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SOME REMARKS ON LOCAL SPECTRAL THEORY

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1. INTRODUCTION

There are several possibilities to obtain a "local" spectral theory for a bounded linear operator T on a (complex) Banach space X. For example, one may consider a suitable unital closed subalgebra $\mathscr{A}$ of $\mathscr{L}(X)$ (the algebra of all bounded linear operators on X) or of the Calkin algebra $Q(X) := \mathscr{L}(X)/\mathscr{K}(X)$ such that T resp. $T + \mathscr{K}(X)$ is an element of $\mathscr{A}$. If $\mathscr{A}$ is semi-simple then this algebra $\mathscr{A}$ can be represented as an algebra of vector valued functions (in the sense of G. R. Allan [8]) over the compact Hausdorff space $\mathfrak{M}_Z$ of all extendable maximal ideals of some unital central subalgebra Z of $\mathscr{A}$, where Z is endowed with some norm $|\cdot|_Z$ such that $(Z,|\cdot|_Z)$ is a Banach algebra. We refer to [8] for more details. This method, which gives a local spectral theory for T in the framework of Banach algebra theory, has been successfully applied to the study of Toeplitz operators, Hankel operators, and others (see for example [17, 18, 19, 30, 31]).

Another kind of local spectral theory is obtained if one considers the spectral decomposition properties of T with respect to the underlying Banach space X. We refer to the monographs [14, 22, 35] for this "spatial" local spectral theory. Recall [14, 3] that an operator $T \in \mathscr{L}(X)$ is *decomposable* if and only if for each open covering $\{U_1, U_2\}$ of the complex plane $\mathbf{C}$ there are closed invariant subspaces Y_1, Y_2 for T with $X = Y_1 + Y_2$ and such that the spectrum $\mathrm{sp}(T, Y_j)$ of T/Y_j is contained in U_j ($j = 1,2$). This class of operators is rather large. It contains the class of spectral operators (in the sense of [22]), the class of Riesz operators and more generally the class of $\mathfrak{A}$-decomposable operators in the sense of I. Colojoară and C. Foiaş [14]. However, there are some important operators of mathematical analysis such as Toeplitz operators with continuous (or more general with quasicontinuous) symbol which are not decomposable in spite of the fact that they have rich functional calculi modulo the ideal of compact operators. Therefore, a "spatial" local spectral theory modulo $\mathscr{K}(X)$, i.e. for elements of $Q(X)$ seems to be of some interest. In order to introduce such a theory we have to represent $Q(X)$ as an algebra of bounded linear operators on some suitable Banach space. A canonical construction of such a representation has already successfully been used in the Fredholm theory (cf. [13, 23]). Denote by $\ell^\infty(X)$ the Banach space (or the normed space, if X is only normed) of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of X, endowed with the norm $\|\cdot\|_\infty$ given by $\|x\|_\infty := \sup_{n\in\mathbf{N}} \|x_n\|$ and write T_∞ for the operator induced by T on $\ell^\infty(X)$ (defined by $T_\infty x := (Tx_n)_{n=1}^\infty$ for $x = (x_n)_{n=1}^\infty \in \ell^\infty(X)$). The set $\mathrm{pc}(X)$ of all precompact sequences of elements of X is a closed subspace of $\ell^\infty(X)$ which is invariant for T_∞. We write $X_q := \ell^\infty(X)/\mathrm{pc}(X)$ and denote by T_q the operator induced by T_∞ on X_q. The mapping $T \to T_q$ is a unital homomorphism from $\mathscr{L}(X)$ to $\mathscr{L}(X_q)$ with kernel $\mathscr{K}(X)$ and induces a norm decreasing monomorphism from $Q(X)$ to $\mathscr{L}(X_q)$ (see [13] for details).

Let us remark that for Hilbert spaces H this induced monomorphism is even an isometry. Indeed, if T is an arbitrary operator in $\mathscr{L}(H)$, then

$$\|T^* + \mathscr{K}(H)\|_{Q(H)}\cdot\|T + \mathscr{K}(H)\|_{Q(H)} = \|T^*T + \mathscr{K}(H)\|_{Q(H)} =$$
$$= r(T^*T + \mathscr{K}(H)) \leqslant r((T^*T)_q) \leqslant \|(T^*T)_q\| \leqslant$$
$$\leqslant \|(T^*)_q\|\cdot\|T_q\| \leqslant \|T^* + \mathscr{K}(H)\|_{Q(H)}\cdot\|T_q\|$$

which implies $\|T + \mathscr{K}(H)\|_{Q(H)} \leqslant \|T_q\|$. Here, r denotes the spectral radius.

If X is again an arbitrary Banach space, then an operator $T \in \mathscr{L}(X)$ will be called *essentially decomposable* if T_q is decomposable. As we shall see, this class of operators contains the class of decomposable operators, the class of Toeplitz operators with quasicontinuous symbols and many other operators.

This paper is organized as follows: In the following part we include some elementary properties of the functor q and some related functors. Especially, we prove that each decomposable operator is essentially decomposable. In the third section we prove some decomposability criteria using the above mentioned Banach algebra localization methods. This has application to matrices of $\mathfrak{A}$-decomposable operators and is related to questions considered in [9, 14, 21, 22]. In the fourth section we give some first applications of the "essential" local spectral theory, and the last part contains a characterization of local type operators.

2. SOME BASIC PROPERTIES OF THE FUNCTOR q

Let $T\colon X \to Y$ be a continuous linear operator. As before, T induces a bounded linear operator $T_\infty\colon \ell^\infty(X) \to \ell^\infty(Y)$ with $T_\infty(\mathrm{pc}(X)) \subset \mathrm{pc}(Y)$. Thus, the operator $T_q\colon X_q \to Y_q$ defined by $T_q(x + \mathrm{pc}(X)) := (Tx_n)_{n=1}^\infty + \mathrm{pc}(Y)$ for $x = (x_n)_{n=1}^\infty \in \ell^\infty(X)$ is a bounded linear operator. Moreover, $\|T_q\| \leqslant \|T\|$.

2.1. LEMMA. *For a bounded linear operator $T\colon X \to Y$ are equivalent.*

(a) *The range* $\mathrm{R}(T)$ *of T is closed.*

(b) *For each precompact sequence* $y = (Tx_n)_{n=1}^\infty$ *in* $\mathrm{R}(T)$ *there exists a precompact sequence* $u = (u_n)_{n=1}^\infty$ *in* X *with* $T_\infty u = y$ *and we have* $(x_n + \mathrm{N}(T))_{n=1}^\infty \in \mathrm{pc}(X/\mathrm{N}(T))$, *where* $\mathrm{N}(T)$ *denotes the kernel of* T.

(c) $\mathrm{N}(T_q) = (\ell^\infty(\mathrm{N}(T)) + \mathrm{pc}(X))/\mathrm{pc}(X)$.

(d) $\mathrm{R}(T_q) = (\ell^\infty(R(T)) + \mathrm{pc}(Y))/\mathrm{pc}(Y)$.

(e) *The range of the transposed operator* $T' \in \mathscr{L}(Y', X')$ *is closed in* X'.

Proof. The equivalence of (a) and (e) is well known. We shall prove $(a) \Rightarrow (b) \Rightarrow$ $\Rightarrow$ (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (d) $\Rightarrow$ (e).

"(a) $\Rightarrow$ (*b*)" If $\mathrm{R}(T)$ is closed then $\mathrm{R}(T)$ is topologically isomorphic to $X/\mathrm{N}(T)$ by means of $x + \mathrm{N}(T) \mapsto Tx$ for $x \in X$. Since it is well known that precompact sequences in a quotient space admit a precompact lifting (as they are contained in the closed convex hull of a null sequence), we obtain (b).

"(b) $\Rightarrow$ (c)" From the definition of T_q we conclude that

$$(\ell^\infty(\mathrm{N}(T)) + \mathrm{pc}(X))/\mathrm{pc}(X) \subset \mathrm{N}(T_q).$$

Assume now that (b) holds and fix an arbitrary $x = (x_n)_{n=1}^\infty + \mathrm{pc}(X) \in \mathrm{N}(T_q)$. Then $(Tx_n)_{n=1}^\infty \in \mathrm{pc}(Y)$ so that by (b) we find $u = (u_n)_{n=1}^\infty \in \mathrm{pc}(X)$ such that $T(x_n - u_n) = 0$ for all $n \in \mathbf{N}$. Clearly $y := (x_n - u_n)_{n=1}^\infty \in \ell^\infty(\mathrm{N}(T))$ and hence

$$x = y + u + \mathrm{pc}(X) \in (\ell^\infty(\mathrm{N}(T)) + \mathrm{pc}(X))/\mathrm{pc}(X).$$

This proves (c).

"(c) $\Rightarrow$ (a)" The mapping $S: X/\mathrm{N}(T) \to Y$ given by $S(x + \mathrm{N}(T)) := Tx$ for $x \in X$ is injective and satisfies $\mathrm{R}(S) = \mathrm{R}(T)$. Assume now that (c) holds and that $\mathrm{R}(T)$ is not closed, i.e. that S is not bounded from below. Then there exists a sequence $(x_n + \mathrm{N}(T))_{n=1}^\infty$ in $X/\mathrm{N}(T)$ with $S(x_n + \mathrm{N}(T)) = Tx_n \to 0$ for $n \to \infty$ and $\|x_n + \mathrm{N}(T)\| = 1$ (in $X/\mathrm{N}(T)$) for all $n \in \mathbf{N}$. As the sequence $(x_n + \mathrm{N}(T))_{n=1}^\infty$ is bounded in $X/\mathrm{N}(T)$ we can find a sequence $u = (u_n)_{n=1}^\infty \in \ell^\infty(X)$ such that $Tx_n = Tu_n$ for all $n \in \mathbf{N}$. Because of $Tu_n \to 0$ for $n \to \infty$ we have $u + \mathrm{pc}(X) \in \mathrm{N}(T_q)$. By (c) we find sequences $v = (v_n)_{n=1}^\infty \in \ell^\infty(\mathrm{N}(T))$ and $w = (w_n)_{n=1}^\infty \in \mathrm{pc}(X)$ with $u = v + w$. Therefore, $Tw_n = Tv_n + Tw_n = Tx_n \to 0$ for $n \to \infty$, w has a convergent subsequence $(w_{n_k})_{k=1}^\infty$ with limit $a \in X$. Then $Ta = \lim_{k\to\infty} Tw_{n_k} = 0$ and thus $a \in \mathrm{N}(T)$. This shows that $w_{n_k} + \mathrm{N}(T) \to 0$ in $X/\mathrm{N}(T)$ for $k \to \infty$ in contradiction to $\|w_{n_k} + \mathrm{N}(T)\| = \|x_{n_k} + \mathrm{N}(T)\| = 1$ in $X/\mathrm{N}(T)$ for all $k \in \mathbf{N}$. Hence (c) implies (a).

"(a) $\Rightarrow$ (d)" Assume that $\mathrm{R}(T)$ is closed. From the definition of T_q we conclude that $\mathrm{R}(T_q) \subset (\ell^\infty(\mathrm{R}(T)) + \mathrm{pc}(Y))/\mathrm{pc}(Y)$. Conversely, if $y = (Tx_n)_{n=1}^\infty \in \ell(\mathrm{R}(T))$, then by the open mapping theorem there is a bounded sequence $u = (u_n)_{n=1}^\infty$ in X with $T_\infty u = y$. Hence, $y + \mathrm{pc}(Y) = T_q(u + \mathrm{pc}(X)) \in \mathrm{R}(T_q)$ and (d) is proved.

"(d) $\Rightarrow$ (e)" Suppose that (d) holds and assume that $\mathrm{R}(T')$ is not closed. Then there exists a sequence $y' = (y_n')_{n=1}^\infty$ in Y' such that $\|y_n' + \mathrm{N}(T')\| =$

$= \operatorname{dist}(y'_n, N(T')) = 1$ for all $n \in \mathbf{N}$ and such that $T'y'_n \to 0$ in X'. The sequence $(y'_n + N(T'))_{n=1}^{\infty}$ is bounded in $Y'/N(T')$ and hence has a bounded lifting so that, without loss of generality, we may suppose that $y' \in \ell^{\infty}(Y')$. Let us now prove that $y' \notin \mathrm{pc}(Y')$. Indeed, if $y' \in \mathrm{pc}(Y')$ then y' would have a convergent subsequence $(y'_{n_j})_{j=1}^{\infty}$, $y'_{n_j} \to a' \in Y'$ for $j \to \infty$. But then $0 = \lim_{j\to\infty} T'y'_{n_j} = T'a'$, so that $a' \in N(T')$ in contradiction to $\operatorname{dist}(y'_{n_j}, N(T')) = 1$ for all $j \in \mathbf{N}$. Now also $(y'_n + N(T'))_{n=1}^{\infty} \notin \mathrm{pc}(X'/N(T'))$, otherwise we could find a precompact lifting and we just proved that this is impossible. Hence, there exists an $\varepsilon > 0$ and a subsequence $(y'_{n_k})_{k=1}^{\infty}$ of y' such that

$$\|y'_{n_k} - (y'_{n_m} + N(T'))\| \geqslant \varepsilon \quad \text{for all } k, m \in \mathbf{N} \text{ with } k \neq m.$$

Furthermore we have $R(T)^{\perp} = N(T')$ and therefore an isometric isomorphism form $R(T)'$ onto $Y'/N(T')$. Hence there are $x_{k,m} \in X$ with $\|Tx_{k,m}\| \leqslant 1$ such that

$$|\langle y'_{n_k} - y'_{n_m}, Tx_{k,m}\rangle| \geqslant \frac{\varepsilon}{2} \tag{1}$$

$(k, m \in \mathbf{N}, k \neq m)$. By (d) there exist $u_{k,m} \in X$, $v_{k,m} \in Y$ for $k, m \in \mathbf{N}$, $k \neq m$, such that $Tx_{k,m} = Tu_{k,m} + v_{k,m}$, $\sup\{\|u_{k,m}\|;\ k, m \in \mathbf{N},\ k \neq m\} < \infty$, and such that the double sequence $(v_{k,m})$ is precompact. The functions y'_{n_k} are equicontinuous and uniformly bounded on the compact set $K := \overline{\{v_{k,m} \mid k, m \in \mathbf{N}, k \neq m\}}$, so that by the Arzela-Ascoli theorem there exists a subsequence $(y'_{n_{k_r}})_{r=1}^{\infty}$ which is uniformly convergent on K. Hence there is some $N_1 \in \mathbf{N}$ such that

$$|\langle y'_{n_{k_r}} - y'_{n_{k_p}}, v_{k_r, k_p}\rangle| < \frac{\varepsilon}{6} \tag{2}$$

for all $r, p \geqslant N_1$ with $r \neq p$. On the other hand there exists some $N \geqslant N_1$ such that for all $r, p \geqslant N$ we have

$$|\langle y'_{n_{k_r}} - y'_{n_{k_p}}, Tu_{k_r, k_p}\rangle| = |\langle T'(y'_{n_{k_r}} - y'_{n_{k_p}}), u_{k_r, k_p}\rangle| < \frac{\varepsilon}{6}. \tag{3}$$

Here we used $T'y'_{n_{k_r}} \to 0$ for $r \to \infty$ and the fact that the double sequence (u_{k_r, k_p}) is bounded. From (1), (2), and (3) we now obtain the contradiction $\frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}$. Hence T' must have closed range.

Our first corollary of 2.1 may also be proved by means of [23].

2.2. COROLLARY. *Let Y be a closed subspace of the Banach space X. Then $q_X(Y) := (\ell^\infty(Y) + \mathrm{pc}(X))/\mathrm{pc}(X)$ is a closed subspace of X_q which is topologically isomorphic to Y_q. Moreover, if $T \in \mathscr{L}(X)$ such that Y is invariant for T then $q_X(Y)$ is invariant for T_q and* $\mathrm{sp}(T_q, q_X(Y)) \subset \mathrm{sp}(T, Y)$.

Proof. Let $\pi: X \to X/Y$ be the canonical epimorphism. By 2.1 ((a) $\Rightarrow$ (c)) we obtain $\mathrm{N}(\pi_q) = (\ell^\infty(Y) + \mathrm{pc}(X))/\mathrm{pc}(X) = q_X(Y)$ (because of $\mathrm{N}(\pi) = Y$). Hence $q_X(Y)$ is closed. Moreover, if Y is invariant for $T \in \mathscr{L}(X)$, then obviously Y_q is invariant for T_q. If $z \notin \mathrm{sp}(T, Y)$ then $zI - T|Y$ and therefore also $(z - T)_q|q_X(Y)$ is surjective. If $(y_n)_{n=1}^\infty + \mathrm{pc}(X) \in \mathrm{N}((zI - T)_q|q_X(Y))$ with $(y_n) \in \ell^\infty(Y)$ then $((zI - T)y_n)_{n=1}^\infty \in \mathrm{pc}(X) \cap \ell^\infty(Y) = \mathrm{pc}(Y)$. Hence, because of $y_n = ((zI - T) \mid Y)^{-1}((zI - T)y_n)$ for all $n \in \mathrm{N}$ and because of the continuity of $((zI - T) \mid Y)^{-1}$ on Y we conclude that $(y_n)_{n=1}^\infty \in \mathrm{pc}(Y) \subset \mathrm{pc}(X)$. Therefore $(zI - T)_q|q_X(Y)$ is invertible and $z \notin \mathrm{sp}(T_q, q_X(Y))$. Finally, the mapping defined by $y + \mathrm{pc}(Y) \to y + \mathrm{pc}(X)$ for $y \in \ell^\infty(Y)$ is a continuous isomorphism from Y_q onto $q_X(Y)$ which must be topological as $q_X(Y)$ is closed.

In the last section the following corollary will be needed:

2.3. COROLLARY. *Let X and Z be Banach spaces and let Y be a closed linear subspace of Z. Then there is a constant $C = C(Y, Z) > 0$ such that for all $T \in \mathscr{L}(X, Y)$ we have $\|T_q\| \leqslant C \cdot \|(j \circ T)_q\|$ where $j : Y \to Z$ is the canonical inclusion mapping.*

Proof. By Corollary 2.2 there exists a constant $C = C(Y, Z) > 0$ such that for all $y = (y_n)_{n=1}^\infty \in \ell^\infty(Y)$ we have

$$\|y + \mathrm{pc}(Y)\|_{Y_q} \leqslant C\|y + \mathrm{pc}(Z)\|_{Z_q}.$$

Hence, for all $x = (x_n)_{n=1}^\infty \in \ell^\infty(X)$ and all $T \in \mathscr{L}(X, Y)$ we have

$$\|T_q(x + \mathrm{pc}(X))\|_{Y_q} = \|(Tx_n)_{n=1}^\infty + \mathrm{pc}(Y)\|_{Y_q} \leqslant$$
$$\leqslant C\|(Tx_n)_{n=1}^\infty + \mathrm{pc}(Z)\|_{Z_q} = \|(j \circ T)_q(x + \mathrm{pc}(X))\|_{Z_q}$$

and the result follows.

2.4. COROLLARY. ([13], Theorem 2). *For a bounded linear operator $T: X \to Y$ are equivalent:*

(a) $T \in \varphi_+(X, Y)$, i.e. $\mathrm{R}(T)$ *is closed and* $\dim \mathrm{N}(T) < \infty$.
(b) T_q *is injective.*
(c) T_q *is bounded below.*

Proof. 1) If $T \in \varphi_+(X, Y)$ then, by 2.1 and 2.2, $\mathrm{R}(T_q)$ is closed and $\mathrm{N}(T_q) = (\ell^\infty(\mathrm{N}(T)) + \mathrm{pc}(X))/\mathrm{pc}(X)$ is the nullspace, as $\ell^\infty(\mathrm{N}(T)) = \mathrm{pc}(\mathrm{N}(T)) \subset \mathrm{pc}(X)$ because of $\dim \mathrm{N}(T) < \infty$. This proves that T_q is injective and bounded below.

2) Obviously (c) implies (b). Suppose now that T_q is injective. Because of $(\ell^\infty(\mathrm{N}(T)) + \mathrm{pc}(X))/\mathrm{pc}(X) \subset \mathrm{N}(T_q)$ this implies $\ell^\infty(\mathrm{N}(T)) \subset \mathrm{pc}(X)$ and therefore

$\dim N(T) < \infty$. Moreover, by 2.1 ((c) ⇒ (a)), $R(T)$ is closed and therefore $T \in \varphi_+(X, Y)$.

2.5. COROLLARY. [24]. *For a bounded linear operator $T: X \to Y$ are equivalent:*

(a) $T \in \varphi_-(X, Y)$, i.e. $R(T)$ *is closed and of finite codimension in* Y.

(b) T_q *is surjective.*

Proof. Suppose that $T \in \varphi_-(X, Y)$. Then there exists a finite dimensional subspace M of Y such that $Y = R(T) \oplus M$ topologically. We then have

$$\ell^\infty(Y) = \ell^\infty(R(T)) + \ell^\infty(M) = \ell^\infty(R(T)) + \mathrm{pc}(M) \subset$$

$$\subset \ell^\infty(R(T)) + \mathrm{pc}(Y) \subset \ell^\infty(Y),$$

as $\dim M < \infty$. Hence, by 2.1 ((a) ⇒ (d)),

$$R(T_q) = (\ell^\infty(R(T)) + \mathrm{pc}(Y))/\mathrm{pc}(Y) = Y_q.$$

Conversely, suppose now that $R(T_q) = Y_q$. Then

$$Y_q = R(T_q) \subset (\ell^\infty(R(T)) + \mathrm{pc}(Y))/\mathrm{pc}(Y) \subset Y_q. \tag{4}$$

From 2.1 ((d) ⇒ (a)) we conclude that $R(T)$ is closed. We still have to prove that $\dim Y/R(T) < \infty$. We prove this by showing that $\ell^\infty(Y/R(T)) \subset \mathrm{pc}(Y/R(T))$. For that take an arbitrary bounded sequence $(y_n + R(T))_{n=1}^\infty$ in $Y/R(T)$. This can be lifted to a bounded sequence in Y, so that we may suppose from the beginning that $y := (y_n)_{n=1}^\infty \in \ell^\infty(Y)$. Because of (4) there are sequences $(Tu_n)_{n=1}^\infty \in \ell^\infty(R(T))$ and $v := (v_n)_{n=1}^\infty \in \mathrm{pc}(Y)$ such that $y = (Tu_n)_{n=1}^\infty + v$, so that $(y_n + R(T))_{n=1}^\infty = (v_n + R(T))_{n=1}^\infty$ is precompact in $Y/R(T)$.

2.6. COROLLARY. [13]. *A bounded linear operator $T: X \to Y$ is a Fredholm operator if and only if $T_q: X_q \to Y_q$ is invertible.*

2.7. COROLLARY. *Every decomposable operator is essentially decomposable.*

Proof. Let $T \in \mathscr{L}(X)$ be decomposable. Consider an open covering $\{U_1, U_2\}$ of **C**. Then by the decomposability of T, there exist invariant subspaces X_1, X_2 of T such that $X = X_1 + X_2$ and $\mathrm{sp}(T, X_j) \subset U_j$ $(j = 1,2)$. Then $\ell^\infty(X_j)$ $(j = 1,2)$ are closed subspaces of $\ell^\infty(X)$. By applying the open mapping theorem to the natural map from $X_1 \oplus X_2 \to X$, we get $\ell^\infty(X) = \ell^\infty(X_1) + \ell^\infty(X_2)$. Then by 2.2, $q_X(X_j)$ are closed subspaces of X_q and $\mathrm{sp}(T_q, q_X(X_j)) \subset \mathrm{sp}(T, X_j) \subset U_j$ $(j = 1,2)$. Also

$$X_q = \ell^\infty(X)/\mathrm{pc}(X) = (\ell^\infty(X_1) + \ell^\infty(X_2))/\mathrm{pc}(X) =$$

$$= [(\ell^\infty(X_1) + \mathrm{pc}(X))/\mathrm{pc}(X)] + [(\ell^\infty(X_2) + \mathrm{pc}(X))/\mathrm{pc}(X)] = q_X(X_1) + q_X(X_2).$$

Hence T_q is decomposable i.e. T is essentially decomposable.

If $T \in \mathscr{L}(X)$, then for each open set Ω in $\mathbf{C}$, a mapping $\alpha_\Omega(T): H(\Omega, X) \to$ $\to H(\Omega, X)$ (the space of X-valued analytic functions on Ω) is given by

$$(\alpha_\Omega(T)g)(z) := (z - T)g(z)$$

for $g \in H(\Omega, X)$, $z \in \Omega$. Recall that T has the *single valued extension property* (SVEP) if and only if $\alpha_\Omega(T)$ is injective for every open Ω in $\mathbf{C}$. We say that $T \in \mathscr{L}(X)$ has the *essential SVEP* if T_q has the SVEP.

2.8. REMARK. The class of essentially decomposable operators is strictly larger than the class of decomposable operators. For let T be the adjoint of the unilateral shift on $H = \ell^2(\mathbf{N})$. Then T is essentially decomposable as $T + \mathscr{K}(H)$ and hence T_q is $\mathscr{C}(\mathbf{C})$-scalar in the sense of [14]. But T does not have the SVEP [14] and therefore T is not decomposable.

2.9. REMARK. Neither the SVEP implies nor is implied by the essential SVEP.

Proof. That essential SVEP does not imply the SVEP can be seen from the example in 2.8.

To show that the SVEP does not imply the essential SVEP, consider the operator $T = \bigoplus_{n=2}^{\infty} Q_n$ on $X = \bigoplus_{n=2}^{\infty} \mathbf{C}^n$ (the Hilbert space direct sum) as in [14], p. 25, where $Q_n =$ $= (q_{i,j})_{i,j=1}^n$ with $q_{i,i+1} = 1$ for $i = 1, \ldots, n-1$ and $q_{i,j} = 0$ otherwise. Then T has the SVEP. Consider the sequence $\{f_j\}$ of analytic X-valued functions defined on $G := \{z : |z| \leqslant 1/2\}$ by $f_j(z) := \bigoplus_{n=2}^{\infty} f_{n,j}(z)$ where

$$f_{n,j}(z) := \begin{cases} (1, z, \ldots, z^{n-1})^t & \text{for } j = n \\ 0 & \text{for } j \neq n \end{cases} \qquad (z \in G).$$

Then for no z in G, $\{f_j(\lambda)\}$ is in $\mathrm{pc}(X)$. But $(zI - T)f_j(z) = (zI_j - Q_j)f_{j,j}(z) \to 0$ as $j \to \infty$ for every $z \in G$. Hence $(zI_q - T_q)f(z) = 0$ in X_q where $f(z) = \{f_j(z)\} +$ $+ \mathrm{pc}(X)$. But $f(z) \neq 0$ in X_q for $z \in G$. So T_q does not have the SVEP or T does not have the essential SVEP.

For $T \in \mathscr{L}(X)$, one may also consider the space $X_a := \ell^\infty(X)/c_0(X)$, where $c_0(X)$ is the space of all null sequences in X, and the induced operator T_a on X_a.

An operator $T \in \mathscr{L}(X)$ is called *asymptotically decomposable* if T_a is decomposable. As in the case of essential decomposability, it can be proved that every decomposable operator is asymptotically decomposable. We do not know in general, whether this class is strictly larger than the class of decomposable operators, but in the special case when X is reflexive and separable, both these classes coincide. For the proof of this fact we shall first prove a general result.

2.10. PROPOSITION. *Let $T \in \mathcal{L}(X)$ be decomposable and Y be a closed invariant subspace for T. Then $T \mid Y$ has the SVEP and the spectral spaces $Y_{T\mid Y}(F)$ (F a closed subset of $\mathbf{C}$) are closed in Y.*

Proof. It is obvious that $T \mid Y$ has the SVEP and for a closed subset F of $\mathbf{C}$, $Y_{T\mid Y}(F) \subset Y \cap X_T(F)$. Let y be a limit point of $Y_{T\mid Y}(F)$. Then there exists a sequence $\{y_n\}$ in $Y_{T\mid Y}(F)$ such that $y_n \to y$. Let λ_0 be a point in $F' := \mathbf{C}\setminus F$. Choose an open neighbourhood U_{λ_0} of λ_0 with $U_{\lambda_0} \cap F = \emptyset$. Now since $y_n \in Y_{T\mid Y}(F)$, $y_n = (\lambda - T)y_n(\lambda)$ where $y_n(\cdot)$ is a Y-valued analytic function defined on $\rho_{T\mid Y}(y_n) \supset U_{\lambda_0}$. Also $y \in X_T(F)$ (note that $X_T(F)$ is closed as T is decomposable). Hence there exists an X-valued analytic function $y(\cdot)$ on U_{λ_0} such that $y = (\lambda - T)y(\lambda)$ $(\lambda \in U_{\lambda_0})$. But $y_n \to y$ i.e. $(\lambda - T)y_n(\lambda) \to (\lambda - T)y(\lambda)$ for $\lambda \in U_{\lambda_0}$. Since T is decomposable, $y_n(\lambda) \to y(\lambda)$ uniformly on compact subsets of U_{λ_0} ([3], Proposition 1). But $y_n(\lambda) \in Y$ and Y is closed. Therefore $y(\lambda) \in Y$ with $y = (\lambda - T)y(\lambda)$ $(\lambda \in U_{\lambda_0})$, which implies that $\lambda_0 \in \rho_{T\mid Y}(y)$ or $\sigma_{T\mid Y}(y) \subset F$ i.e. $y \in Y_{T\mid Y}(F)$, which completes the proof.

2.11. PROPOSITION. *Let $T \in \mathcal{L}(X)$ be asymptotically decomposable. If X is reflexive and separable then T is decomposable.*

Proof. Since X is isometrically embedded in X_a and $T_a \mid X = T$, by 2.10 T has the SVEP and the spaces $X_T(F)$ are closed in X. So it remains to show that for each open covering $\{U_1, U_2\}$ of $\mathrm{sp}(T)$, each x in X has a decomposition $x = x_1 + x_2$ with $\sigma_T(x_j) \subset U_j$ $(j = 1,2)$.

Let $\{U_j : j = 1,2\}$ be an open covering of $\mathrm{sp}(T)$. Choose closed subsets F_j of $\mathbf{C}$ such that $\{\mathrm{int}\, F_j : j = 1,2\}$ still covers $\mathrm{sp}(T)$ and $F_j \subset U_j$ $(j = 1,2)$. Let $x \in X$. Then by the decomposability of T_a, there exist sequences $\{x_j^n\}$ $(j = 1,2)$ in $\ell^\infty(X)$ and $\{c_n\}$ in $c_0(X)$ such that

$$x = x_1^n + x_2^n + c_n \quad (n \in \mathbf{N})$$

and

$$\sigma_{T_a}(\{x_j^n\} + c_0(X)) \subset \mathrm{int}\, F_j \quad (j = 1,2).$$

(Here we make the usual identification of an element x in X with the coset $\{x\} + c_0(X)$ in X_a corresponding to the constant sequence $\{x\}$.)

Let $\{f_j^n\}_{n\in\mathbf{N}}$ and $\{a_j^n\}_{n\in\mathbf{N}}$ be $\ell^\infty(X)$ and $c_0(X)$-valued analytic functions on $F' = \mathbf{C}\setminus F_j$ such that

$$x_j^n = (\lambda - T)f_j^n(\lambda) + a_j^n(\lambda) \quad (\lambda \in F_j').$$

Since X is reflexive and separable, its unit ball is compact and metrizable in the weak topology. Hence $\{x_j^n\}$ has a weakly convergent subsequence, say $\{x_j^{n,k}\}$. Suppose $y_j^k := x_j^{n,k} \to x_j$ weakly. Then $x = x_1 + x_2$. The proof is complete if we show that $\sigma_T(x_j) \subset F_j \subset U_j$. Let $g_j^k := f_j^{n,k}$ and define the linear transformations $S_j^k: (X', \|\cdot\|) \to H(F_j')$ by

$$(S_j^k x')(z) := x'(g_j^k(z)) \quad (x' \in X',\ z \in F_j').$$

Then $\{S_j^k\}_{k\in\mathbf{N}}$ is a family of continuous linear transformations. Also since $\{g_j^k\}$ is an $\ell^\infty(X)$-valued analytic function on F_j', it is bounded on compact subsets of F_j'. Therefore $\{S_j^k x'\}_{k\in\mathbf{N}}$ is bounded in $H(F_j')$ and hence is relatively compact by Montel's theorem, for each $x' \in X'$. So $\{S_j^k\}$ is equicontinuous by the Banach-Steinhaus theorem.

Let $\{x_m'\}_{m\in\mathbf{N}}$ be a countable dense set in X' (X' is separable as X is separable and reflexive). Now $\{S_j^k x_m'\}$ being relatively compact, has a convergent subsequence for each m. By a diagonalization, there exists a convergent sub-subsequence of $\{S_j^k x_m'\}$ which we shall again denote by $\{S_j^k x_m'\}$. Since $\{S_j^k\}$ is equicontinuous and $\{x_m'\}$ is dense in X', $S_j^k: X' \to H(F_j')$ converges pointwise. Hence, by the Banach-Steinhaus theorem, there exists a continuous linear map $S_j: X' \to H(F_j')$ such that $S_j^k x' \to$ $\to S_j x'$ for each $x' \in X'$.

For a fixed $z \in F_j'$, let $\varphi_j^z: X' \to \mathbf{C}$ be defined by $\varphi_j^z(x') = (S_j x')(z)$. Then $\varphi_j^z \in X'' = X$ (because X is reflexive). Let $g_j(z) = \varphi_j^z$. Then g_j is a X-valued weakly analytic function on F_j' and hence is strongly analytic. Also $g_j^k(z) \to g_j(z)$ weakly for each $z \in F_j'$ and hence $(\lambda - T)g_j^k(\lambda) \to (\lambda - T)g_j(\lambda)$ for every λ in F_j'. Consequently, for $x' \in X'$,

$$x'(x_j) = \lim x'(y_j^k) =$$

$$= \lim[x'((\lambda - T)g_j^k(\lambda) + a_j^{n,k}(\lambda))] = x'((\lambda - T)g_j(\lambda))$$

(as $a_j^{n,k}(\lambda) \to 0$) for each λ in F_j'.

So $x_j = (\lambda - T)g_j(\lambda)$ for each λ in F_j' by the Hahn-Banach theorem. Therefore $\sigma_T(x_j) \subset F_j$, which completes the proof.

3. DECOMPOSABILITY CRITERIA VIA THE LOCALIZATION METHOD OF ALLAN

In this section we use the methods of Allan [8] to obtain the (essential) decomposability for matrices of certain (essentially) commuting (essentially) decomposable operators. In fact our results may also be used for wider classes of operators. First we recall some definitions.

3.1. DEFINITION. Let K be a compact Hausdorff space and let $\mathscr{A}$ be a subalgebra (not necessarily closed) of the algebra $C(K)$ of all continuous complex valued functions on K.

(a) $\mathscr{A}$ is called *normal* if for each finite open covering $\{U_1, \ldots, U_n\}$ of K there are $f_1, \ldots, f_n \in \mathscr{A}$ such that $f_1 + \ldots + f_n \equiv 1$ and $\operatorname{supp} f_j \subset U_j$ for $j = 1, \ldots, n$.

(b) $\mathscr{A}$ is called *spectrally closed* if for each finite system $f = (f_1, \ldots, f_m)$ in $\mathscr{A}$ and each $w \in \mathbf{C}^m \setminus f(K)$ there are $u_1, \ldots, u_m \in \mathscr{A}$ such that $\sum_{j=1}^{m} (w_j - f_j)u_j \equiv 1$ on K.

3.2. EXAMPLES. (a) Every unital regular semi-simple commutative Banach algebra is normal and spectrally closed when considered as an algebra of continuous functions on its maximal ideal space (cf. [4], Example 4.4 (e)).

(b) Let Ω be a bounded open subset of $\mathbf{C}^n$, $K := \bar{\Omega}$ and fix a totally disconnected compact subset Λ of Ω. Then the algebra $\mathscr{A}_\Lambda$ of all continuous functions f on K which are analytic in some neighbourhood U_f of Λ (depending on f) is normal and spectrally closed in $C(K)$. Notice that there is no complete algebra norm on $\mathscr{A}_\Lambda$.

We shall need:

3.3. THEOREM. *Let $\Phi: \mathscr{A} \to \mathscr{B}$ be a unital homomorphism from some normal spectrally closed algebra $\mathscr{A}$ on some compact Hausdorff space K to some unital Banach algebra $\mathscr{B}$. Then the family of all closed subsets F of K such that Φ vanishes on $K \setminus F$ (*i.e. *such that $\Phi(f) = 0$ for all $f \in \mathscr{A}$ with* $\operatorname{supp} f \cap F = \emptyset$*) has a unique minimal element which will be denoted by* $\operatorname{supp}\Phi$. *For $t \in \operatorname{supp}\Phi$ the closure M_t of $\{\Phi(a) \mid a \in \mathscr{A},\ a(t) = 0\}$ is a maximal ideal in $\mathfrak{R} := \overline{\Phi(\mathscr{A})}$. Moreover the mapping $t \to M_t$ is a homeomorphism from* $\operatorname{supp}\Phi$ *onto the maximal ideal space $\Delta(\mathfrak{R})$ of $\mathfrak{R}$. If $\hat{}: \mathfrak{R} \to C(\Delta(\mathfrak{R}))$ denotes the Gelfand transform then $\Phi(a)\hat{}(M_t) = a(t)$ for all $a \in \mathscr{A}$, $t \in$* $\operatorname{supp}\Phi$.

Proof. We write $Z := (a)_{a \in \mathscr{A}}$ and $\Phi(Z) := (\Phi(a))_{a \in \mathscr{A}}$. As $\Phi(\mathscr{A})$ is dense in $\mathfrak{R}$ there is a natural homeomorphism $\varphi: \Delta(\mathfrak{R}) \to \sigma_{\mathfrak{R}}(\Phi(Z))$ from $\Delta(\mathfrak{R})$ onto the joint spectrum $\sigma_{\mathfrak{R}}(\Phi(z))$ of $\Phi(Z)$ in $\mathfrak{R}$ which is given by $\varphi(M) := \Phi(Z)\hat{}(M) := (\Phi(a)\hat{}(M))_{a \in \mathscr{A}}$ for $M \in \Delta(\mathfrak{R})$. Moreover, by [4], Theorem 4.5 in connection with Example 4.4.(d), $\operatorname{supp}\Phi$ exists and the mapping $\psi: \operatorname{supp}\Phi \to \mathbf{C}^{\mathscr{A}}$ given by $\psi(t) := Z(t) := (a(t))_{a \in \mathfrak{A}}$ is a homeomorphism from $\operatorname{supp}\Phi$ onto $\sigma_{\mathfrak{R}}(\Phi(Z))$. Thus, $\varphi^{-1} \circ \psi: \operatorname{supp}\Phi \to \Delta(\mathfrak{R})$ is a homeomorphism with the property that $\Phi(a)\hat{}(\varphi^{-1}(\psi(t))) = a(t)$ for all $t \in \operatorname{supp}\Phi$, $a \in \mathscr{A}$. Hence, $M_t \subset \varphi^{-1}(\psi(t))$ for all $t \in \operatorname{supp}\Phi$. Conversely, as $\Phi(\mathscr{A}) = \{\Phi(a) \mid a(t) = 0\} + \mathbf{C}1$ is dense in $\mathfrak{R}$, the closed ideal M_t is of codimension 1 in $\mathfrak{R}$ and therefore coincides with $\varphi^{-1}(\psi(t))$.

Suppose now that $\mathfrak{X}$ is a Banach space and that $\Phi: \mathscr{A} \to \mathscr{L}(\mathfrak{X})$ is a unital homomorphism, where $\mathscr{A}$ is as before. For a closed subset F of K we define

$$\mathfrak{X}_\Phi(F) := \cap \{\mathrm{N}(\Phi(a)) : a \in \mathscr{A},\ \operatorname{supp} a \cap F = \emptyset\}.$$

Then $\mathfrak{X}_\Phi(F)$ is a closed subspace of $\mathfrak{X}$ which is invariant for all $\Phi(a)$, $a \in \mathscr{A}$.

3.4. LEMMA. *For all closed $F \subset K$ and all $a \in \mathscr{A}$ we have* $\operatorname{sp}(\Phi(a), \mathfrak{X}_\Phi(F)) \subset$ $\subset a(F)$ *and hence for the spectral radius:*

$$r(\Phi(a) \mid \mathfrak{X}_\Phi(F)) \leqslant \sup_{t \in F} |a(t)|.$$

Proof. $\Phi_F: \mathscr{A} \to \mathscr{L}(\mathfrak{X}_\Phi(F))$ with $\Phi_F(a) := \Phi(a) \mid \mathfrak{X}_\Phi(F)$ is a unital homomorphism. Again we may apply Theorem 4.5 and Example 4.4. (d) of [4] and obtain for the joint Taylor spectrum ([4, 33, 35]) of the system $(\Phi_F(a))_{a \in \mathscr{A}}$ with respect to $\mathfrak{X}_\Phi(F)$:

$$\operatorname{sp}((\Phi(a))_{a \in \mathscr{A}}, \mathfrak{X}_\Phi(F)) = \operatorname{sp}((\Phi_F(a))_{a \in \mathscr{A}}, \mathfrak{X}_\Phi(F)) =$$

$$= \{(a(t))_{a \in \mathscr{A}} \mid t \in \operatorname{supp} \Phi_F\} \subset \{(a(t))_{a \in \mathscr{A}} : t \in F\},$$

where the last inclusion follows from the fact that Φ_F vanishes on $K \setminus F$ by the definition of $\mathfrak{X}_\Phi(F)$. Because of the projection property of the Taylor spectrum ([4]) this implies

$$\operatorname{sp}(\Phi(a), \mathfrak{X}_\Phi(F)) \subset a(F)$$

for all $a \in \mathscr{A}$.

Let us now consider the following situation. $\Phi: \mathscr{A} \to \mathscr{B}$ is a unital homomorphism from a spectrally closed normal algebra on a compact Hausdorff space K to a unital Banach algebra $\mathscr{B}$ such that $\Phi(\mathscr{A})$ is contained in the centre $\mathfrak{Z}(\mathscr{B})$ of $\mathscr{B}$. As $\mathscr{A}$ is normal we see from Theorem 3.3 that $\mathfrak{R} = \overline{\Phi(\mathscr{A})}$ is a regular commutative Banach algebra (not necessarily semi-simple) contained in $\mathfrak{Z}(\mathscr{B})$. Hence, by [8], Theorem 2.13, each maximal ideal M_t, $t \in \operatorname{supp} \Phi$, is extendable in $\mathscr{B}$ (i.e. is contained in some proper bi-ideal of $\mathscr{B}$). For $t \in \operatorname{supp} \Phi$, let $\mathfrak{I}(t)$ be the closed bi-ideal in $\mathscr{B}$ generated by M_t. This is a proper bi-ideal in $\mathscr{B}$ (cf. [8], Theorem 2.13) and coincides with the closure in $\mathscr{B}$ of the set of all finite sums of the form $\sum_{j=1}^{m} b_j \Phi(a_j)$, where $b_1, \ldots, b_m \in \mathscr{B}$, $a_1, \ldots, a_m \in \mathscr{A}$, and $a_j(t) = 0$ for $j = 1, \ldots, m$. As in [8] we define the *local* algebra $\mathscr{B}_t$ to be the quotient algebra $\mathscr{B}/\mathfrak{I}(t)$ endowed with its quotient norm and denote by $\pi_t : \mathscr{B} \to \mathscr{B}/\mathfrak{I}(t) = \mathscr{B}_t$ the canonical epimorphism. If $b \in \mathscr{B}$ then we write simply $\sigma_t(b)$ for the spectrum of $\pi_t(b)$ in the Banach algebra $\mathscr{B}_t$. Notice that for all $a \in \mathscr{A}$ we have $\pi_t(\Phi(a)) = a(t) \cdot \pi_t(1)$. We refer to [8] for more details on this localization method.

3.5. LEMMA. *Suppose that $\mathfrak{X}$ is a Banach space and that $\Psi: \mathscr{B} \to \mathscr{L}(\mathfrak{X})$ is a continuous unital homomorphism.*

(a) *For all closed $F \subset K$ the space $\mathfrak{X}_{\Psi \circ \Phi}(F)$ is invariant for all $\Psi(b)$, $b \in \mathscr{B}$.*

(b) *If $b \in \mathscr{B}$, $t \in \operatorname{supp} \Phi$, and $z \in \mathbf{C}$ are such that $z \notin \sigma_t(b)$ then there exists a closed neighbourhood U of t such that $z \notin \operatorname{sp}(\Psi(b), \mathfrak{X}_{\Psi \circ \Phi}(U))$.*

(c) *If* $b \in \mathscr{B}$ *and if* $F, H \subset K$ *are closed with* $F \subset H$ *then* $\operatorname{sp}(\Psi(b), \mathfrak{X}_{\Psi\circ\Phi}(F)) \subset \subset \operatorname{sp}(\Psi(b), \mathfrak{X}_{\Psi\circ\Phi}(H))$.

Proof. (a) is an immediate consequence of the definition of $\mathfrak{X}_{\Psi\circ\Phi}(F)$ and of the fact, that the range of Φ is contained in the centre of $\mathscr{B}$.

(b) Because of $z \notin \sigma_t(b)$ there exists some $u \in \mathscr{B}$ such that $(z-b)u - 1 \in \mathfrak{J}(t)$ and $u(z-b) - 1 \in \mathfrak{J}(t)$. Hence, by the definition of $\mathfrak{J}(t)$ and because of the continuity of Ψ, there are finitely many $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathscr{B}, f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathscr{A}$ with $f_j(t) = 0 = g_i(t)$ for $j = 1, \ldots, n$, $i = 1, \ldots, m$ such that

(1)
$$\left\| (z - \Psi(b))\Psi(u) - 1 - \sum_{j=1}^{n} \Psi(x_j)\Psi(\Phi(f_i)) \right\| = \left\| \Psi\left((z-b)u - 1 - \sum_{j=1}^{n} x_j\Phi(f_i) \right) \right\| < \frac{1}{2},$$

and

(1′)
$$\left\| \Psi(u)(z - \Psi(b)) - 1 - \sum_{i=1}^{m} \Psi(y_i)\,\Psi(\Phi(g_i)) \right\| < \frac{1}{2}.$$

As the functions $f_1, \ldots, f_n, g_1, \ldots, g_m$ are continuous at t, there exists a closed neighbourhood U of t such that

(2)
$$\max_{1\leqslant j\leqslant n} \sup_{s\in U} |f_j(s)| < \varepsilon, \quad \max_{1\leqslant i\leqslant m} \sup_{s\in U} |g_i(s)| < \delta$$

where $\varepsilon > 0$ and $\delta > 0$ are chosen in such a way that

(3)
$$\varepsilon \sum_{j=1}^{n} \|\Psi(x_j)\| < \frac{1}{4} \quad \text{and} \quad \delta \sum_{i=1}^{m} \|\Psi(y_i)\| < \frac{1}{4}.$$

We define now $f_0 :\equiv 1$ and $x_0 := (z-b)u - 1 - \sum_{j=1}^{n} x_j\Phi(f_j)$. With this notation we have for all $k \in \mathbf{N}$,

$$((z - \Psi(b))\Psi(u) - 1)^k = \left(\sum_{j=0}^{n} \Psi(x_j\,\Phi(f_j)) \right)^k = \sum_{\mu\in M(n,k)} \prod_{i=1}^{k} \Psi(x_{\mu(i)}\Phi(f_{\mu(i)})),$$

where $M(n, k)$ is the set of all functions from $\{1, \ldots, k\}$ to $\{0,1, \ldots, n\}$. As the range of Φ is contained in $\mathfrak{Z}(\mathscr{B})$ we obtain

$$\begin{aligned}((z-\Psi(b))\Psi(u)-1)^k &= \sum_{\mu\in M(n,k)}\left(\prod_{i=1}^{k}\Psi(x_{\mu(i)})\right)\prod_{i=1}^{k}\Psi(\Phi(f_{\mu(i)})) = \\ &= \sum_{p=0}^{k}\sum_{\mu\in M_p}\left(\prod_{i=1}^{k}\Psi(x_{\mu(i)})\right)(\Psi\circ\Phi)\left(\prod_{i=1}^{k}f_{\mu(i)}\right)\end{aligned} \tag{4}$$

where M_p is the set of all $\mu \in M(n, k)$ with the property that $\{i: \mu(i) = 0\}$ has p elements ($p = 0,1, \ldots, k$). Because of (2) and Lemma 3.4, there exists some constant $C \geqslant 1$ such that for all $j = 1, \ldots, n$ and $q \in \mathbf{N}_0$,

$$\|\Psi(\Phi(f_j^q)) \mid \mathfrak{X}_{\Psi\circ\Phi}(U)\| = \|(\Psi(\Phi(f_j)) \mid \mathfrak{X}_{\Psi\circ\Phi}(U))^q\| \leqslant C\cdot\varepsilon^q.$$

Now, for $\mu \in M_p$ $(1 \leqslant p \leqslant k)$,

$$(\Psi\circ\Phi)\left(\prod_{i=1}^{k}f_{\mu(i)}\right) = (\Psi\circ\Phi)(f_1^{\alpha_1}\cdot \ldots \cdot f_n^{\alpha_n}) = \prod_{j=1}^{n}\Psi(\Phi(f_j^{\alpha_j}))$$

with certain $\alpha_1, \ldots, \alpha_n \in \mathbf{N}_0$ having the property that $\alpha_1 + \ldots + \alpha_n = k - p$. Hence,

$$\begin{aligned}\left\|(\Psi\circ\Phi)\left(\prod_{i=1}^{k}f_{\mu(i)}\right)\middle|\,\mathfrak{X}_{\Psi\circ\Phi}(U)\right\| &\leqslant \prod_{j=1}^{n}\|\Psi(\Phi(f_j^{\alpha_j}))\mid\mathfrak{X}_{\Psi\circ\Phi}(U)\| \leqslant \\ &\leqslant C^n\prod_{j=1}^{n}\varepsilon^{\alpha_j} = C^n\varepsilon^{k-p} = C^n\prod_{i=1}^{k}\varepsilon_{\mu(i)}\end{aligned}$$

with $\varepsilon_0 := 1$ and $\varepsilon_j := \varepsilon$ for $j = 1, \ldots, n$. Thus we obtain from (4) by restricting this equation to $\mathfrak{X}_{\Psi\circ\Phi}(U)$,

$$\begin{aligned}&\|[(z-(\Psi(b)\mid\mathfrak{X}_{\Psi\circ\Phi}(U)))\,(\Psi(u)\mid\mathfrak{X}_{\Psi\circ\Phi}(U))-1]^k\| \leqslant \\ &\leqslant C^n\sum_{p=0}^{k}\sum_{\mu\in M_p}\left(\prod_{i=1}^{k}\|\Psi(x_{\mu(i)})\|\right)\prod_{i=1}^{k}\varepsilon_{\mu(i)} = C^n\left(\sum_{j=0}^{n}\varepsilon_j\|\Psi(x_j)\|\right)^k = \\ &= C^n\left(\|\Psi(x_0)\| + \varepsilon\sum_{j=1}^{n}\|\Psi(x_j)\|\right)^k \leqslant C^n\left(\frac{1}{2}+\frac{1}{4}\right)^k = C^n\left(\frac{3}{4}\right)^k\end{aligned}$$

for all $k \in \mathbf{N}$, where we used (1) and (3). This implies

$$r((z-(\Psi(b)\mid\mathfrak{X}_{\Psi\circ\Phi}(U)))\,(\Psi(u)\mid\mathfrak{X}_{\Psi\circ\Phi}(U))-1) \leqslant \frac{3}{4} < 1$$

so that $(z - (\Psi(b) \mid \mathfrak{X}_{\Psi\circ\Phi}(U)) (\Psi(u) \mid \mathfrak{X}_{\Psi\circ\Phi}(U))$ is invertible in $\mathscr{L}(\mathfrak{X}_{\Psi\circ\Phi}(U))$. Therefore, $z - (\Psi(b) \mid \mathfrak{X}_{\Psi\circ\Phi}(U))$ has a right inverse in $\mathscr{L}(\mathfrak{X}_{\Psi\circ\Phi}(U))$. In the same way we see (by means of (1') and the right inequalities in (2) and (3)) that $z - (\Psi(b) | \mathfrak{X}_{\Psi\circ\Phi}(U))$ also has a left inverse in $\mathscr{L}(\mathfrak{X}_{\Psi\circ\Phi}(U))$. This shows that $z \notin \mathrm{sp}(\Psi(b), \mathfrak{X}_{\Psi\circ\Phi}(U))$.

(c) Put $T := (\Psi(\Phi(a)))_{a \in \mathscr{A}}$. By [4], Theorem 4.5 in connection with Example 4.4 (d) and Corollary 3.5, the system T is decomposable and $\mathfrak{X}_{\Psi\circ\Phi}(F)$ is a spectral maximal space for T contained in $\mathfrak{X}_{\Psi\circ\Phi}(H)$. Moreover, if $z \in \mathbf{C} \setminus \mathrm{sp}(L(b), \mathfrak{X}_{\Psi\circ\Phi}(H))$ then $U := (z - L(b) \mid \mathfrak{X}_{\Psi\circ\Phi}(H))^{-1}$ commutes with $L(b) \mid \mathfrak{X}_{\Psi\circ\Phi}(H)$. Hence, by [4], Lemma 3.4, $\mathfrak{X}_{\Psi\circ\Phi}(F)$ is invariant for U. This shows that $z \notin \mathrm{sp}\,(L(b), \mathfrak{X}_{\Psi\circ\Phi}(F))$.

3.6. THEOREM. *Suppose that $\mathfrak{X}$ is a Banach space and that $\Psi : \mathscr{B} \to \mathscr{L}(\mathfrak{X})$ is a continuous unital homomorphism. If $b \in \mathscr{B}$ is such that for all $t \in \operatorname{supp} \Phi$ the local spectrum $\sigma_t(b)$ at t has no interior points then $\Psi(b)$ has the single valued extension property.*

Proof. Suppose that $f: G \to \mathfrak{X}$ is an analytic $\mathfrak{X}$-valued function on some open subset G of $\mathbf{C}$ such that $(z - \Psi(b))f(z) \equiv 0$ on G. We have to prove that $f \equiv 0$ on G. Without loss of generality we may assume that G is connected. If $t \in \operatorname{supp} \Phi$ then, by the fact that the interior of $\sigma_t(b)$ is empty, we find some $z_t \in G$ such that $z_t \notin \sigma_t(b)$. Hence, by Lemma 3.5, there is some closed neighbourhood U_t of t such that $z_t \notin \mathrm{sp}(\Psi(b), \mathfrak{X}_{\Psi\circ\Phi}(U_t))$. As $\operatorname{supp} \Phi$ is compact, there are finitely many $t_1, \ldots, t_n \in \operatorname{supp} \Phi$ such that $\operatorname{supp} \Phi \subset \bigcup_{j=1}^{n} \operatorname{int} U_{t_j}$. We write $U_j := U_{t_j}$, $\mathfrak{X}_j := \mathfrak{X}_{\Psi\circ\Phi}(U_j)$, and $z_j := z_{t_j}$ for $j = 1, \ldots, n$. As $\mathscr{A}$ is normal there are $a_0, a_1, \ldots, a_n \in \mathscr{A}$ with $\operatorname{supp} a_0 \cap \operatorname{supp} \Phi = \emptyset$, $\operatorname{supp} a_j \subset U_j$ $(j = 1, \ldots, n)$, and $\sum_{j=0}^{n} a_j \equiv 1$ on K. It follows that $1 = \sum_{j=0}^{n} \Phi(a_j) = \sum_{j=1}^{n} \Phi(a_j)$ in $\mathscr{B}$ and hence

$$f(z) = \sum_{j=1}^{n} \Psi(\Phi(a_j))f(z) \quad \text{for all } z \in G.$$

Notice that for all $z \in G$ and $j = 1, \ldots, n$ we have

$$f_j(z) := \Psi(\Phi(a_j))f(z) \in \mathfrak{X}_j,$$

so that f_j is an analytic function on G with values in $\mathfrak{X}_j$. Moreover, $(z - \Psi(b))f_j(z) \equiv \Psi(\Phi(a_j))\,(z - \Psi(b))f(z) \equiv 0$ on G, as b commutes with $\Phi(a_1), \ldots, \Phi(a_n)$. Now $z_j \notin \mathrm{sp}(\Psi(b), \mathfrak{X}_j)$, so that in a neighbourhood V_j of z the operator $z - (\Psi(b) \mid \mathfrak{X}_j)$ has an inverse $(z - (\Psi(b) \mid \mathfrak{X}_j))^{-1} \in \mathscr{L}(\mathfrak{X}_j)$. Thus,

$$f_j(z) = (z - (\Psi(b) \mid \mathfrak{X}_j))^{-1}(z - \Psi(b))f_j(z) \equiv 0$$

on V_j and hence $f_j \equiv 0$ on all of G by the identity theorem. It follows that $f = f_1 + \ldots + f_n$ vanishes identically on G and the proof of the theorem is complete.

We are now able to prove the above mentioned decomposability criterium.

3.7. THEOREM. *Let $\mathfrak{X}$ be a Banach space and suppose that $\Psi: \mathscr{B} \to \mathscr{L}(\mathfrak{X})$ is a continuous unital homomorphism. If b is an element of $\mathscr{B}$ such that for all $t \in \operatorname{supp} \Phi$ the local spectrum $\sigma_t(b)$ of b at t is totally disconnected, then $\Psi(b)$ is decomposable.*

Proof. Let $\mathfrak{M} := \{x \in \mathscr{B} : xb = bx\}$ be the commutant algebra of b in $\mathscr{B}$. $L: \mathscr{B} \to \mathscr{L}(\mathscr{B})$, $x \to L(x)$, is a unital homomorphism, so that, by Theorem 3.6, $L(b)$ and hence also $L(b) \mid \mathfrak{M}$ has the single valued extension property. Hence, for all $x \in \mathfrak{M}$ the local spectrum $\tau(x) := \sigma(x; L(b), \mathfrak{M})$ of $L(b)$ at x with respect to $\mathfrak{M}$ is defined and has the following properties (cf. [14], p. 1):

$$\forall\, x, y \in \mathfrak{M} \quad \forall\, w \in \mathbf{C}: \tau(xy) \subset \tau(x) \cap \tau(y), \;\; \tau(x+y) \subset \tau(x) \cup \tau(y), \;\; \tau(wx) \subset \tau(x),$$

$$\forall\, x \in \mathfrak{M} \;:\; \tau(x) = \emptyset \Leftrightarrow x = 0,$$

$$\forall\, x \in \mathfrak{M} \quad \forall\, z \notin \tau(x) \quad \exists\, x(z) \in \mathfrak{M} \;:\; (z - b)x(z) = x.$$

We want to use Theorem 1.4 of [5] and have therefore to prove that $(\mathfrak{M}, b, \tau)$ is a spectral triple in the sense of [5], Definition 1.3. The only property which remains to be proved is

(i) For every finite open covering $\{U_1, \ldots, U_n\}$ of $\mathbf{C}$ there are $x_1, \ldots, x_n \in \mathfrak{M}$ such that $\tau(x_j) \subset \overline{U}_j$ $(j = 1, \ldots, n)$ and $x_1 + \ldots + x_n = 1$. In order to prove (i) let $\{U_1, \ldots, U_n\}$ be an arbitrary finite open covering of $\mathbf{C}$.

(a) First fix an arbitrary $t \in K$. As $\sigma_t(b)$ is totally disconnected there exist pairwise disjoint closed sets $F_1, \ldots, F_n \subset \mathbf{C}$ with $F_j \subset U_j$ $(j = 1, \ldots, n)$ and $\sigma_t(b) \subset \bigcup_{j=1}^{n} F_j$. Then there are open sets $G_1, \ldots, G_n \subset \mathbf{C}$ with $F_j \subset G_j \subset \overline{G}_j \subset U_j$ $(j = 1, \ldots, n)$ and such that $\overline{G}_i \cap \overline{G}_j = \emptyset$ for $i \neq j$. We put $G := G_1 \cup \ldots \cup G_n$ and $D := \{w \in \mathbf{C} : |w| \leqslant \|b\|\}$. If $z \in D \setminus G$ then, by Lemma 3.5. (b), there is a closed neighbourhood W_z of t such that $z \notin \operatorname{sp}(L(b), \mathscr{B}_{L\circ\Phi}(W_z))$. Then $\{V_z : z \in D \setminus G\}$, where $V_z := \mathbf{C} \setminus \operatorname{sp}(L(b), \mathscr{B}_{L\circ\Phi}(W_z))$ for $z \in D \setminus G$, is an open covering of the compact set $D \setminus G$ which has a finite subcovering $\{V_{z_1}, \ldots, V_{z_m}\}$. The set $W(t) := \bigcap_{j=1}^{m} W_{z_j}$ is a closed neighbourtood of t. Moreover, by Lemma 3.5. (c), we have

$$\operatorname{sp}(L(b), \mathscr{B}_{L\circ\Phi}(W(t))) \subset \bigcap_{j=1}^{m} \operatorname{sp}(L(b), \mathscr{B}_{L\circ\Phi}(W_{z_j})) \subset G \cap D \subset D,$$

using the construction of W_{z_j} and the fact that

$$\operatorname{sp}(L(b), \mathscr{B}_{L\circ\Phi}(W_{z_j})) \subset \{w \in \mathbf{C} : |w| \leqslant \|L(b) \mid \mathscr{B}_{L\circ\Phi}(W_{z_j})\|\}$$

and $\|L(b) \mid \mathscr{B}_{L\circ\Phi}(W_{z_j})\| \leqslant \|b\|$. It follows from the analytic functional calculus that there exist idempotents $P_1, \ldots, P_n \in \mathscr{L}(\mathscr{B}_{L\circ\Phi}(W(t)))$ in the bicommutant of $L(b) \mid \mathscr{B}_{L\circ\Phi}(W(t))$ such that $P_1 + \ldots + P_n = 1$ and $\operatorname{sp}(L(b), P_i(\mathscr{B}_{L\circ\Phi}(W(t)))) \subset G_i$ for $i = 1, \ldots, n$.

(b) Fix now $x \in \mathscr{B}_{L\circ\Phi}(W(t)) \cap \mathfrak{M}$. In this step we prove that for $i = 1, \ldots, n$, $P_i x \in \mathfrak{M}$ and $\tau(P_i x) \subset \overline{G_i} \subset U_i$.

We define $R: \mathscr{B} \to \mathscr{L}(\mathscr{B})$ by $R(u)y := yu$ for $u, y \in \mathscr{B}$. Then $\mathscr{B}_{L\circ\Phi}(W(t))$ is invariant for $R(b)$. Indeed, if $y \in \mathscr{B}_{L\circ\Phi}(W(t))$ and if $a \in \mathscr{A}$ such that $\operatorname{supp} a \cap W(t) = \emptyset$ then

$$L(\Phi(a))\,(R(b)y) = \Phi(a)yb = 0$$

so that $R(b)y \in \mathscr{B}_{L\circ\Phi}(W(t))$ by the definition of $\mathscr{B}_{L\circ\Phi}(W(t))$. Moreover $R(b) \mid \mathscr{B}_{L\circ\Phi}(W(t))$ commutes with $L(b)$. Hence, using the fact that P_i is in the bicommutant of $L(b) \mid \mathscr{B}_{L\circ\Phi}(W(t))$ we see that

$$b(P_i x) = L(b)P_i x = P_i(L(b)x) = P_i(bx) =$$

$$= P_i(xb) = P_i(R(b)x) = R(b)\,(P_i x) = (P_i x)b,$$

i.e, $P_i x \in \mathfrak{M}$. Moreover, $P_i(\mathscr{B}_{L\circ\Phi}(W(t)))$ is invariant for $R(b)$. For $z \in \mathbf{C}\setminus\overline{G_i}$ we define

$$f(z) := (z - L(b) \mid P_i(\mathscr{B}_{L\circ\Phi}(W(t))))^{-1}P_i x.$$

As $P_i x \in \mathfrak{M}$ and as $(z - L(b) \mid P_i(\mathscr{B}_{L\circ\Phi}(W(t))))^{-1}$ is in the bicommutant of $L(b) \mid P_i(\mathscr{B}_{L\circ\Phi}(W(t)))$ it follows easily that $f(z) \in \mathfrak{M}$ for all $z \in \mathbf{C}\setminus\overline{G}_i$. Hence, $\tau(P_i x) = \sigma(P_i x;\, L(b), \mathfrak{M}) \subset \overline{G_i} \subset U_i$.

(c) By compactness of $\operatorname{supp}\Phi$ there are finitely many $t_1, \ldots, t_k$ such that

$$\operatorname{supp}\Phi \subset \bigcup_{j=1}^{k} \operatorname{int} W(t_j).$$

As $\mathscr{A}$ is normal there are $a_0, a_1, \ldots, a_k \in \mathscr{A}$ such that $\operatorname{supp} a_0 \cap \operatorname{supp}\Phi = \emptyset$, $\operatorname{supp} a_j \subset \operatorname{int} W(t_j)$ for $j = 1, \ldots, k$ and $a_0 + a_1 + \ldots + a_k \equiv 1$ on K. Then

$$1 = \Phi(a_0) + \Phi(a_1) + \ldots + \Phi(a_k) = \Phi(a_1) + \ldots + \Phi(a_k)$$

and $\Phi(a_j) \in \mathfrak{M} \cap \mathscr{B}_{L\circ\Phi}(W(t_j))$ for $j = 1, \ldots, k$. By (a), (b) there are $x_{i,j} \in \mathfrak{M}$ ($i = 1, \ldots, n$, $j = 1, \ldots, k$) such that $\Phi(a_j) = x_{1,j} + \ldots + x_{n,j}$ and $\tau(x_{i,j}) \subset U_i$.

Therefore $1 = u_1 + \ldots + u_n$, where $u_i := \sum_{j=1}^{k} x_{i,j}$ and $\tau(u_i) \subset U_i$ for $i = 1, \ldots, n$. This proves that condition (i) is fulfilled. Hence $(\mathfrak{M}, b, \tau)$ is a spectral triple in the sense of [5] and it follows from Theorem 1.4 in [5] that $\Psi(b)$ is decomposable. This completes the proof of the theorem.

We now want to give some examples of situations where the Theorems 3.6 and 3.7 can be applied. First we apply Theorem 3.6 to a class of pseudo-differential operators studied in [30, 31]. We shall need some notations. We write PC for the unital Banach algebra of piecewise continuous functions on $\mathbf{R}$ generated by the characteristic functions of the intervals $[t, \infty)$, $t \in \mathbf{R}$. For $a \in PC \otimes PC$ and $f \in \mathscr{H} :=$ $:= L^2(\mathbf{R})$ one defines

$$(\mathrm{Op}\,(a)f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(x, y)\, \mathrm{e}^{\mathrm{i}xy}\, Ff(y)\, \mathrm{d}y \quad (x \in \mathbf{R})$$

where Ff is the Fourier transform of f. The C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ generated by $\{\mathrm{Op}(a) : a \in PC \otimes PC\}$ is denoted by $\Psi(PC, PC)$. Let also $\Psi(C, C)$ be the C^*-subalgebra of $\Psi(PC, PC)$ generated by all $a \in C \otimes C$, where C is the algebra of all continuous functions on $\mathbf{R}$ possessing equal limits at $+\infty$ and $-\infty$. Then $\mathscr{K}(\mathscr{H}) \subset \Psi(C, C)$ and $\mathscr{A} := \Psi(C, C)/\mathscr{K}(\mathscr{H})$ is a commutative C^*-subalgebra of the centre of $\mathscr{B} := \Psi(PC, PC)/\mathscr{K}(\mathscr{H})$. $\mathscr{A}$ is isometric $*$-isomorphic to $C(K)$, where $K = \mathbf{R}^* \times \mathbf{R}^* \setminus \mathbf{R} \times \mathbf{R}$. Here $\mathbf{R}^*$ is the one-point compactification of $\mathbf{R}$. The local algebras $\mathscr{B}_t$, $t \in K$, of $\mathscr{B}$ over $\mathscr{A}$ have been computed by S. C. Power [30, 31] who also proved that for all $t \in K$ the local spectrum $\sigma_t(\mathrm{Op}(a))$ of $\mathrm{Op}(a)$, $a \in PC \otimes PC$, at t consists only of a finite number of possibly degenerated parabolic arcs. Let now $\Psi : \mathscr{B} \to \mathscr{L}(\mathscr{H}_q)$ be the monomorphism induced by the homomorphism $T \to T_q$ from $\Psi(PC, PC)$ to $\mathscr{H}_q$. Then we obtain from Theorem 3.6:

3.8. COROLLARY. *For all $a \in PC \otimes PC$, the operator* $\mathrm{Op}(a)$ *has the essential single valued extension property.*

Notice that the interior of the essential spectrum of Op (a) may be nonempty.

Recall that a C^*-algebra $\mathscr{B}$ is called n-homogeneous if all irreducible representations of $\mathscr{B}$ are n-dimensional ($n \in \mathbf{N}$).

3.9. COROLLARY. *Let $\mathscr{H}$ be a Hilbert space and suppose that $\mathscr{B}$ is a unital n-homogeneous C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ (resp. of $Q(\mathscr{H})$). Then each $T \in \mathscr{B}$ (resp. each $T \in \mathscr{L}(\mathscr{H})$ with $T + \mathscr{K}(\mathscr{H}) \in \mathscr{B}$) is decomposable (resp. essentially decomposable).*

Proof. Let $\mathscr{A}$ be the centre of $\mathscr{B}$. It follows from Theorem 5 and the proof of Theorem 8 in [34] that $\mathscr{A}$ is $*$-isomorphic to the algebra $C(\Omega)$ of all continuous functions on the structure space Ω of $\mathscr{B}$ and that (with this identification) for each $t \in \Omega$ we have a $*$-isomorphism from $\mathscr{B}_t$ onto the algebra M_n of all complex $n \times n$-matrices. Hence, if $T \in \mathscr{B}$ (resp. $T \in \mathscr{L}(\mathscr{H})$ such that $T + \mathscr{K}(\mathscr{H}) \in \mathscr{B}$) then $\sigma_t(T)$ (resp. $\sigma_t(T + \mathscr{K}(\mathscr{H})))$ is finite and thus totally disconnected. Let now $\Psi: \mathscr{B} \to \mathscr{L}(\mathscr{H})$ be the canonical inclusion (resp. let $\Psi: \mathscr{B} \to \mathscr{L}(\mathscr{H}_q)$ be the canonical monomorphism induced by the homomorphism $S \to S_q$) then we see from Theorem 3.7 that T (resp. T_q) is decomposable.

The most trivial type of n-homogeneous unital C^*-algebras are C^*-algebras which are $*$-isomorphic to $C(K) \otimes M_n$ for some compact Hausdorff space K. The centre of $C(K) \otimes M_n$ is just $C(K) \otimes 1_n \simeq C(K)$, where 1_n is the unit of M_n.

3.10. Corollary. *Let $\mathscr{T}_n(QC)$ be the C^*-algebra of all Toeplitz operators T_φ on $\mathscr{H} := \bigoplus_{j=1}^{n} H^2(\mathbf{T})$ such that the symbol φ is in $QC \otimes M_n$. Here $QC := H^\infty + C(\mathbf{T}) \cap \overline{H^\infty + C(\mathbf{T})}$ is the commutative unital C^*-algebra of quasi-continuous functions on $\mathbf{T}$. Then it follows from* [19], Corollary 3.3, *that $\mathscr{T}_n(QC)/\mathscr{K}(\mathscr{H}) \cong QC \otimes M_n$, so that this algebra is of the above-mentioned type. From* Corollary 3.9 *we obtain that each $T_\varphi \in T_n(QC)$ is essentially decomposable.*

Corresponding results follow for Toeplitz operators on the boundary of pseudo-convex domains with continuous symbols [12, 36]. Sometimes compact operator valued symbols are also considered [15]. In this case one has an exact sequence

$$0 \to \mathscr{K}(\mathscr{H}) \xrightarrow{i} \mathscr{B} \xrightarrow{\sigma} \mathscr{A} \hat{\otimes}_\alpha (\mathscr{K}(H_2) \oplus \mathbf{C}1) \to 0$$

where H_1 and H_2 are Hilbert spaces, $\mathscr{H} := H_1 \hat{\otimes} H_2$, the Hilbert space tensor product; $\mathscr{B}$, a C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ with $\mathscr{K}(\mathscr{H}) \subset \mathscr{B}$, $\mathscr{A}$ is a commutative C^*-algebra, α is the unique C^*-norm on $\mathscr{A} \otimes (\mathscr{K}(H_2) \oplus \mathbf{C}1)$, i is the inclusion map and σ the symbol map.

3.11. Corollary. *In the above situation all operators in $\mathscr{B}$ are essentially decomposable.*

Proof. $\mathscr{B}/\mathscr{K}(\mathscr{H})$ is topologically isomorphic to $C(K)\hat{\otimes}_\alpha(\mathscr{K}(H_2) \oplus \mathbf{C}1) \cong C(K, \mathscr{K}(H_2) \oplus \mathbf{C}1)$, where K is the maximal ideal space of $\mathscr{A}$. $C(K) \otimes 1$ is then contained in the centre of $C := C(K, \mathscr{K}(H_2) \oplus \mathbf{C}1)$ and for each $t \in K$, the local algebra C_t is $\mathscr{K}(H_2) \oplus \mathbf{C}1$. Hence for each $c \in C$, $\sigma_t(c)$ is totally disconnected (notice that $\pi_t(c)$ is of the form $zI + T$ with compact operator T and that the spectrum of this operator in $\mathscr{K}(H_2) \oplus \mathbf{C}1$ is the same as in $\mathscr{L}(H_2)$). Now the result follows from 3.7 with $\Psi: C \to \mathscr{B}/\mathscr{K}(\mathscr{H}) \to \mathscr{L}(\mathscr{H}_q)$ being the composition of the natural homomorphisms.

Recall that a bounded linear operator T on a Hilbert space $\mathfrak{H}$ is called (essentially) n-normal ($n \in \mathbf{N}$) if T is unitarily equivalent to an operator matrix of the form $(T_{i,j})_{i,j=1,2,\ldots,n}$ on $\bigoplus_{j=1}^{n} H$ for some Hilbert space H, where the operators

T_{ij}, $i, j = 1,2, \ldots, n$ are mutually (essentially) commuting (essentially) normal operators.

3.12. COROLLARY. *Every (essentially) n-normal operator is (essentially) decomposable.*

Proof. Let $\mathscr{A}$ be the commutative C^*-subalgebra of $\mathscr{L}(H)$ (resp. $Q(H)$) generated by $\{T_{ij};\ i, j = 1,2, \ldots, n\}$ (resp. $\{T_{ij} + \mathscr{K}(H);\ i,j = 1,2, \ldots, n\}$) and put $\mathscr{B} := \mathscr{A} \otimes M_n$. Then $\mathscr{B}$ is of the above-mentioned type and T (resp. $T + \mathscr{K}(\mathfrak{H})$) is an element of a C^*-subalgebra of $\mathscr{L}(\mathfrak{H})$ (resp. $Q(\mathfrak{H})$) which is isomorphic to $\mathscr{B}$. Hence it follows from 3.9 that T is (essentially) decomposable.

3.13. COROLLARY. *Let T be an algebraically n-normal operator on a Hilbert space $\mathfrak{H}$*, i.e. *an operator such that all irreducible $*$-representations of the C^*-algebra $C^*(T)$, generated by T in $\mathscr{L}(\mathfrak{H})$, are of dimension $\leqslant n$. There T is decomposable.*

Proof. It is well-known (cf. [29]) that T is similar to a direct sum of the form $\bigoplus_{j=1}^{n} T_j$ where T_j are j-normal. Now the statement follows from 3.11 and [14], Proposition 1.8, p. 34.

REMARK. K. R. Davidson introduced in [16] the class of essentially spectral operators on a Hilbert space $\mathfrak{H}$. These are operators $T \in \mathscr{L}(\mathfrak{H})$ which are similar to an operator of the form $N + Q$ where N is essentially normal, Q is essentially quasinilpotent, and $NQ - QN$ is compact. By 3.11, N_q is decomposable and Q_q is a quasinilpotent operator on $\mathfrak{H}_q$ commuting with N_q. By [14], Theorem 2.2.1, $T_q = N_q + Q_q$ is decomposable. Thus the class of essentially spectral operators is contained in the class of essentially decomposable operators.

If $\mathscr{A}$ is a normal, spectrally closed subalgebra of $C(K)$ for some compact Hausdorff space K and if $\varphi_1 : \mathscr{A} \to \mathscr{L}(\mathfrak{X})$ is a unital homomorphism, then for $n \in \mathbf{N}$, we define $\Phi_n : M_n(\mathscr{A}) = \mathscr{A} \otimes M_n \to \mathscr{L}(\mathfrak{X}^n)$ by

$$\Phi_n(a)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \Phi_1(a_{11}) & \ldots & \Phi_1(a_{1n}) \\ \vdots & & \vdots \\ \Phi_1(a_{n1}) & \ldots & \Phi_1(a_{nn}) \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

for $a = (a_{ij})_{i,j=1,\ldots,n} \in M_n(\mathscr{A})$ and $x_1, \ldots, x_n \in \mathfrak{X}$. Moreover let $\Phi : \mathscr{A} \to \mathscr{L}(\mathfrak{X}^n)$ be the mapping given by $\Phi(a) := \Phi_n(a \otimes 1_n)$, for $a \in \mathscr{A}$, where 1_n is the unit of M_n. Then Φ is a unital homomorphism and the range of Φ is contained in the centre of $\mathscr{B} := \overline{\Phi_n(M_n(\mathscr{A}))}$. With these notations we have:

3.14. COROLLARY. *For all $a \in M_n(\mathscr{A})$ the operator $\Phi_n(a) \in \mathscr{L}(\mathfrak{X}^n)$ is decomposable.*

Proof. Fix $a = (a_{ij})_{i,j=1,\ldots,n} \in M_n(\mathscr{A})$. We want to apply Theorem 3.7 to the element $b := \Phi_n(a) \in \mathscr{B}$ and the inclusion mapping $\Psi : \mathscr{B} \to \mathscr{L}(\mathfrak{X}^n)$. To this purpose we have to prove that for each $t \in \operatorname{supp} \Phi$ the local spectrum $\sigma_t(b)$ of b at t is totally disconnected. Hence, fix an arbitrary $t \in \operatorname{supp} \Phi$. Then $S_t := \{z \in \mathbf{C} \mid \det(z - a(t)) = 0\}$ is finite. If $z \notin S_t$, then $\det(z - a(s)) \neq 0$ for all s in some closed neighbourhood U of t. As $\mathscr{A}$ is normal, there is some $h \in \mathscr{A}$ such that $\operatorname{supp} h \subset U$ and such that $h \equiv 1$ in some neighbourhood V of t $(V \subset U)$. It follows from Cramer's rule that there exists some $\tilde{a} \in M_n(\mathscr{A})$ such that for all $s \in U$ we have

$$(z - a(s))^{-1} = \frac{1}{\det(z - a(s))} \tilde{a}(s).$$

As $\mathscr{A}$ is spectrally closed, the function $f: K \to \mathbf{C}$ given by

$$f(s) := \begin{cases} h(s)/\det(z - a(s)) & \text{for } s \in U \\ 0 & \text{for } s \in K \setminus U \end{cases}$$

belongs to $\mathscr{A}$. Hence $g := (f \otimes 1_n)\tilde{a} \in M_n(\mathscr{A})$ and $(z - a)g = h \otimes 1_n = g(z - a)$. Hence,

$$(z - \Phi_n(a))\Phi_n(g) - 1 = \Phi_n(g)(z - \Phi_n(a)) - 1 =$$
$$= \Phi(h) - 1 = \Phi(h - 1) \in \mathfrak{J}(t).$$

This shows that $(z - \pi_t(\Phi_n(a)))^{-1}$ exists in $\mathscr{B}_t$, i.e. that $z \notin \sigma_t(\Phi_n(a))$. It follows that $\sigma_t(\Phi_n(a)) \subset S_t$ is finite and hence totally disconnected. By 3.7 the operator $\Phi_n(a) \in \mathscr{L}(\mathfrak{X}^n)$ is decomposable.

REMARK. Matrix operators of the kind $\Phi_n(a)$, $a \in M_n(\mathscr{A})$, have been studied by several authors [9, 14, 21, 22]. N. Dunford has considered operators of the form

$$A = \int_\Omega a(t)E(\mathrm{d}t),$$

where E is a countably additive spectral measure defined on a σ-field Σ of subsets of a set Ω and with values in $\mathscr{L}(\mathscr{H})$, $\mathscr{H}$ a Hilbert space, and where $a \in M_n(\mathscr{A})$, $\mathscr{A}$ being the commutative C^*-algebra of all E-essentially bounded complex valued functions on Ω. Especially, $\mathscr{A}$ is of the type $C(K)$ and $A = \Phi_n(a)$ with

$$\Phi(f) := \int_\Omega f(t)E(\mathrm{d}t) \quad \text{for } f \in \mathscr{A}.$$

It has been proved in [14], Theorem 6.4.4, that every operator of the kind $\Phi_2(a)$, $a \in M_2(\mathscr{A})$, is of class $C^1(\mathbf{C})$ and its spectral distribution is given by

$$(*) \qquad f \to \int_\Omega f(a(t))E(\mathrm{d}t).$$

C. Apostol [9] gave an example with $\Omega = \mathrm{sp}(S, \mathscr{H}) \subset \mathbf{C}$, S a scalar type spectral operator with spectral measure E and $a \in M_3(\mathscr{A})$ even being analytic in a neighbourhood of the spectrum of S such that the mapping (*) does not define a spectral distribution. However, he proved for analytic $a \in M_n(\mathscr{A})$ that the operator $\Phi_n(\dot{a})$ is always $\mathfrak{A}$-decomposable for some suitable topologically admissible algebra $\mathfrak{A}$. In the more general situation of arbitrary operators $\Phi_n(a)$, $a \in \mathscr{A}$, Corollary 3.14 (or Corollary 3.9) now gives the (slightly weaker) result that these operators are decomposable.

4. SOME APPLICATIONS OF ESSENTIAL LOCAL SPECTRAL THEORY

In this section we give some first applications of the essential local spectral theory.

4.1. PROPOSITION. *Let $\mathfrak{X}$ be a Banach space and let $T \in \mathscr{L}(\mathfrak{X})$.*

(a) *If $T \in \Phi^-(\mathfrak{X})$ and if T has the essential* SVEP *then T is Fredholm.*

(b) *If $T \in \Phi^+(\mathfrak{X})$ and if T' (the transposed operator) has the essential* SVEP *then T is a Fredholm operator.*

Proof. (a) If $T \in \Phi^-(\mathfrak{X})$ then T_q is surjective by Corollary 2.5. Since T_q also has the SVEP it follows from [37] that T_q must be injective. Hence $T \in \Phi(\mathfrak{X})$ by Corollary 2.5.

(b) $T \in \Phi^+(\mathfrak{X})$ implies that $T' \in \Phi^-(\mathfrak{X}')$. As T' has the essential SVEP we conclude from (a) that $T' \in \Phi(\mathfrak{X}')$. Hence also $T \in \Phi(\mathfrak{X})$.

4.2. COROLLARY. *If $a \in PC \otimes PC$ then the operator* Op(a) *(*cf. Section 3*) is Fredholm if and only if it is semi-Fredholm.*

Proof. By 3.8 the operator Op(a) has the essential SVEP. Hence, if $\mathrm{Op}(a) \in \Phi^-(L^2(\mathrm{R}))$ then Op(a) is Fredholm by Proposition 4.1. Suppose now that $\mathrm{Op}(a) \in \Phi^+(L^2(\mathrm{R}))$. An elementary computation shows for the transposed operator that

$$\mathrm{Op}(a)' = F\,\mathrm{Op}(\tilde{a})F^{-1}$$

where $F \in \mathscr{L}(L^2(\mathbf{R}^n))$ is the Fourier transform and $\tilde{a}(x,y) = a(y,x)$ for $x, y \in \mathbf{R}$. Hence $\tilde{a} \in PC \otimes PC$ and $\mathrm{Op}(\tilde{a}) \in \Phi^-(L^2(\mathbf{R}))$ as $\mathrm{Op}(a)' \in \Phi^-(L^2(\mathbf{R}))$. By the first part of the proof this implies $\mathrm{Op}(\tilde{a}) \in \Phi(L^2(\mathbf{R}))$ and therefore $\mathrm{Op}\,(a)' \in \Phi(L^2(\mathrm{R}))$, i.e. $\mathrm{Op}\,(a) \in \Phi(L^2(\mathbf{R}))$.

4.3. PROPOSITION. *Let $\mathfrak{X}$ be a Banach space and let $T \in \mathscr{L}(\mathfrak{X})$ be an essential decomposable operator. If T is semi-Fredholm then T is actually a Fredholm operator*

Proof. Suppose first that $T \in \Phi^-(\mathfrak{X})$. As T_q is decomposable, T_q has the SVEP. Hence it follows from Proposition 4.1 that T is Fredholm. If $T \in \Phi^+(\mathfrak{X})$ then T_q is injective with closed range (by Corollary 2.4) and decomposable. Hence the transposed operator $(T_q)'$ is surjective and by [25] decomposable. Thus $(T_q)'$ has the SVEP. By the above-mentioned result of Vrbová [37] we obtain that $(T_q)'$ is invertible, i.e. that T_q is invertible. By Corollary 2.5, T is a Fredholm operator.

4.4. COROLLARY. *If T is a Toeplitz operator in $\mathcal{T}_n(QC)$ (cf.* Corollary 3.10*) then T is Fredholm if and only if T is semi-Fredholm.*

This follows immediately from Proposition 4.3 and Corollary 3.10. A corresponding result is true for Toeplitz operators with continuous $n \times n$-matrix symbols on strongly pseudoconvex domains in $\mathbf{C}^n$ and for pseudodifferential operators with continuous $n \times n$-matrix symbols as in [15, 20]. From 4.3 and Corollary 3.11 we obtain:

4.5. COROLLARY. *If an essentially n-normal operator is semi-Fredholm it is already a Fredholm operator.*

For decomposable operators, Proposition 4.3 can be sharpened:

4.6. PROPOSITION. *Let $T \in \mathcal{L}(\mathfrak{X})$ be a decomposable operator on a Banach space $\mathfrak{X}$. If $z \in$ sp$(T, \mathfrak{X})$ is such that $z - T$ is semi-Fredholm, then z is an isolated point of* sp$(T, \mathfrak{X})$ *and $z - T$ is a Fredholm operator of index* 0.

Proof. By Proposition 4.3 and Corollary 2.7, $z - T$ is a Fredholm operator. It is a well known fact (cf. [10, 11]) that then there is an $\varepsilon > 0$ and an analytic projection valued function $P: U_\varepsilon(z) \to \mathcal{L}(\mathfrak{X})$ such that $\mathrm{N}(P(w)) = \mathrm{N}(w - T)$ for $0 < |z - w| < \varepsilon$ and such that $w - T$ is Fredholm for all $w \in U_\varepsilon(z)$. As T has the SVEP we conclude that $1 - P(w) \equiv 0$ on $U_\varepsilon(z)$, i.e. that $w - T$ is injective for $0 < |z - w| < \varepsilon$. Applying the same argument to the transposed $w - T'$ of $w - T$ (notice that T' is also decomposable by [25], Theorem 2) we see that $w - T'$ is also injective for $0 < |z - w| < \varepsilon'$ where $\varepsilon \geqslant \varepsilon' > 0$. Hence, for $0 < |z - w| < \varepsilon'$ the operator $w - T$ is invertible, i.e. z is an isolated point of sp $(T, \mathfrak{X})$. Moreover ind$(z - T) = 0$, as the index of the function $w \to w - T$ is constant on $U_\varepsilon(z)$.

Proposition 4.6 generalizes corresponding statements for (pre-) spectral operators as given in [26, 28].

5. A CHARACTERIZATION OF LOCAL TYPE OPERATORS

Local type operators occur in a natural way in the theory of singular integral operators (see for example [27], p. 400 ff.). We shall give the definition in an abstract situation. We shall constantly use the definitions and results of [4]. Let X and Y be two Banach spaces and suppose that $T = (T_\lambda)_{\lambda \in \Lambda}$ and $S = (S_\lambda)_{\lambda \in \Lambda}$ are

$(\mathscr{A}, Z = (Z_\lambda)_{\lambda\in\Lambda}, \tau)$- resp. $(\mathscr{B}, W = (W_\lambda)_{\lambda\in\Lambda}, \sigma)$-scalar systems in $\mathscr{L}(X)$ resp. $\mathscr{L}(Y)$ with $(\mathscr{A}, Z, \tau)$-spectral representation $\Phi: \mathscr{A} \to \mathscr{L}(X)$ for T and $(\mathscr{B}, W, \sigma)$-spectral representation $\Psi: \mathscr{B} \to \mathscr{L}(Y)$ (see [4], Definition 4.3). A linear mapping $L: X \to Y$ is called *of (T, S)-local type* if for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$ with $\tau(a) \cap \sigma(b) = \emptyset$ the operator $\Psi(b)L\Phi(a) \in \mathscr{L}(X, Y)$ is compact. It has been shown in [7] that local type operators are in many cases continuous or have at least some continuity properties. In the following we shall only consider bounded local type operators. The following lemma shows that our definition of a (T, S)-local type operator does not depend on the special choice of the spectral triples $(\mathscr{A}, Z, \tau)$, $(\mathscr{B}, W, \sigma)$ and of the spectral representations Φ, Ψ.

5.1. LEMMA. (a) *The systems $T_q := (T_{\lambda,q})_{\lambda\in\Lambda}$ and $S_q := (S_{\lambda,q})_{\lambda\in\Lambda}$ are decomposable on X_q resp. Y_q with spectral capacities (denoted by $\mathscr{E}_T^q$ resp. $\mathscr{E}_S^q$) not depending on the special choice of $(\mathscr{A}, Z, \tau)$, $(\mathscr{B}, W, \sigma)$, Φ, and Ψ.*

(b) *$L \in \mathscr{L}(X, Y)$ is of (T, S)-local type if and only if $L_q\mathscr{E}_T^q(F) \subset \mathscr{E}_S^q(F)$ holds for all closed subsets of $\mathbf{C}^\Lambda$.*

Proof. (a) is a consequence of [4], Theorem 4.5 and Corollary 3.3(a). Notice, that also by [4], Theorem 4.5, the (unique) spectral capacity $\mathscr{E}_T^q$ for T_q is given by

$$(*) \qquad \mathscr{E}_T^q(F) = \bigcap \{N(\Phi(a)_q) : a \in \mathscr{A} \text{ and } \tau(a) \cap F = \emptyset\}$$

for closed $F \subset \mathbf{C}^\Lambda$. The corresponding statement holds for S_q and $\mathscr{E}_S^q$.

(b) Let $L \in \mathscr{L}(X, Y)$ be of (T, S)-local type and $F \subset \mathbf{C}^\Lambda$ be closed. Recall that $\mathscr{E}_T^q(F) = \mathscr{E}_T^q(H)$ where $H := F \cap \operatorname{sp}(T_q, X_q)$ (cf. [4], Corollary 3.3 (b)). Fix now an arbitrary vector $x \in \mathscr{E}_T^q(F) = \mathscr{E}_T^q(H)$ and let b be an element of $\mathscr{B}$ with $\sigma(b) \cap \cap F = \emptyset$. As $\mathbf{C}^\Lambda$ is completely regular and H is compact, there are open sets U, $V \subset \mathbf{C}^\Lambda$ such that $U \cup V = \mathbf{C}^\Lambda$, $H \subset U$, $H \cap V = \emptyset$ and $\sigma(b) \cap U = \emptyset$. By the definition of an $(\mathscr{A}, Z, \tau)$-spectral triple there are $a, c \in \mathscr{A}$ such that $a + c = 1$, $\tau(a) \subset U$, and $\tau(c) \subset V$. Because of $x \in \mathscr{E}_T^q(H)$ and $\tau(c) \cap H = \emptyset$ we obtain $x = = \Phi(a)_q x + \Phi(c)_q x = \Phi(a)_q x$. Therefore

$$\Psi(b)_q L_q x = \Psi(b)_q L_q \Phi(a)_q x = (\Psi(b)L\Phi(a))_q x = 0$$

as the operator $\Psi(b)L\Phi(a)$ is compact because of $\sigma(b) \cap \tau(a) = \emptyset$. Thus $L_q x \in \in \bigcap \{N(\Psi(b)_q) : b \in \mathscr{B},\ \sigma(b) \cap F = \emptyset\} = \mathscr{E}_S^q(F)$. This proves $L_q(\mathscr{E}_T^q(F)) \subset \mathscr{E}_S^q(F)$ for all closed $F \subset \mathbf{C}^\Lambda$.

Conversely, suppose that $L_q(\mathscr{E}_T^q(F)) \subset \mathscr{E}_S^q(F)$ holds for all closed $F \subset \mathbf{C}^\Lambda$. Fix $a \in \mathscr{A}$, $b \in \mathscr{B}$ such that $\tau(a) \cap \sigma(b) = \emptyset$. Notice, that $R(\Phi(a)_q) \subset \mathscr{E}_T^q(\tau(a))$. This follows from $(*)$ and the definition of an $(\mathscr{A}, Z, \tau)$-spectral triple. Hence, for each $x \in X_q$ we have

$$(\Psi(b)L\Phi(a))_q x = \Psi(b)_q L_q \Phi(a)_q x = 0$$

because of $L_q\Phi(a)_q x \in L_q\mathscr{E}^q_T(\tau(a)) \subset \mathscr{E}^q_S(\tau(a))$ and $\sigma(b) \cap \tau(a) = \emptyset$. This shows that $\Psi(b)L\Phi(a)$ is a compact operator. Hence, L must be of (T, S)-local type.

5.2. Lemma. *Let $L \in \mathscr{L}(X, Y)$ be an operator such that $S_\lambda L - LT_\lambda$ is a compact operator for all $\lambda \in \Lambda$. Then L is of (T, S)-local type.*

Proof. For $x \in X_q$ we denote the local spectrum of T_q at x with respect to X_q by $\sigma(x; T_q, X_q)$ (cf. [4], p. 86, for the definition). Now, $L_q T_{\lambda,q} = S_{\lambda,q} L_q$ holds for all $\lambda \in \Lambda$. Hence, it follows from the definition of the local spectrum that $\sigma(L_q x; S_q, Y_q) \subset \sigma(x; T_q, X_q)$ for all $x \in X_q$. Because of

$$\mathscr{E}^q_T(F) = X_{q,T_q}(F) = \{x \in X_q \mid \sigma(x; T_q, X_q) \subset F\}$$

and also $\mathscr{E}^q_S(F) \subset Y_{q,S_q}(F)$ (by [4], Lemma 3.8) this implies $L_q\mathscr{E}^q_T(F) \subset \mathscr{E}^q_S(F)$ for all closed $F \subset \mathbf{C}^\Lambda$ so that L is of (T, S)-local type by Lemma 5.1.

Recall that a bounded linear operator A on a Banach space X is said to be *hermitian-equivalent* if $\sup_{t \in \mathbf{R}} \|e^{itA}\| < \infty$. $C \in \mathscr{L}(X)$ is called *normal-equivalent* if $C = A + iB$ with commuting hermitian-equivalent operators $A, B \in \mathscr{L}(X)$.

5.3. Theorem. *Let $T = (T_\lambda)_{\lambda\in\Lambda}$ and $S = (S_\lambda)_{\lambda\in\Lambda}$ be $(\mathscr{A}, Z, \tau)$-resp. $(\mathscr{B}, W, \sigma)$--scalar systems in $\mathscr{L}(X)$ resp. $\mathscr{L}(Y)$ and suppose that for all $\lambda \in \Lambda$ the operators $T_{\lambda,q}$, $S_{\lambda,q}$ are normal-equivalent in $\mathscr{L}(X_q)$ resp. $\mathscr{L}(Y_q)$. For $L \in \mathscr{L}(X, Y)$ are equivalent:*

(a) *L is of (T, S)-local type.*

(b) *For all $\lambda \in \Lambda$ the operator $S_\lambda L - LT_\lambda \in \mathscr{L}(X, Y)$ is compact.*

Proof. Because of Lemma 5.2 we have only to prove that (a) implies (b). Hence, let $L \in \mathscr{L}(X, Y)$ be of (T, S)-local type. By Lemma 5.1 the system T_q and S_q are decomposable with spectral capacities $\mathscr{E}^q_T$ and $\mathscr{E}^q_S$. By [4], Corollary 3.6, also the operators $T_{\lambda,q}$, $S_{\lambda,q}$ $(\lambda \in \Lambda)$ are decomposable and their spectral capacities $\mathscr{E}^q_{T,\lambda}$, $\mathscr{E}^q_{S,\lambda}$ are given by $\mathscr{E}^q_{T,\lambda}(F) = \mathscr{E}^q_T(P_\lambda^{-1}(F))$ for all closed $F \subset \mathbf{C}$ and $\lambda \in \Lambda$. Here, $P_\lambda \colon \mathbf{C}^\Lambda \to \mathbf{C}$ is the canonical projection onto the component with index λ. Because of this and Lemma 5.1, we conclude that for all closed $F \subset \mathbf{C}$, $\lambda \in \Lambda$ we have

$$L_q\mathscr{E}^q_{T,\lambda}(F) \subset \mathscr{E}^q_{S,\lambda}(F).$$

As $\mathscr{E}^q_{T,\lambda}(F) = X_{q,T_{\lambda,q}}(F)$ and $\mathscr{E}^q_{S,\lambda}(F) = Y_{q,S_{\lambda,q}}(F)$ we may apply [14], Theorem 2.3.3, and obtain for all $\lambda \in \Lambda$

$$\lim_{n\to\infty} \|C(S_{\lambda,q}, T_{\lambda,q})^n L_q\|^{1/n} = 0,$$

where $C(S_{\lambda,q}, T_{\lambda,q}) \in \mathscr{L}(\mathscr{L}(X_q, Y_q))$ is defined by $C(S_{\lambda,q}, T_{\lambda,q})A := S_{\lambda,q}A - AT_{\lambda,q}$ for $A \in \mathscr{L}(X_q, Y_q)$. As the operators $S_{\lambda,q}$, $T_{\lambda,q}$ are normal equivalent this implies (by [2], Theorem 1) that

$$(S_\lambda L - LT_\lambda)_q = S_{\lambda,q}L_q - L_qT_{\lambda,q} = 0$$

i.e. that $S_\lambda L - LT_\lambda$ is compact for all $\lambda \in \Lambda$.

Before giving some applications of this characterization theorem for the usual spaces of function occurring in the theory of singular integral operators, we have to investigate certain multiplication operators on these spaces.

A bounded linear operator A on a Banach space X is said to be *essentially of class* (C) if there exists a continuous unital homomorphism Φ: $C(\mathbf{C}) \to \mathscr{L}(X_q)$ such that $\Phi(\mathrm{id}_{\mathbf{C}}) = A_q$. Using the fact that $\operatorname{supp}\Phi = \operatorname{sp}(A_q, X_q)$ is compact, it is easy to see that for each real valued $f \in C(\mathbf{C})$ the operator $\Phi(f)$ is hermitian equivalent in $\mathscr{L}(X_q)$. Because of $A_q = \Phi(\mathrm{Re} \circ \mathrm{id}_{\mathbf{C}}) + \mathrm{i}\Phi(\mathrm{Im} \circ \mathrm{id}_{\mathbf{C}})$ this implies that A_q is normal-equivalent in $\mathscr{L}(X_q)$.

5.4. PROPOSITION. *Consider the following situations.*

(a) *Ω is a compact Hausdorff space, $\mathscr{A} = C(\Omega)$, and*

(a.1) *$X := C(\Omega)$, or*

(a.2) *X is a closed ideal in $C(\Omega)$, or*

(a.3) *$X := L^p(\Omega, \mu)$, where μ is a positive (not necessarily finite) measure on Ω and $1 \leqslant p \leqslant \infty$.*

(b) *(Ω, d) is a compact metric space, $0 < \lambda \leqslant 1$, and $\mathscr{A} := X := H^\lambda(\Omega)$ is the Banach algebra of Hölder continuous functions*

$$H^\lambda(\Omega) = \left\{ f \in C(\Omega) ;\ |f|_\lambda := \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)^\lambda} ;\ x, y \in \Omega,\ x \neq y \right\} < \infty \right\}$$

endowed with the norm $\|\cdot\|_\lambda$ given by $\|f\|_\lambda := \|f\|_{\sup} + |f|_\lambda$ for $f \in H^\lambda(\Omega)$.

(c) *G is a bounded open subset of $\mathbf{R}^n$, $\Omega := \overline{G}$, $\mathscr{A} := C^\infty(\overline{G})$ (the algebra of all those $f \in C^\infty(G)$ with the property that for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0^n$ the partial derivatives*

$$\mathrm{D}^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \qquad (|\alpha| := \alpha_1 + \ldots + \alpha_n)$$

have continuous extensions to $\overline{G} = \Omega$), and

(c.1) *$X := W_0^{p,k}(G)$ with $k \in \mathbf{N}_0$ and $1 \leqslant p < \infty$ (see [1] for the notations and theory concerning Sobolev spaces), or*

(c.2) *$X := C_0^{k+\lambda}(G)$ the completion of the set $C_c^\infty(G)$ of all those $f \in C^\infty(G)$ having compact support in G with respect to the norm $\|\cdot\|_{k,\lambda}$ given by*

$$\|f\|_{k,\lambda} := \sum_{|\alpha| \leqslant k} \|\mathrm{D}^\alpha f\|_{\sup} + \sum_{|\alpha| = k} |D^\alpha f|_\lambda$$

with $k \in \mathbf{N}_0$ and $0 \leqslant \lambda < 1$ (and $|\cdot|_\lambda \equiv 0$ for $\lambda = 0$), or (c.3) *G has the cone property and* $X := W^{p,k}(G)$ (cf. [1] *for the definition) with* $k \in \mathbf{N}_0$, $1 \leqslant p < \infty$, *or*

(c.4) *G is a $C^{k+\lambda}$-regular domain and* $\overline{X} := C^{k+\lambda}(G)$ (cf. [38] *for the definitions) with* $k \in \mathbf{N}_0$ *and* $0 \leqslant \lambda < 1$.

(d) *G is an unbounded open subset of* $\mathbf{R}^n$, *Ω is the closure of G in the one-point compactification* $\mathbf{R}^n_* = \mathbf{R}^n \cup \{\infty\}$ *of* $\mathbf{R}^n$, *$\mathscr{A}$ is the algebra of restrictions to Ω of all those* $f \in C(\mathbf{R}^n_*)$ *which are C^∞ on $\mathbf{R}^n$ and constant in a neighbourhood U_f of ∞, and*

(d.1) $X := C_0^k(G)$ *with* $k \in \mathbf{N}_0$, *or*

(d.2) $X := W_0^{p,k}(G)$ *with* $k \in \mathbf{N}_0$, $1 \leqslant p < \infty$, *or*

(d.3) *Ω has the cone property and* $X := W^{p,k}(\Omega)$ *with* $1 \leqslant p < \infty$ *and* $k \in \mathbf{N}_0$.

(e) $k \in \mathbf{N}_0$, *Ω is a m-dimensional compact C^k-manifold,* $\mathscr{A} := C^\infty(\Omega)$, *and*

(e.1) $X := C^k(\Omega)$, *or*

(e.2) $X := W^{p,k}(\Omega)$ *with* $1 \leqslant p < \infty$.

In all these situations the natural multiplication operators $M_X(a)$, $a \in \mathscr{A}$, given by $M_X(a)x := ax$ for $a \in \mathscr{A}$, $x \in X$, are essentially of class (C) *so that for all $a \in \mathscr{A}$ the operator $M_X(a)_q$ is normal-equivalent in X_q.*

Proof. The proof for the situations in (a) is trivial as in these cases already the operators $M_X(a)$, $a \in \mathscr{A} = C(\Omega)$, are of class (C), so that also $M_X(a)_q$ must be of class (C).

(b) We write $D := \{(t, t) \mid t \in \Omega\}$ and $C_b(\Omega \times \Omega \setminus D)$ for the Banach space of all bounded continuous functions on $\Omega \times \Omega \setminus D$ endowed with the supnorm. For $f \in H^\lambda(\Omega)$ let $\Delta_f \in C_b(\Omega \times \Omega \setminus D)$ be the function given by

$$\Delta_f(s, t) := \frac{f(s) - f(t)}{d(s, t)^\lambda} \quad \text{for } (s, t) \in \Omega \times \Omega \setminus D.$$

Then the mapping $f \to J(f) := (f, \Delta_f)$ defines an isometric isomorphism J from $X := H^\lambda(\Omega)$ onto a closed linear subspace Y_0 of $Y := C(\Omega) \oplus C_b(\Omega \times \Omega \setminus D)$. Fix now $a \in \mathscr{A} = H^\lambda(\Omega)$. Then $M_X(a) = J^{-1}T(a)J$ and hence $\|M_X(a)_q\| = \|T(a)_q\|$ where $T(a) \in \mathscr{L}(Y_0)$ is the restriction to Y_0 of the operator $S(a) \in \mathscr{L}(Y)$ given by

$$S(a)(f, g) := (af, (a \otimes 1) g + (1 \otimes f)\Delta_a) \quad \text{for } (f, g) \in Y.$$

Here $a \otimes 1$, $1 \otimes f \in C_b(\Omega \times \Omega \setminus D)$ are defined by $(a \otimes 1)(s, t) := a(s)$, $(1 \otimes f)(s, t) := f(t)$ for $(s, t) \in \Omega \times \Omega \setminus D$. Thus $S(a) = M_Y(a) + K(a)$ where $M_Y(a)(f, g) := (af, (a \otimes 1)g)$ and $K(a)(f, g) := (0, (1 \otimes f)\Delta_a)$ for $(f, g) \in Y$. Let us show that $K(a) \mid Y_0 \in \mathscr{L}(Y_0, Y)$ is a compact operator. We have $K(a) \mid Y_0 = H(a)L$ where the continuous linear operators $L : Y_0 \to C(\Omega)$ and $H(a) : C(\Omega) \to Y$ are given by

$L(f, \Delta_f) := f$ for $f \in H^\lambda(\Omega)$ and $H(a)h := (0, (1 \otimes h)\Delta_a)$ for $h \in C(\Omega)$. By the Arzela--Ascoli theorem, the operator L and hence also $K(a) \mid Y_0$ is compact. Denote by $j: Y_0 \to Y$ the inclusion mapping and let C be the constant given by Corollary 2.3. Then we obtain because of $(K(a) \mid Y_0)_q = 0$,

$$\|M_X(a)_q\| = \|T(a)_q\| \leqslant C\|(j \circ T(a))_q\| = C\|(M_Y(a) \mid Y_0)_q + (K(a) \mid Y_0)_q\| =$$

$$= C\|(M_Y(a) \mid Y_0)_q\| \leqslant C\|M_Y(a)\| \leqslant C\|a\|_{\sup}.$$

This shows that the unital homomorphism $a \to M_X(a)_q$ from the dense subalgebra $\mathscr{A} = H^\lambda(\Omega)$ of $C(\Omega)$ is continuous with respect to the supnorm and hence may be extended to a continuous unital homomorphism $\Phi: C(\Omega) \to \mathscr{L}(X_q)$. For $a \in \mathscr{A}$ we define now $\Psi_a: C(\mathbf{C}) \to \mathscr{L}(X_q)$ by $\Psi_a(f) := \Phi(f \circ a)$. Then Ψ_a is a continuous unital homomorphism with $\Psi_a(\mathrm{id}_{\mathbf{C}}) = \Phi(a) = M_X(a)_q$. This shows that $M_X(a)$ is essentially of class (C).

(c), (d). In these cases we define $Jf := (f_\alpha)_{|\alpha| \leqslant k}$ for $f \in X$. Then J is an isometric isomorphism from X onto a closed subspace Y_0 of $Y := \bigoplus_{|\alpha| \leqslant k} X_\alpha$, where for $|\alpha| \leqslant k$,

$$X_\alpha := \begin{cases} L^p(G) \text{ in the cases (c. 1), (c.3), (d.2), (d.3)} \\ C(\Omega) \text{ in the cases } \lambda = 0 \text{ of (c.2) and (c.4) and in the case (d.1)} \\ H^\lambda(\Omega) \text{ in the cases } \lambda > 0 \text{ of (c.2) and (c.4).} \end{cases}$$

Fix now an arbitrary function $a \in \mathscr{A}$. Then we have $M_X(a) = J^{-1}T(a)J$ where (by the Leibniz rule)

$$T(a)(D^\alpha f)_{|\alpha| \leqslant k} = \Big(\sum_{\beta \leqslant \alpha} \tbinom{\alpha}{\beta}(D^\beta a)\cdot(D^{\alpha-\beta} f)\Big)_{|\alpha| \leqslant k}$$

for $y = (y_\alpha)_{|\alpha| \leqslant k} \in Y$. We have

$$S(a) = M_Y(a) + \sum_{0 < |\gamma| \leqslant k} \sum_{0 < \beta \leqslant \gamma} K_{\beta,\gamma},$$

where, for $y = (y_\alpha)_{|\alpha| \leqslant k}$, $M_Y(a)y := (ay_\alpha)_{|\alpha| \leqslant k}$ and for $0 < \beta \leqslant \gamma$, $0 < |\gamma| \leqslant k$, $K_{\beta,\gamma}y := (y_\alpha)_{|\alpha| \leqslant k}$ with

$$g_\alpha := \begin{cases} 0 & \text{for } \alpha \neq \gamma \\ \tbinom{\alpha}{\beta}(D^\beta a)y_{\gamma-\beta} & \text{for } \alpha = \gamma. \end{cases}$$

We shall show that the operators $K_{\beta,\gamma} \mid Y_0 : Y_0 \to Y$ are compact. To see this we consider the following factorisation of $K_{\beta,\gamma} \mid Y_0$:

$$\begin{array}{ccccc} Y_0 & & \xrightarrow{K_{\beta,\gamma} \mid Y_0} & & Y \\ \downarrow{\scriptstyle J^{-1}} & & & & \uparrow{\scriptstyle A} \\ X & \xrightarrow[L]{} & Y_1 & \xrightarrow[R]{} & Y_2 \end{array} .$$

Here the space Y_1 is given as

$$Y_1 := \begin{cases} W_0^{p,1}(G) & \text{in the cases (c.1) and (d.2)} \\ W^{p,1}(G) & \text{in the cases (c.3) and (d.3)} \\ C_0^{1+\lambda}(G) & \text{in the case (c.2)} \\ C^{1+\lambda}(\overline{G}) & \text{in the case (c.4)} \\ C_0^1(G_0) & \text{in the case (d.1)} \end{cases}$$

where in the case (d.1), we chose $G_0 := G \cap U$ for some bounded open $U \subset \mathbf{R}^n$ containing $\operatorname{supp} D^\beta a$ (notice that $\beta > 0$ and that a is constant in a neighbourhood of ∞). The continuous linear mapping $L: X \to Y_1$ is defined by $Lf := D^\beta f$ in the cases of (c) and in the cases of (d) by $Lf := bD^\beta f$ where $b \in C^\infty(\mathbf{R}^n)$ has compact support (contained in U in the case (d.1)) satisfying $b \equiv 1$ in a neigbourhood of $\operatorname{supp} D^\beta a$.

The Banach space Y_2 and the mapping $R: Y_1 \to Y_2$ are defined as follows:

$$Y_2 := \begin{cases} L^p(G) & \text{in the cases (c.1), (c.3)} \\ C^{0+\lambda}(\Omega) & \text{in the cases (c.2), (c.4)} \\ C_0(G_0) & \text{in the case (d.1)} \end{cases}$$

and in these cases, $R: Y_1 \to Y_2$ is the canonical inclusion mapping which is compact by [1], Theorem 6.2, resp. by [38], Satz 8 on p. 262. In the remaining (d.2) and (d.3) let $U \subset \mathbf{R}^n$ be a bounded neighbourhood of $\operatorname{supp} b$, write $G_0 := G \cap U$, $Y_2 := L^p(G_0)$, and let $R: Y_1 \to Y_2$ be the inclusion mapping $Y_1 \to L^p(G)$ followed by the restriction mapping $L^p(G) \to L^p(G_0)$. By [1], Theorem 6.2, this operator is compact in both cases.

Finally, in all the situations of (c) and (d), let $A: Y_2 \to Y$ be the continuous mapping defined by $Ah := (y_\alpha)_{|\alpha| \leqslant k}$ with

$$Y_\alpha := \begin{cases} 0 & \text{for } \alpha \neq \gamma \\ \binom{\alpha}{\beta}(D^\beta a)\cdot h & \text{for } \alpha = \gamma\,. \end{cases}$$

Here we use the fact that in all cases Y_2 is in a natural way a closed subspace of X_γ.

It follows that in all the cases of (c) and (d) the operators $K_{\beta,\gamma} \mid Y_0$ $(0 < |\gamma| \leqslant k,$ $0 < \beta \leqslant \gamma)$ are compact, so that $(S(a) \mid Y_0)_q = (M_Y(a) \mid Y_0)_q$ and hence $\|(S(a) \mid Y_0)_q\| = \|(M_Y(a) \mid Y_0)_q\|$. In the situations of (d), (c.1), and (c.3) and for $\lambda = 0$ in (c.2) and (c.4) we have therefore

$$\|(S(a) \mid Y_0)_q\| \leqslant \|M_Y(a)\| \leqslant \|a\|_{\sup}\,.$$

In the remaining cases i.e. for $0 < \lambda < 1$ in (c.2), (c.4), we observe that $M_Y(a) = \bigoplus_{|\alpha| \leqslant k} M_{X_\alpha}(a)$. From this we conclude that (with a constant $C_1 \geqslant 1$):

$$\|(M_Y(a) \mid Y_0)_q\| \leqslant \|M_Y(a)_q\| \leqslant C_1 \sup_{|\alpha| \leqslant k} \|M_{X_\alpha}(a)_q\|.$$

By the proof of (b) we have $\|M_{X_\alpha}(a)\|_q \leqslant \|a\|_{\sup}$ so that also in this case we have $\|(S(a) \mid Y_0)_q\| \leqslant C_2\|a\|_{\sup}$. Write C for the constant given by Corollary 2.3 for the inclusion $j\colon Y_0 \to Y$. Then we obtain for all $a \in \mathscr{A}$,

$$\|M_X(a)_q\| = \|T(a)_q\| \leqslant C\|(j \circ T(a))_q\| = C\|(S(a) \mid Y_0)_q\| \leqslant C_1 C\|a\|_{\sup}.$$

As in the proof of (b) we conclude from this that $M_X(a)$ is essentially of class (C).

(e) We fix a finite atlas $\{(U_1, h_1), \ldots, (U_r, h_r)\}$ for the m-dimensional compact C^k-manifold Ω. Then there are $b_1, \ldots, b_r$, $d_1, \ldots, d_r \in C^k(\Omega)$ with $\operatorname{supp} b_j \subset \operatorname{supp} d_j \subset U_j$, $d_j \equiv 1$ in a neighbourhood of $\operatorname{supp} b_j$ $(j = 1, \ldots, r)$ and $b_1 + \ldots + b_r \equiv 1$, $d_1 + \ldots + d_r \equiv 1$ on Ω. Hence, for $a \in \mathscr{A}$, we have

$$M_X(a) = \sum_{j=1}^{r} M_X(b_j a).$$

Fix now $j \in \{1, \ldots, r\}$ and consider the following commutative diagramm of Banach spaces and bounded linear mappings

$$\begin{array}{ccc} X & \xrightarrow{M_X(b_j a)} & X \\ {\scriptstyle L}\downarrow & & \uparrow{\scriptstyle R} \\ X_0 & \xrightarrow[M_{X_0}(a h_j^{-1})]{} & X_0 \end{array}$$

where

$$X_0 := \begin{cases} C_0^k(h_j(U_j)) & \text{in the case (e.1)} \\ W_0^{p,k}(h_j(U_j)) & \text{in the case (e.2),} \end{cases}$$

$L\colon X \to X_0$ is the operator given by $Lf := (b_j \circ h_j^{-1}) \cdot (f \circ h_j^{-1})$ for $f \in X$, and $R\colon X_0 \to X$ is defined by $Rg := d_j \cdot (g \circ h_j)$ for $g \in X_0$. Now we are in the situation of (c.1) resp. (c.2), as the atlas can be chosen in such a way that $h_i(U_i) \subset \mathbf{R}^m$ is bounded for $i = 1, \ldots, r$ and conclude that $\|M_{X_0}(a \circ h_j^{-1})_q\| \leqslant C_j\|a \circ h_j^{-1}\|_{\sup} \leqslant C_j\|a\|_{\sup}$, where C_j is a constant not depending on a. Hence, $\|M_X(b_j a)_q\| \leqslant C_j\|R_q\| \cdot \|a\|_{\sup} \cdot \|L_q\| \leqslant C_j'\|a\|_{\sup}$ for all $a \in \mathscr{A}$ with a constant $C_j' > 0$.

We obtain now for all $a \in \mathscr{A}$

$$\|M_X(a)_q\| \leqslant \sum_{j=1}^{r} \|M_X(b_j a)_q\| \leqslant \sum_{j=1}^{r} C_j'\|a\|_{\sup}.$$

As in the proof of (b) one concludes now that $M_X(a)$ is essentially of class (C) for all $a \in \mathscr{A}$.

5.5. COROLLARY. *Let Ω be a compact Hausdorff space and let μ, v be two positive Borel measures on Ω, $1 \leqslant p, r \leqslant \infty$. For a bounded linear operator $L: L^p(\Omega, \mu) \to L^r(\Omega, v)$ are equivalent:*

(i) *L is of local type*, i.e. *for all $a, b \in \mathscr{A} := C(\Omega)$ such that* $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$ *the operator $M_Y(a)LM_X(b)$ is compact where $M_Y(a)$ resp. $M_X(b)$ is the operator of multiplication with a resp. b on $Y := L^r(\Omega, v)$ resp. $X := L^p(\Omega, \mu)$.*

(ii) *For all $a \in \mathscr{A} := C(\Omega)$ the operator $M_Y(a)L - LM_X(a)$ is compact.*

Proof. Denote by $\varphi: \Omega \to \mathbf{C}^{\mathscr{A}}$ the mapping $t \to (a(t))_{a \in \mathscr{A}}$. As $\mathscr{A}$ separates the points of Ω this is a homeomorphism from Ω onto $\varphi(\Omega)$. For $a \in \mathscr{A}$ we write $\tau(a) := \operatorname{supp}(a \circ \varphi^{-1})$. Moreover we put $Z := (Z_a)_{a \in \mathscr{A}}$ with $Z_a := a$ for $a \in \mathscr{A}$. Then, by [4], Example 4.4. (d), $(\mathscr{A}, Z, \tau)$ is a spectral triple as $\mathscr{A}$ is a spectrally closed normal subalgebra of $C(\Omega)$ (even $\mathscr{A} = C(\Omega)$) and $T := (M_X(a))_{a \in \mathscr{A}}$ resp. $S := (M_Y(a))_{a \in \mathscr{A}}$ are $(\mathscr{A}, Z, \tau)$-scalar systems on X resp. Y. Moreover we have for $a, b \in \mathscr{A}$:

$$\operatorname{supp} a \cap \operatorname{supp} b = \emptyset \Leftrightarrow \tau(a) \cap \tau(b) = \emptyset.$$

It follows that L is of local type in the sense of (i) if and only if L is of (T, S)-local type. Because of Proposition 5.4. (a) and Theorem 5.3, this is equivalent to (ii).

5.6. COROLLARY. *Let Ω be a compact m-dimensional C^k-manifold, $1 \leqslant p, s < \infty$, and $i, j \in \mathbf{N}_0$ with $i, j \leqslant k$. Let X be one of the spaces $W^{p,i}(\Omega)$, $C^i(\Omega)$ and let Y be $W^{s,j}(\Omega)$ or $C^j(\Omega)$. For $L \in \mathscr{L}(X, Y)$ are equivalent:*

(i) *L is of local type*, i.e. *for all $a, b \in \mathscr{A} := C^\infty(\Omega)$, with* $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, *the operator $M_Y(a)LM_X(b)$ is compact.*

(ii) *For all $a \in \mathscr{A} := C^\infty(\Omega)$ the operator $M_Y(a)L - LM_X(a)$ is compact.*

(iii) *For all $a \in C^k(\Omega)$ the operator $M_Y(a)L - LM_X(a)$ is compact.*

Proof. The equivalence of (i) and (ii) follows in the same way as in the proof of Corollary 5.5. The equivalence of (ii) and (iii) is obvious as $C^\infty(\Omega)$ is dense in $C^k(\Omega)$ and the mapping $a \to M_Y(a)L - LM_X(a)$ is continuous with respect to the $C^k(\Omega)$-topology.

5.7. COROLLARY. *Let $G \subset \mathbf{R}^n$ or $G \subset \mathbf{C}^n (\simeq \mathbf{R}^{2n})$ be bounded and open and let X and Y be two spaces as listed in* 5.4. (c). *For $L \in \mathscr{L}(X, Y)$ are equivalent:*

(i) *L is of local type*, i.e. *for all $a, b \in \mathscr{A} := C^\infty(\bar{G})$ with* $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$ *the operator $M_Y(a) LM_X(b)$ is compact.*

(ii) *For all $a \in \mathscr{A} := C^\infty(\bar{G})$ the operator $M_Y(a)L - LM_X(a)$ is compact.*

(iii) *For $j = 1, \ldots, n$ the operator $M_Y(\pi_j)L - LM_X(\pi_j)$ is compact. Here, $\pi_j: \bar{G} \to \mathbf{C}$ is the coordinate function defined by $\pi_j(z) := z_j$ for $z = (z_1, \ldots, z_n) \in \bar{G}$.*

Proof. The equivalence of (i) and (ii) is proved in the same way as in the proof of Corollary 5.5. Write now $\pi := (\pi_1, \ldots, \pi_n)$. Then $(\mathscr{A}, \pi, \text{supp})$ is a spectral triple (by [4], Example 4.4. (a)) and $T := (M_X(\pi_j))_{j=1}^n$ and $S := (M_Y(\pi_j))_{j=1}^n$ are $(\mathscr{A}, \pi, \text{supp})$-scalar systems on X resp. Y. Obviously, L is of local type in the sense of (i) if and only if it is of (T, S)-local type. Because of Proposition 5.4. (c) and Theorem 5.3 this is equivalent to (ii).

5.8. COROLLARY. *Let $G \subset \mathbf{R}^n$ or $G \subset \mathbf{C}^n \simeq \mathbf{R}^{2n}$ be an open and unbounded set and let X, Y be two spaces as listed in* 5.4. (d). *For $T \in \mathscr{L}(X, Y)$ are equivalent:*

(i) *L is of local type*, i.e. *for all $a, b \in \mathscr{A}$ ($\mathscr{A}$ as in* 5.4. (d)*) with* $\text{supp}\, a \cap \text{supp}\, b = \emptyset$ *the operator $M_Y(a)LM_X(b)$ is compact.*

(ii) *For all $a \in \mathscr{A}$ the operator $M_Y(a)L - LM_X(a)$ is compact.*

(iii) *For $j = 0, 1, \ldots, n$ the operator $M_Y(\varphi_j)L - LM_X(\varphi_j)$ is compact, where $\varphi_j\colon G \to \mathbf{C}$ is defined by $\varphi_0(z) := \exp(-|z|^2)$ and $\varphi_j(z) := z_j\varphi_0(z)$ for $z = (z_1, \ldots, z_n) \in G$.*

Proof. Again the equivalence of (i) and (ii) is obtained as in the proof of Corollary 5.5. The functions φ_j may be extended with all their derivatives to the closure Ω of G in $\mathbf{R}^n \cup \{\infty\}$ resp. $\mathbf{C}^n \cup \{\infty\}$ by defining $D^\alpha\varphi_j(\infty) = 0$ for all $\alpha \in \mathbf{N}_0^n$ and $j = 0,1, \ldots, n$. Moreover $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ separate the points of Ω. For $j = 0, \ldots, n$ we write $Z_j\colon \mathbf{C}^{n+1} \to \mathbf{C}^n$ for the coordinate function defined by $Z_j(w) := w_j$ for $w = (w_j)_{j=0}^n \in \mathbf{C}^{n+1}$. Then $(C^\infty(\mathbf{C}^{n+1}), Z = (Z_j)_{j=0}^n, \text{supp})$ is a spectral triple (cf. [4], Example 4.4. (a)) and $T = (M_X(\varphi_j))_{j=0}^n$, $S = (M_Y(\varphi_j))_{j=1}^n$ are in the natural way $(C^\infty(\mathbf{C}^{n+1}), Z, \text{supp})$-scalar systems on X resp. Y and it is easy to see that L satisfies (i) if and only if L is of (T, S)-local type.

Let us now prove that the operators $M_X(\varphi_j)_q$, $M_Y(\varphi_j)_q$, $j = 0,1,\ldots,n$ are normal equivalent. Indeed, it follows from the proof of (c), (d) and (b) in 5.4 that the mapping $a \to M_X(a)_q$ from $\mathscr{A}$ to $\mathscr{L}(X_q)$ has an extension to a continuous unital homomorphism $\Phi\colon C(\Omega) \to \mathscr{L}(X_q)$. As the mapping $a \to M_X(a)$ is continuous on $\mathscr{A}_0 := \{a \in \mathscr{A} \mid a(\infty) = 0\}$ with respect to the topology of $\mathscr{S}(\mathbf{R}^n)$ (resp. $\mathscr{S}(\mathbf{C}^n) = \mathscr{S}(\mathbf{R}^n)$) we conclude that $\Phi(\varphi_j) = M_X(\varphi_j)_q$ for $j = 0,1, \ldots, n$. Hence it follows as in the proof of 5.4. (b) that $M_X(\varphi_j)$ is essentially of class (C) and therefore $M_X(\varphi_j)_q$ is normal equivalent in $\mathscr{L}(X_q)$ for $j = 0,1, \ldots, n$. In the same way we see that the operators $M_Y(\varphi_0)_q, \ldots, M_Y(\varphi_n)_q$ are normal equivalent.

Now we conclude from Theorem 5.3 that (i) must be equivalent to (ii).

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