation retract of $\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$. As a consequence,

$$\pi_n(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})) \simeq \pi_n(\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})), \quad n \geqslant 0.$$

Since $\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$ is homotopy equivalent to the space of Fredholm operators of index zero [10], which is homotopy equivalent to BU (the classifying space of vector bundles with group $U(\infty) = \lim U(n)$) [1], [10], we obtain

$$\pi_n(\mathscr{U}_0(\mathscr{A})) = \pi_n(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1+\mathscr{K})) \simeq \pi_n(BU) = \begin{cases} 0 & \text{if } n = 2m - 1 \\ \mathbf{Z} & \text{if } n = 2m,\ m \geqslant 1 \end{cases}$$

[15, p. 215].

Case 1. Let $T = B \otimes 1$, $B \in \mathscr{L}(\mathbf{C}^n)$. If $B = A \otimes 1_k$, where $A \in \mathscr{L}(\mathbf{C}^r)$ is irreducible, then $B \otimes 1 \simeq A \otimes 1_k \otimes 1$ is unitarily equivalent to $A \otimes 1$. Therefore we can assume that $B = A$ is irreducible. By using a faithful unital $*$-representation of the finite dimensional C^*-algebra $C^*(\tilde{T})$, and Proposition 2, one can show that $\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) = \{(1_n \otimes U)^\sim : U \in \mathscr{U}(\mathscr{H})\}$. Therefore

$$\pi_n(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) = \pi_n(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})), \quad n \geqslant 1.$$

Let $[\gamma]$ be a generator of $\pi_2(\mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K})) \simeq \mathbf{Z}$:

$$\gamma : ([0, 1] \times [0, 1], \partial([0, 1] \times [0, 1])) \to \mathscr{U}(\mathscr{H})/\mathscr{U}(\mathscr{H}) \cap (1 + \mathscr{K}).$$

If $\gamma(s, t) = (U(s, t))^{\sim}$, then $\gamma^{(n)}(s, t) = (U^{(n)}(s, t))^{\sim} = (I_n \otimes U(s, t))^{\sim}$. Thus, the inclusion $\iota_0\colon \mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \to \mathscr{U}_0(\mathscr{A})$ induces the homomorphism "multiplication by n": $\iota_{0*}\colon \pi_2(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) \to \pi_2(\mathscr{U}_0(\mathscr{A}), \tilde{1})$, $\iota_{0*}(z) = nz$ for every z in $\pi_2(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1})$. It follows that in the exact homotopy sequence

$$\begin{array}{ccccc} \dots \xrightarrow{\partial} \pi_2(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) & \xrightarrow{\iota_{0*}} & \pi_2(\mathscr{U}_0(\mathscr{A}), \tilde{1}) & \xrightarrow{p_{0*}} & \pi_2(\mathscr{U}_0(\mathscr{A})/\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p_0(\tilde{1})) \to 0 \\ \downarrow \simeq & & \downarrow \simeq & & \\ \mathbf{Z} & & \mathbf{Z} & & \end{array} \tag{5.3}$$

we have $\pi_2(\mathscr{U}_0(\mathscr{A})/\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p_0(\tilde{1})) \simeq \mathbf{Z}/n\mathbf{Z} = \mathbf{Z}_n$.

This implies that for $n > 1$ the exact homotopy sequence cannot split because $\mathbf{Z}_n$ cannot be a direct summand of $\mathbf{Z}$. Therefore p_0 does not have a global cross section unless $n = 1$, that is, unless $\tilde{T}$ is a multiple of $\tilde{1}$.

Case 2. Assume that $B = \bigoplus_{i=1}^{m} B_i$, $m \geqslant 2$, where the operators B_i (acting on finite dimensional spaces) are irreducible and disjoint. Let $\mathscr{U}^0(\tilde{T})$ be the pathwise connected component of $\tilde{1}$ in $\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$. If $\tilde{V}$ is in $\mathscr{U}^0(\tilde{T})$, then by using a faithful unital *-representation of $C^*(\tilde{T})$ and Proposition 3, we can show that $\tilde{V} = \bigoplus_{i=1}^{m} \tilde{V}_i$, where the V_i 's are Fredholm operators such that $\sum_{i=1}^{m} \operatorname{ind} V_i = 0$. Let $\alpha\colon [0, 1] \to \mathscr{U}^0(\tilde{T})$ be a path from $\tilde{1}$ to $\tilde{V}$. Then by applying to $\alpha(t)$ the same argument as to $\tilde{V}$, we obtain $\alpha(t) = \bigoplus_{i=1}^{m} \alpha_i(t)$. Since $\alpha_i(0) = \tilde{1}_i$, $\tilde{V}_i = \alpha_i(1)$ is in $\mathscr{U}^0(B_i \otimes 1)^{\sim}$ for each i, $i = 1, 2, \dots, m$. Therefore

$$\mathscr{U}^0(\tilde{T}) = \left\{ \bigoplus_{i=1}^{m} (I_{n_i} \otimes U_i)^{\sim} : U_i \in \mathscr{U}(\mathscr{H}_i) \right\}$$

and

$$\pi_2(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), 1) = \pi_2(\mathscr{U}^0(\tilde{T})) = \bigoplus_{i=1}^{m} \pi_2(\mathscr{U}(\mathscr{H}_i)/\mathscr{U}(\mathscr{H}_i) \cap (1 + \mathscr{K})) \simeq \mathbf{Z}^{(m)}.$$

But $\mathbf{Z}^{(m)}$ cannot be a direct summand of $\mathbf{Z}$ for $m \geqslant 2$, so the exact homotopy sequence

$$\begin{array}{ccccc} \dots \xrightarrow{\partial} \pi_2(\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) & \xrightarrow{\iota_{0*}} & \pi_2(\mathscr{U}_0(\mathscr{A}), \tilde{1}) & \xrightarrow{p_{0*}} & \pi_2(\mathscr{U}_0(\mathscr{A})/\mathscr{U}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p_0(\tilde{1})) \to 0 \\ \downarrow \simeq & & \downarrow \simeq & & \\ \mathbf{Z}^{(m)} & & \mathbf{Z} & & \end{array} \tag{5.4}$$

cannot split. This completes the proof of the theorem. ▨

The results in this section have an interesting consequence. D. Deckard and L. A. Fialkow [2] showed that if τ has a local cross section φ, then it is possible to choose φ in such a way that whenever two operators T_1, T_2 in $\mathscr{U}(T)$ have a compact difference $T_1 - T_2$, then $\varphi(T_1) - \varphi(T_2)$ is also compact.

If $T \neq \lambda 1$, then the analogous statement is false for global cross sections. Namely, if τ has a global cross section φ and $T \neq \lambda 1$, then it is impossible to choose φ in such a way that compact differences are preserved, because then there would be a global cross section in the Calkin algebra, in contradiction with Theorem 11.

PART II.

6. A CHARACTERIZATION OF THE SIMILARITY ORBIT AS A HOMOGENEOUS SPACE

THEOREM 12 (L.A.Fialkow—D.A.Herrero [6]). *An operator T in $\mathscr{L}(\mathscr{H})$ has a local similarity cross section if and only if it is similar to a nice Jordan operator. Furthermore, in this case the continuous bijection γ that makes the diagram*

$$\begin{array}{ccc} \mathscr{G}(\mathscr{H}) & \xrightarrow{\;p\;} & \mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T) \\ {\scriptstyle s}\downarrow & \swarrow {\scriptstyle \gamma} & \\ \mathscr{S}(T) & & \end{array}$$

commutative is actually a homeomorphism, and a local cross section $(\varphi, \mathscr{B})$ can be constructed so that if T_1, T_2 are in $\mathscr{S}(T)$ and $T_1 - T_2 \in \mathscr{K}$, then $\varphi(T_1) - \varphi(T_2) \in \mathscr{K}$.

REMARK. If T is similar to a nice Jordan operator, Theorem 12 implies that $\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$ is pathwise connected. Moreover, since $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$ is a closed subgroup of $\mathscr{G}(\mathscr{H})$, the existence of a local cross section for p implies that

$$\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T) \xrightarrow{\;i\;} \mathscr{G}(\mathscr{H}) \xrightarrow{\;p\;} \mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$$

is a fibre bundle [14, p. 57] (i is the inclusion map).

Also, via γ, the existence of a global cross section for s is equivalent to the existence of a global cross section for p.

7. CHARACTERIZATION OF $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$

Let T be similar to a nice Jordan operator J. Then $T = W^{-1}JW$ for some W in $\mathscr{G}(\mathscr{H})$ and

$$\mathscr{A}'(T) = \{W^{-1}AW : A \in \mathscr{A}'(J)\}.$$

Therefore $\mathscr{A}'(T) \simeq \mathscr{A}'(J)$, so it is enough to characterize $\mathscr{A}'(J)$. Let $\mathscr{H} = \bigoplus_{j=1}^{k} \mathscr{H}_j$, $Q = \bigoplus_{j=1}^{k} q_{n_j}^{(\alpha_j)} \in \mathscr{L}(\mathscr{H})$ ($q_{n_j}^{(\alpha_j)} \in \mathscr{L}(\mathscr{H}_j)$, $1 \leq n_1 < n_2 < \ldots < n_k$, $1 \leq \alpha_j \leq \infty$, for each $j = 1, 2, \ldots, k$). Each $\mathscr{H}_j$ can be decomposed as a direct sum $\mathscr{H}_j = \bigoplus_{l=1}^{n_j} \mathscr{H}_j^l$ (with $\dim \mathscr{H}_j^l = \alpha_j$) in such a way that the matrix $(Q_{j,rs})_{r,s=1}^{n_j}$ of $q_{n_j}^{(\alpha_j)}$ with respect to this decomposition satisfies $Q_{j,r,r+1} = 1_{\alpha_j}$ for $1 \leq r \leq n_j - 1$ and $Q_{j,rs} = 0$ for any other pair (r, s).

Let $A \in \mathscr{L}(\mathscr{H})$ and let $(A_{ij,rs})$ be the matrix of A with respect to the decomposition $\mathscr{H} = \bigoplus_{j=1}^{k} \left(\bigoplus_{l=1}^{n_j} \mathscr{H}_j^l \right) (A_{ij,rs} : \mathscr{H}_j^s \to \mathscr{H}_i^r)$. Straightforward computations show that the commutant $\mathscr{A}'(Q)$ is the set of all the operators A in $\mathscr{L}(\mathscr{H})$ which satisfy the following conditions:

a) If $j \geq i$, then $A_{ij,1s} = A_{ij,2,s+1} = \ldots = A_{ij,n_j+1-s,n_j}$ for $n_j + 1 - n_i \leq s \leq n_j$ and $A_{ij,rs} = 0$ for all $r > s + n_i - n_j$.

b) If $j < i$, then $A_{ij,1s} = A_{ij,2,s+1} = \ldots = A_{ij,n_j+1-s,n_j}$ for $1 \leq s \leq n_j$ and $A^i{}_{j,rs} = 0$ for all $r > s$, and

c) $A_{ij,1s} : \mathscr{H}_j^s \to \mathscr{H}_i^1$ is an arbitrary bounded linear mapping.

Let $A_{ij,s} = A_{ij,1s}$ and let $\Delta(A) = \bigoplus_{j=1}^{k} A_{jj,1}^{(n_j)}$.

The above notation is used in the following:

LEMMA 13. *If* $Q = \bigoplus_{j=1}^{k} q_{n_j}^{(\alpha_j)}$, *then*

$$\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q) = \left\{ A = (A_{ij,rs}) \in \mathscr{A}'(Q) : \Delta(A) = \bigoplus_{j=1}^{k} A_{jj,1}^{(n_j)} \text{ is invertible} \right\}.$$

LEMMA 14. *$\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$ is homotopy equivalent to a product*

$$\times \{\mathscr{G}(\mathbf{C}^{\alpha_j}) : 1 \leq j \leq k,\ \alpha_j < \infty\}.$$

Proof. By Lemma 13, $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$ is the set of matrices of operators $(A_{ij,rs})$ in $\mathscr{A}'(Q)$ with $\Delta(A) = \bigoplus_{j=1}^{k} A_{jj,1}^{(n_j)}$ invertible. Define

$$h : \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q) \times [0, 1] \to \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$$

by $h(A, t) = \Delta(A) + t(A - \Delta(A))$. Clearly, $h(A, t)$ is in $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$ for each t in $[0, 1]$ and h is continuous.

Since $h(A, 1) = A$ and $h(A, 0) = \Delta(A)$ for all A in $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$, we see that the set of invertible matrices of the form $\Delta(A)$ is homotopy equivalent to $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$.

Next, observe that the mapping

$$d: \times\{\mathscr{G}(\mathbf{C}^{\alpha_j}) : 1 \leqslant j \leqslant k\} \to \{\Delta(A) : A \text{ in } \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)\}$$

(if $\alpha_j = \infty$, then $\mathbf{C}^{\alpha_j}$ must be understood as $\mathscr{H}$) defined by

$$d(A_1, \ldots, A_k) = \bigoplus_{j=1}^{k} A_j^{(n_j)} = \bigoplus_{j=1}^{k} A_j \otimes 1_{n_j}$$

is a homotopy equivalence.

Since $\mathscr{G}(\mathscr{H})$ is contractible [11], we are done. ▨

COROLLARY 15. $\pi_n(\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)) = \oplus\{\pi_n(\mathscr{G}(\mathbf{C}^{\alpha_j})) : 1 \leqslant j \leqslant k,\ \alpha_j < \infty\}$. *In particular,*

$$\pi_1(\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)) = \mathbf{Z}^{(\beta)},$$

where $\beta = \operatorname{card}\{j : \alpha_j < \infty\}$, *and* $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(Q)$ *is pathwise connected.*

Proof. This follows from the fact that $\pi_1(\mathbf{C}^{\alpha_j}) = \mathbf{Z}$. ▨

PROPOSITION 16. *Let* $J = \bigoplus_{i=1}^{n} [\lambda_i 1_i + Q_i]$ *be a nice Jordan operator* (λ_1, $\lambda_2, \ldots, \lambda_n$ *are distinct complex numbers) with* $Q_i = \bigoplus_{j=1}^{m_i} q_{k_{ij}}^{(\alpha_{ij})}$, $1 \leqslant k_{i1} < k_{i2} < \ldots < k_{im_i}$ *and* $\alpha_{ij} = \infty$ *for at most one value of* j *for each* i, $1 \leqslant i \leqslant n$. *Then* $\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(J)$ *is homotopy equivalent to a product*

$$\times\{\mathscr{G}(\mathbf{C}^{\alpha_{ij}}) : \alpha_{ij} < \infty,\ 1 \leqslant j \leqslant m_i,\ 1 \leqslant i \leqslant n\}.$$

Proof. The fact that $\mathscr{A}'(\lambda_i 1_i + Q_i) = \mathscr{A}'(Q_i)$ and $\sigma(\lambda_i 1_i + Q_i) \cap \cap\, \sigma(\lambda_h 1_h + Q_h) = \emptyset$ for $i \neq h$ imply [8] that

$$\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(J) \simeq \times_{i=1}^{n} \mathscr{G}(\mathscr{H}_i) \cap \mathscr{A}'(Q_i).$$

Now the result follows from Lemma 14. ▨

8. GLOBAL SIMILARITY CROSS SECTIONS

THEOREM 17. *Let* T *be similar to a nice Jordan operator. The fibre bundle*

$$\mathscr{G}(T) \xrightarrow{i} \mathscr{G}(\mathscr{H}) \xrightarrow{p} \mathscr{G}(\mathscr{H})/\mathscr{G}(T),$$

where $\mathscr{G}(T) = \mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(T)$, has a global cross section if and only if T is similar to a very nice Jordan operator, that is, T is similar to an operator of the form $\bigoplus_{j=1}^{m} (\lambda_j + q_{k_j} \otimes 1)$, where $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct complex numbers and q_k is the $k \times k$ Jordan cell in $\mathscr{L}(\mathbf{C}^k)$.

Proof. Assume that T is similar to a very nice Jordan operator. By Proposition 16, $\mathscr{G}(T)$ is homotopy equivalent to the product of m copies of $\mathscr{G}(\mathscr{H})$; therefore it is contractible. By substituting $\mathscr{G}(\mathscr{H})$ for $\mathscr{U}(\mathscr{H})$ in the proof of Theorem 4, we obtain that $\mathscr{G}(\mathscr{H})/\mathscr{G}(T)$ is weakly homotopy equivalent to a contractible space. By Lemma 5, $(\mathscr{G}(\mathscr{H}), \mathscr{G}(T))$ is a relative CW-complex, whence $\mathscr{G}(\mathscr{H})/\mathscr{G}(T)$ is a CW-complex. Therefore $\mathscr{G}(\mathscr{H})/\mathscr{G}(T)$ is actually contractible [14]. As in the proof of Theorem 4, it follows that p has a global cross section.

If T is similar to a nice Jordan operator $J = \bigoplus_{i=1}^{n} \left(\lambda_i + \bigoplus_{j=1}^{m} q_{k_{ij}}^{(\alpha_{ij})}\right)$ with some $\alpha_{ij} < \infty$ (i.e., J is not very nice), then Corollary 16 states that $\pi_1(\mathscr{G}(T)) = \mathbf{Z}^{(\beta)}$, where $\beta = \sum_{i=1}^{n} \operatorname{card}\{j : \alpha_{ij} < \infty\} \geq 1$.

If we substitute $\mathscr{G}(\mathscr{H})$ for $\mathscr{U}(\mathscr{H})$ and β for m in the proof of Theorem 7, we obtain that p does not have a global cross section. ▨

In the finite dimensional case, we have the following result:

THEOREM 18. *Let $T \in \mathscr{L}(\mathbf{C}^n)$. The fibre bundle*

$$\mathscr{G}(T) \xrightarrow{i} \mathscr{G}(\mathbf{C}^n) \xrightarrow{p} \mathscr{G}(\mathbf{C}^n)/\mathscr{G}(T),$$

where $\mathscr{G}(T) = \mathscr{G}(\mathbf{C}^n) \cap \mathscr{A}'(T)$, has a global cross section if and only if T is a multiple of the identity.

Proof. Let us assume that T is similar to $J = \bigoplus_{i=1}^{p} \left[\gamma_i 1_i + \bigoplus_{j=1}^{m_i} q_{k_{ij}}^{(\alpha_{ij})}\right]$, where $\lambda_1, \lambda_2, \dots, \lambda_p$ are distinct complex numbers and $1 \leq k_{i1} < \dots < k_{im_i}$. If there are two or more terms of the form $q_{k_{ij}}^{(\alpha_{ij})}$ then $\pi_1(\mathscr{G}(T)) = \pi_1(\mathscr{G}(J))$ has at least two direct summands isomorphic with $\mathbf{Z}$. Therefore the exact homotopy sequence

$$\dots \to \pi_1(\mathscr{G}(T)) \xrightarrow{i_*} \pi_1(\mathscr{G}(\mathbf{C}^n)) \simeq \mathbf{Z} \xrightarrow{p_*} \pi_1(\mathscr{G}(\mathbf{C}^n)/\mathscr{G}(T)) \to 0$$

cannot split. This implies that p does not have a global cross section.

Now assume that T is similar to $\lambda 1_{km} + q_k^{(m)}$. Let $\mathscr{G}(q_k^{(m)})$ denote the intersection $\mathscr{G}(\mathbf{C}^{km}) \cap \mathscr{A}'(\lambda 1_{km} + q_k^{(m)}) = \mathscr{G}(\mathbf{C}^{km}) \cap \mathscr{A}'(q_k^{(m)})$. By Corollary 15, $\pi_1(\mathscr{G}(q_k^{(m)})) \simeq \simeq \pi_1(\mathscr{G}(\mathbf{C}^m)) \simeq \mathbf{Z}$.

Let $[\gamma_m]$ be the generator of $\pi_1(U(m)) = \pi_1(\mathscr{G}(\mathbf{C}^m))$ defined in Lemma 8, and let $d:\mathscr{G}(\mathbf{C}^m) \to \mathscr{G}(q_k^{(m)})$ be the mapping defined in Lemma 14. Since $d(\gamma_m(t)) = \operatorname{diag}(\gamma_m(t),\ldots,\gamma_m(t))$ is homotopic with $(\gamma_{mk}(t))^{(k)}$, we have $i_* d_*[\gamma_m] = k[\gamma_{mk}]$. A repetition of the argument in the proof of Lemma 8 (with q replaced by k) shows that p does not have a global cross section unless $k = 1$. ▨

9. SIMILARITY CROSS SECTIONS IN THE CALKIN ALGEBRA

With the same notation as in Section 5, let $\mathscr{G}(\mathscr{A})$ denote the group of invertible elements of $\mathscr{A}$ and let $\mathscr{G}_0(\mathscr{A})$ be the pathwise connected component of $\tilde{1}$ in $\mathscr{G}(\mathscr{A})$. Let

$$S(\tilde{T}) = \{\tilde{W}^{-1}\tilde{T}\tilde{W} : \tilde{W} \in \mathscr{G}(\mathscr{A})\}$$

and let $s: \mathscr{G}(\mathscr{A}) \to S(\tilde{T})$ be defined by $s(\tilde{W}) = \tilde{W}^{-1}\tilde{T}\tilde{W}$, $\tilde{W} \in \mathscr{G}(\mathscr{A})$.

The mapping $s: \mathscr{G}(\mathscr{A}) \to \mathscr{S}(\tilde{T})$ has a local cross section if and only if T is similar to a compact perturbation of a nice Jordan operator [6]. As in Section 5, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{G}(\mathscr{A}) & \xrightarrow{\;p\;} & \mathscr{G}(\mathscr{A})/\mathscr{G}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \\ {\scriptstyle s}\downarrow & \swarrow{\scriptstyle \varphi} & \\ \mathscr{S}(\tilde{T}) & & \end{array}$$

where φ is continuous and bijective.

When $\tilde{T} = \lambda\tilde{1}$, s and p obviously have global cross sections ($\mathscr{G}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) = \mathscr{G}(\mathscr{A})$, and $\mathscr{S}(\tilde{T}) = \{\tilde{T}\}$). We will prove that p does not have a global cross section if $\tilde{T}$ is not a multiple of the identity. It will follow that s does not have one either, as in Section 5.

Consider the fibre bundle

$$\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \xrightarrow{i_0} \mathscr{G}_0(\mathscr{A}) \xrightarrow{p_0} \mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}).$$

An argument completely analogous to the one employed in Section 5 shows that p has a global cross section only if the exact homotopy sequence of $p_0 : \mathscr{G}_0(A) \to \mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$,

$$\ldots \xrightarrow{\partial} \pi_n(\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}),\ \tilde{1}) \xrightarrow{i_{0*}} \pi_n(\mathscr{G}_0(\mathscr{A}),\ \tilde{1}) \xrightarrow{p_{0*}}$$
$$\xrightarrow{p_{0*}} \pi_n(\mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}),\ p_0(\tilde{1})) \xrightarrow{\partial} \ldots$$

splits.

Since $\mathscr{G}_0(\mathscr{A})$ is isomorphic with $\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})$, we have that

$$\pi_n(\mathscr{G}_0(\mathscr{A})) = \pi_n(\mathscr{G}(\mathscr{H})/\mathscr{G}(\mathscr{H}) \cap (1 + \mathscr{K})) \simeq \pi_n(BU) = \begin{cases} 0, & n \text{ odd}, \\ \mathbf{Z}, & n \text{ even}, \ n \geqslant 1 \end{cases}$$

[15, p. 215] and $\pi_0(\mathscr{G}_0(\mathscr{A})) = 0$.

Let $\mathscr{G}^0(T)$ be the component of $\tilde{1}$ in $\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$. For the computation of $\pi_n(\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) = \pi_n(\mathscr{G}^0(\tilde{T}), \tilde{1})$ $(n \geqslant 1)$ we need the following:

LEMMA 19. *Let* $\pi\colon \mathscr{L}(\mathscr{H}) \to \mathscr{A} = \mathscr{L}(\mathscr{H})/\mathscr{K}$ *be the canonical projection of* $\mathscr{L}(\mathscr{H})$ *onto* $\mathscr{A}$. *If* $J \in \mathscr{L}(\mathscr{H})$ *is a nice Jordan operator, then* $\mathscr{A}'(\tilde{J}) = \pi(\mathscr{A}'(J))$ *and* $\mathscr{G}^0(\tilde{J}) = \pi(\mathscr{G}(\mathscr{H}) \cap \mathscr{A}'(J))$.

PROPOSITION 20. *Let* $T \in \mathscr{A}$ *be similar to* $\pi(\lambda 1 + q_k^{(\infty)})$, $k \geqslant 2$. *Then* $p_0\colon \mathscr{G}_0(\mathscr{A}) \to \mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$ *does not have a global cross section.*

Proof. By Lemmas 13 and 19, if $\tilde{W}$ belongs to $\mathscr{G}^0(\tilde{T})$ then it can be represented as a matrix

$$\tilde{W} = \begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & \cdots & & \tilde{V}_k \\ & \tilde{V}_1 & \tilde{V}_2 & & & \vdots \\ & & \tilde{V}_1 & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & 0 & & & \tilde{V}_1 & \tilde{V}_2 \\ & & & & & \tilde{V}_1 \end{pmatrix}$$

with $\tilde{V}_1 \in \mathscr{G}_0(\mathscr{A}(\mathscr{H}_1))$. As in Lemma 14, it follows that $\mathscr{G}^0(\tilde{T})$ is homotopy equivalent to $\mathscr{G}_0(\mathscr{A}(\mathscr{H}_1))$; the map $\tilde{W} \to \tilde{V}_1$ is a homotopy equivalence and its homotopy inverse is the map $\tilde{V}_1 \to \tilde{V}_1^{(k)}$. This implies that the inclusion map $i_0\colon \mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}) \to \mathscr{G}_0(\mathscr{A})$ induces the homomorphism "multiplication by k" in the second homotopy group. The same argument as in Case 1 of Theorem 11 leads to the conclusion that p_0 does not have a global cross section unless $k = 1$. ▨

PROPOSITION 21. *Let* $\tilde{T} \in \mathscr{A}$ *be similar to* $\tilde{J}$, *where* $J = \bigoplus_{i=1}^{n} (\lambda_i 1_i + q_{k_i}^{(\infty)})$ *is a very nice Jordan operator and* $n \geqslant 2$. *Then* $p_0\colon \mathscr{G}_0(\mathscr{A}) \to \mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$ *does not have a global cross section.*

Proof. Let $J = J_1 \oplus J_2 \oplus J_3$, where $J_1 = \lambda_1 1_1 + q_{k_1}^{(\infty)}$, $J_2 = \lambda_2 1_2 + q_{k_2}^{(\infty)}$ and $J_3 = \bigoplus_{i=3}^{n} (\lambda_i 1_i + q_{k_i}^{(\infty)})$. Then

$$\mathscr{G}^0(\tilde{J}) = \{\tilde{W} \in \mathscr{G}(\mathscr{A}) \cap \mathscr{A}'(\tilde{J}) : \tilde{W} = \tilde{W}_1 \oplus \tilde{W}_2 \oplus \tilde{W}_3,\ \tilde{W}_j \in \mathscr{G}^0(\tilde{J}_j),\ j = 1,2,3\}$$

(see Lemma 19 and the proof of Proposition 20). Thus,

$$\pi_2(\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) \simeq \bigoplus_{j=1}^{3} \pi_2(\mathscr{G}^0(\tilde{J}_j)) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \pi_2(\mathscr{G}^0(\tilde{J}_3)).$$

Since the latter group cannot be a direct summand of $\pi_2(\mathscr{G}_0(\mathscr{A})) \simeq \mathbf{Z}$, it follows that the exact homotopy sequence

$$\ldots \to \pi_2(\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), \tilde{1}) \to \pi_2(\mathscr{G}_0(\mathscr{A}), \tilde{1}) \to \pi_2(\mathscr{G}_0(\mathscr{A})/\mathscr{G}_0(\mathscr{A}) \cap \mathscr{A}'(\tilde{T}), p_0(\tilde{1})) \to$$
$$\to 0 \to 0 \to$$

cannot split. Therefore p_0 does not have a global cross section. ▧

The results of this section can be summarized in the following.

THEOREM 22. *$p: \mathscr{G}(\mathscr{A}) \to \mathscr{G}(\mathscr{A})/\mathscr{G}(\mathscr{A}) \cap \mathscr{A}'(\tilde{T})$ has a global cross section if and only if $\tilde{T} = \lambda\tilde{1}$ for some $\lambda \in \mathbf{C}$.*

As a consequence, we deduce that if an operator T has a global similarity cross section $\varphi: \mathscr{S}(T) \to \mathscr{G}(\mathscr{H})$, then φ cannot preserve compact differences unless T is a multiple of the identity.

REFERENCES

1. ATIYAH, M., *K-theory*, W. A. Benjamin Inc., New York—Amsterdam, 1967.
2. DECKARD, D.; FIALKOW, L. A., Characterization of Hilbert space operators with unitary cross sections, *J. Operator Theory*, **2**(1979), 153—158.
3. ERNEST, J., *Charting the operator terrain*, Memoirs Amer. Math. Soc., N° **171**, Amer. Math. Soc., Providence, R. I., 1976.
4. FIALKOW, L. A., A note on limits of unitarily equivalent operators, *Trans. Amer. Math. Soc.*, **232**(1977), 205—220.
5. FIALKOW, L. A., Similarity cross sections for operators, *Indiana Univ. Math. J.*, **28**(1979), 71—86.
6. FIALKOW, L. A.; HERRERO, D. A., Characterization of Hilbert space operators with similarity cross sections, preprint, 1978.

7. FIALKOW, L. A.; HERRERO, D. A., Characterization of operators having local similarity cross sections, *Notices Amer. Math. Soc.*, **25**(1978), A-357.
8. FIALKOW, L. A.; HERRERO, D. A., Inner derivations with closed range and local similarity cross sections in the Calkin algebra, preprint, 1979.
9. HERRERO, D. A.; SALINAS, N., Analytically invariant and bi-invariant subspaces, *Trans. Amer. Math. Soc.*, **173**(1972), 117–136.
10. KOSCHKORKE, U., Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps, *Proceedings of Symposia in Pure Math.*, Vol. XV, Amer. Math. Soc., Providence, Rhode Island, 1970, 95–134.
11. KUIPER, N. H., The homotopy type of the unitary group of Hilbert space, *Topology*, **3**(1965), 19–30.
12. LUNDELL, A. T.; WEINGRAM, S., *The topology of* CW-*complex*, Van Nostrand Reinhold, New York–Toronto, 1969.
13. MILNOR, J., On spaces having the homotopy of a CW-complex, *Trans. Amer. Math. Soc.*, **90**(1959), 272–280.
14. SPANIER, E. H., *Algebraic topology*, Mc Graw-Hill, New York–Toronto, 1966.
15. SWITZER, R. M., *Algebraic topology–Homotopy and homology*, Springer-Verlag, New York–Heidelberg–Berlin, 1975.

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