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$C_{1\cdot}$-CONTRACTIONS WITH HILBERT-SCHMIDT DEFECT OPERATORS

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1. INTRODUCTION

According to their theory of characteristic functions and functional models for contractions, Sz.-Nagy and Foiaş investigated a contraction T whose *defect operator* $D_T = (I - T^*T)^{1/2}$ is of Hilbert-Schmidt class and whose spectrum does not fill the unit disc (see [5, Chapter VIII]). Such a contraction was called a weak contraction and proved to possess a good structure. In this paper we investigate a contraction T of class $C_{1\cdot}$ whose defect operator D_T is of Hilbert-Schmidt class. We note that such a contraction is a weak contraction if and only if it is of class C_{11} (see [5, Chapter VIII]). Recall that a contraction T is of class $C_{1\cdot}$ if $\lim \|T^n x\| \neq 0$ for every non-zero x, and T is of class $C_{0\cdot}$ if $\lim \|T^n x\| = 0$ for every x. The classes $C_{\cdot 1}$ and $C_{\cdot 0}$ are defined by using T^* instead of T, and $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$ for $\alpha, \beta = 0, 1$.

For a contraction T of class $C_{1\cdot}$, there is an injection X with dense range such that $XT = VX$ for some isometry V (see [5, pp. 71–72] and [4]). In the recent paper [12] Uchiyama proved for a C_{10}-contraction T with Hilbert-Schmidt defect operator D_T that there exist an injection X with dense range and an injection Y such that $XT = SX$ and $TY = YS$ for a unilateral shift S with $\operatorname{ind} S = \operatorname{ind} T$ (for a semi-Fredholm operator A, $\operatorname{ind} A$ denotes Fredholm index). In Section 3 we show for a $C_{1\cdot}$-contraction T with Hilbert-Schmidt defect operator D_T that there exist an injection X with dense range and a sequence of injections $\{Y_n : n = 1, 2, \ldots\}$ such that $XT = VX$, $TY_n = Y_n V$ $(n = 1, 2, \ldots)$ and the span of $\{\operatorname{ran} Y_n : n = 1, 2, \ldots\}$ is the space on which T acts, where V is an isometry with $\operatorname{ind} V = \operatorname{ind} T$, and if T is of class C_{10}, then the isometry V is a unilateral shift. Then we treat three natural weakly closed algebras of operators associated with such T; the first one is $\operatorname{Alg} T$, the weakly closed algebra generated by T and the identity, the second is the double commutant $\{T\}''$, and the third is $\operatorname{Alg}\operatorname{Lat} T$, where $\operatorname{Lat} T$ denotes the class of all T-invariant subspaces and $\operatorname{Alg}\operatorname{Lat} T$ is the algebra consisting of all operators

A for which $\mathrm{Lat}\,T \subseteq \mathrm{Lat}\,A$. Obviously $\mathrm{Alg}\,T \subseteq \{T\}''$ and $\mathrm{Alg}\,T \subseteq \mathrm{Alg}\,\mathrm{Lat}\,T$. An operator T is said to have the *bicommutant property* if $\mathrm{Alg}\,T = \{T\}''$ while T is said to be *reflexive* if $\mathrm{Alg}\,T = \mathrm{Alg}\,\mathrm{Lat}\,T$. Every isometry is reflexive ([2]). Every non-unitary isometry has the bicommutant property while a unitary operator U has this property if and only if it is *reductive*, that is, $\mathrm{Lat}\,U = \mathrm{Lat}\,U^*$ ([9]), and reductive unitary operators were characterized ([13]). In Section 4 we prove the bicommutant property for a $C_{1\cdot}$-contraction T not of class C_{11} whose defect operator D_T is of Hilbert-Schmidt class, and for a C_{11}-contraction, the condition for its bicommutant property can be completely described in terms of its characteristic function. In the final section we establish the reflexivity of every $C_{1\cdot}$-contraction with Hilbert-Schmidt defect operator.

For contractions whose defect operators are of finite rank, these results were proved by Uchiyama ([10] and [11]) and Wu ([14], [15], [16] and [17]). But our proofs are more direct and transparent even in the case of finite rank.

2. PRELIMINARIES

A contraction is completely non-unitary (c.n.u.) if it has no non-trivial unitary direct summand. For a c.n.u. contraction we use the functional model of Sz.-Nagy and Foiaş [5]. All Hilbert spaces are assumed to be separable.

For a Hilbert space $\mathscr{E}$, $L^2(\mathscr{E})$ denotes the Lebesgue space of $\mathscr{E}$-valued, norm-square integrable functions on the unit circle, and $H^2(\mathscr{E})$ is the Hardy subspace of $L^2(\mathscr{E})$. For two Hilbert spaces $\mathscr{E}$ and $\mathscr{E}'$, $L^\infty(\mathscr{E}, \mathscr{E}')$ and $H^\infty(\mathscr{E}, \mathscr{E}')$ denote the Lebesgue and Hardy spaces of operator-valued, bounded functions on the unit circle whose values are operators from $\mathscr{E}$ to $\mathscr{E}'$, respectively. Multiplication on $L^2(\mathscr{E})$ by an operator-function F in $L^\infty(\mathscr{E}, \mathscr{E}')$ is an operator from $L^2(\mathscr{E})$ to $L^2(\mathscr{E}')$, which we denote by the same letter F;

$$(Ff)(e^{it}) = F(e^{it})f(e^{it}) \quad (f \in L^2(\mathscr{E})).$$

An operator-function $F \in L^\infty(\mathscr{E}, \mathscr{E}')$ is in $H^\infty(\mathscr{E}, \mathscr{E}')$ if and only if the multiplication operator F maps the subspace $H^2(\mathscr{E})$ of $L^2(\mathscr{E})$ into the subspace $H^2(\mathscr{E}')$ of $L^2(\mathscr{E}')$. Let T be a c.n.u. contraction, and let $\mathscr{D}_T$ denote the closure of the range of the defect operator D_T. The characteristic function Θ_T of T is an operator-function in $H^\infty(\mathscr{D}_T, \mathscr{D}_{T^*})$ whose values are contractions, defined by

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1}D_T]|\mathscr{D}_T \quad (|\lambda| < 1).$$

The (unitarily equivalent) functional model of T is the operator $S(\Theta_T)$ on the Hilbert space

$$H(\Theta_T) = K(\Theta_T) \ominus \{\Theta_T h \oplus \Delta_T h : h \in H^2(\mathscr{D}_T)\},$$

where

$$K(\Theta_T) = H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}, \quad \Delta_T(e^{it}) = (I - \Theta_T(e^{it})^*\Theta_T(e^{it}))^{1/2},$$

defined by

$$S(\Theta_T)(f \oplus g) = P(\chi f \oplus \chi g),$$

where $\chi(e^{it}) = e^{it}$ and P denotes the orthogonal projection of $K(\Theta_T)$ onto $H(\Theta_T)$ (see [5, Chapter VI]). A c.n.u. contraction T is of class $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if and only if its characteristic function Θ_T is *-outer (resp. outer) (see [5, Chapter VI, Proposition 3.5]). Recall that an operator-function Θ in $H^\infty(\mathcal{E}, \mathcal{E}')$ is *outer* if $\Theta H^2(\mathcal{E})$ is dense in $H^2(\mathcal{E}')$, and Θ is *inner* if $\Theta(e^{it})$ is an isometry for almost every t. Θ is **-outer* (resp. **-inner*) if $\tilde{\Theta}$ is outer (resp. inner), where $\tilde{\Theta}$ is an operator-function in $H^\infty(\mathcal{E}', \mathcal{E})$ defined by $\tilde{\Theta}(\lambda) = \Theta(\bar{\lambda})^*$. We also use the *canonical factorization* of an operator-function in $H^\infty(\mathcal{E}, \mathcal{E}')$; an operator-function Θ in $H^\infty(\mathcal{E}, \mathcal{E}')$ admits the canonical factorization $\Theta = \Theta_i\Theta_e$, where Θ_i is inner and Θ_e is outer, and from the canonical factorization of $\tilde{\Theta}$ we obtain the **-canonical factorization* $\Theta = \Theta_{*e}\Theta_{*i}$ of Θ, where Θ_{*i} is *-inner and Θ_{*e} is *-outer (see [5, p. 204]).

The proof of [12, Proposition 2] shows the following lemma for *-outer functions. For completeness, we give its proof here.

LEMMA 1. *If an operator-function $\Theta \in H^\infty(\mathcal{E}, \mathcal{E}')$ is *-outer, then the operator Θ is injective and the pre-image of $H^2(\mathcal{E}')$ under Θ is contained in $H^2(\mathcal{E})$, that is, $f \in L^2(\mathcal{E})$ is mapped into $H^2(\mathcal{E}')$ by Θ only if f is in $H^2(\mathcal{E})$. If Θ is inner in addition, and if $g \in H^2(\mathcal{E}')$ is in* $\operatorname{ran}\Theta$, *then Θ^*g is in $H^2(\mathcal{E})$.*

Proof. Since $\operatorname{ran}\tilde{\Theta}$ is dense in $L^2(\mathcal{E})$, for almost every t $\operatorname{ran}\Theta^*(e^{it})$ is dense in $\mathcal{E}$, so that $\ker\Theta(e^{it}) = \{0\}$ which implies that Θ is injective. Further it follows from the *-outer property of Θ that the image of $L^2(\mathcal{E}') \ominus H^2(\mathcal{E}')$ under Θ^* is dense in $L^2(\mathcal{E}) \ominus H^2(\mathcal{E})$. Therefore if Θf is in $H^2(\mathcal{E}')$, that is, Θf is orthogonal to $L^2(\mathcal{E}') \ominus H^2(\mathcal{E}')$, then f is orthogonal to $L^2(\mathcal{E}) \ominus H^2(\mathcal{E})$, hence f belongs to $H^2(\mathcal{E})$. If Θ is inner in addition and if $g \in H^2(\mathcal{E}')$ is in $\operatorname{ran}\Theta$, then $g = \Theta\Theta^*g$ so that Θ^*g must be in $H^2(\mathcal{E})$ as above. ▨

LEMMA 2. *Let $\Theta = \Theta_2\Theta_1$ be the canonical factorization of $\Theta \in H^\infty(\mathcal{E}, \mathcal{E}')$; $\Theta_1 \in H^\infty(\mathcal{E}, \mathcal{F})$ is outer while $\Theta_2 \in H^\infty(\mathcal{F}, \mathcal{E}')$ is inner. If there is $\Omega \in H^\infty(\mathcal{E}', \mathcal{E})$ and $0 \neq \delta \in H^\infty$ such that*

$$\Omega\Theta = \delta I_{\mathcal{E}}, \tag{1}$$

then the outer part δ_e of δ is a scalar multiple of Θ_1; for some $\Phi \in H^\infty(\mathcal{F}, \mathcal{E})$

$$\Phi\Theta_1 = \delta_e I_{\mathcal{E}} \quad \text{and} \quad \Theta_1\Phi = \delta_e I_{\mathcal{F}}. \tag{2}$$

*If, in addition, Θ is *-outer, then Θ_2 is *-outer too.*

Proof. Let $\Phi = (\Omega\Theta_2)_e$ be the outer part of $\Omega\Theta_2$. According to the uniqueness (up to a constant unitary) of the canonical factorization (see [5, p. 204]), we may assume that (1) implies $\Phi\Theta_1 = \delta_e I_{\mathscr{E}}$, hence $\{\Theta_1\Phi - \delta_e I_{\mathscr{F}}\}\Theta_1 = 0$. Since, Θ_1 being outer, ran Θ_1 is dense, this implies that $\Theta_1\Phi - \delta_e I_{\mathscr{F}} = 0$, proving (2). It follows from (2) that $\tilde{\Phi}\tilde{\Theta}_1 = \tilde{\delta}_e I_{\mathscr{F}}$. Then since $\tilde{\delta}_e I_{\mathscr{F}}$ is outer together with $\delta_e I_{\mathscr{F}}$, $\tilde{\Phi}$ must be outer. If, in addition, Θ is *-outer, $\tilde{\Phi}\tilde{\Theta}$ is outer. Then the relation $\tilde{\Phi}\tilde{\Theta} = \tilde{\Theta}_2\tilde{\delta}_e I_{\mathscr{E}'}$ implies that ran $\tilde{\Theta}_2$ is dense, that is, Θ_2 is *-outer. ▨

Let T be a contraction of class $C_{1\cdot}$ with Hilbert-Schmidt defect operator. Since, for any α with $|\alpha| < 1$,

$$I - T_\alpha^* T_\alpha = S_\alpha^*(I - T^*T)S_\alpha$$

where

$$T_\alpha = (T - \alpha I)(I - \bar{\alpha}T)^{-1} \quad \text{and} \quad S_\alpha = (1 - |\alpha|^2)^{1/2}(I - \bar{\alpha}T)^{-1}$$

(see [5, p. 240]), the operator $T - \alpha I$ is left Fredholm together with T. Also since T is of class $C_{1\cdot}$, $T - \alpha I$ is injective, hence it is left invertible. It follows from Fredholm index theory (see [3, Chapter 5]) that

$$\dim \ker(T - \alpha I)^* = -\operatorname{ind}(T - \alpha I)$$

is invariant for $|\alpha| < 1$. Further ind $T = 0$ if and only if T is a weak contraction.

For operators T_1 and T_2, $T_1 \overset{\text{ci}}{\prec} T_2$ denotes that there exists a family $\{X_\alpha\}$ of injections such that $X_\alpha T_1 = T_2 X_\alpha$ for each α and the span $\bigvee_\alpha \operatorname{ran} X_\alpha$ is the whole space on which T_2 acts. If the family $\{X_\alpha\}$ can be chosen to consist of a single operator, i.e. if there exists an injection X with dense range such that $XT_1 = T_2X$, then T_1 is called a *quasi-affine transform* of T_2, and this relation of T_1 and T_2 is denoted by $T_1 \prec T_2$. And T_1 and T_2 are said to be *completely injection-similar* if $T_1 \overset{\text{ci}}{\prec} T_2$ and $T_2 \overset{\text{ci}}{\prec} T_1$ ([7]).

For a Hilbert space $\mathscr{E}$, let $S_{\mathscr{E}}$ denote the unilateral shift on $H^2(\mathscr{E})$. For a c.n.u. contraction T, let S_{T^*} and U_T denote the unilateral shift on $H^2(\mathscr{D}_{T^*})$ and the unitary operator of multiplication by $\chi(e^{it}) = e^{it}$ on $\overline{\Delta_T L^2(\mathscr{D}_T)}$, respectively.

3. COMPLETE INJECTION-SIMILARITY

In this section we prove the following theorem.

THEOREM 1. *A c.n.u. $C_{1\cdot}$-contraction T with Hilbert-Schmidt defect operator is completely injection-similar to an isometry. More precisely*

$$S_{\mathscr{E}} \oplus U_T \overset{\text{ci}}{\prec} T \prec S_{\mathscr{E}} \oplus U_T$$

where $\mathscr{E}$ is a Hilbert space of dimension $-\operatorname{ind} T$.

To prove this theorem we need some lemmas.

Lemma 3 is a refined version of the result obtained in the proof of [12, Theorem 2] for a $C_{\cdot 0}$-contraction whose point spectrum does not fill the open unit disc.

LEMMA 3. *If T is a c.n.u. $C_{1\cdot}$-contraction with Hilbert-Schmidt defect operator, then for each cpmplex α with $|\alpha| < 1$ there exists an isometry V_α from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$ such that*

(3) $$\ker V_\alpha^* = \ker \Theta_T(\alpha)^*$$

and that $V_\alpha^* \Theta_T$ has a scalar multiple $\delta_\alpha \in H^\infty$;

(4) $$\Omega_\alpha V_\alpha^* \Theta_T = V_\alpha^* \Theta_T \Omega_\alpha = \delta_\alpha I_{\mathscr{D}_T}$$

with some $\Omega_\alpha \in H^\infty(\mathscr{D}_T)$ $(= H^\infty(\mathscr{D}_T, \mathscr{D}_T))$.

Proof. For $|\alpha| < 1$, the operator $T_\alpha = (T - \alpha I)(I - \bar{\alpha} T)^{-1}$ is a contraction and its characteristic function $\Theta_{T_\alpha}(\lambda)$ coincides with $\Theta_T\left(\dfrac{\lambda + \alpha}{1 + \bar{\alpha}\lambda}\right)$ (see [5, p. 240]), that is, there exist unitary operators $A_\alpha \colon \mathscr{D}_{T_\alpha} \mapsto \mathscr{D}_T$ and $B_\alpha : \mathscr{D}_{T_\alpha^*} \mapsto \mathscr{D}_{T^*}$ such that

(5) $$\Theta_{T_\alpha}(\lambda) = B_\alpha^* \Theta_T \left(\frac{\lambda + \alpha}{1 + \bar{\alpha}\lambda} \right) A_\alpha \quad \text{for } |\lambda| < 1.$$

Since the defect operator D_{T_α} of T_α is of Hilbert-Schmidt class together with D_T, the operator

$$I + (T_\alpha^* | \mathscr{D}_{T_\alpha^*}) \Theta_{T_\alpha}(\lambda) = D_{T_\alpha}^2 + \lambda D_{T_\alpha} T_\alpha^* (I - \lambda T_\alpha^*)^{-1} D_{T_\alpha}$$

is of trace class for $|\lambda| < 1$, and it follows by (5) that $I + A_\alpha T_\alpha^* B_\alpha^* \Theta_T(\lambda)$ is of trace class for $|\lambda| < 1$. Let $B_\alpha T_\alpha A_\alpha^* = V_\alpha P_\alpha$ be the polar decomposition of $B_\alpha T_\alpha A_\alpha^*$. Then V_α is an isometry from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$ because T_α is injective, and

$$\ker V_\alpha^* = \ker(A_\alpha T_\alpha^* B_\alpha^*) = \ker(-A_\alpha \Theta_{T_\alpha}(0)^* B_\alpha^*) =$$

$$= \ker(-\Theta_T(\alpha)^*) = \ker \Theta_T(\alpha)^*.$$

Since $I + P_\alpha V_\alpha^* \Theta_T(\lambda)$ $(|\lambda| < 1)$ and $I - P_\alpha$ are of trace class, the identity

$$I + V_\alpha^* \Theta_T(\lambda) = I + P_\alpha V_\alpha^* \Theta_T(\lambda) + (I - P_\alpha) V_\alpha^* \Theta_T(\lambda)$$

shows that $I + V_\alpha^* \Theta_T(\lambda)$ is of trace class for $|\lambda| < 1$. Then there exists an operator-function Ω_α in $H^\infty(\mathscr{D}_T)$ such that

$$\Omega_\alpha(\lambda) V_\alpha^* \Theta_T(\lambda) = V_\alpha^* \Theta_T(\lambda) \Omega_\alpha(\lambda) = \delta_\alpha(\lambda) I_{\mathscr{D}_T} \quad \text{for } |\lambda| < 1,$$

where $\delta_\alpha(\lambda) = \det(-V_\alpha^*\Theta_T(\lambda)) \in H^\infty$ (see [1]). Since T_α is injective and $I - T_\alpha^* T_\alpha$ is of trace class, we have that

$$\delta_\alpha(\alpha) = \det(-V_\alpha^*\Theta_T(\alpha)) = \det(A_\alpha(T_\alpha^* T_\alpha | \mathscr{D}_{T_\alpha})^{1/2} A_\alpha^*) \neq 0,$$

and δ_α is a non-zero function in H^∞. This completes the proof. ▨

We remark that $\ker \Theta_T(\alpha)^*$ has the same dimension $-\operatorname{ind} T$ for every $|\alpha| < 1$. In fact, putting $\lambda = 0$ in (5), we have

$$-T_\alpha | \mathscr{D}_{T_\alpha} = \Theta_{T_\alpha}(0) = B_\alpha^* \Theta_T(\alpha) A_\alpha .$$

Since T_α maps $\mathscr{D}_{T_\alpha}$ into $\mathscr{D}_{T_\alpha^*}$ while it isometrically maps the orthocomplement of $\mathscr{D}_{T_\alpha}$ onto the one of $\mathscr{D}_{T_\alpha^*}$ (see [5, p. 260]), it follows that $\Theta_T(\alpha)$ is left invertible and

$$\dim \ker \Theta_T(\alpha)^* = \dim \ker T_\alpha^* = -\operatorname{ind} T.$$

LEMMA 4. *If a c.n.u. contraction T with Hilbert-Schmidt defect operator is of class $C_{1\cdot}$ but not of class C_{11}, then* $\operatorname{ind} T < 0$ *and there are a Hilbert space $\mathscr{E}$ of dimension* $-\operatorname{ind} T$ *and an operator-function $\Phi \in H^\infty(\mathscr{D}_{T^*}, \mathscr{E})$ that is *-inner and outer such that*

$$\ker \Phi(e^{it}) = \operatorname{ran} \Theta_T(e^{it}) \quad \text{a.e. } t \tag{6}$$

and

$$\operatorname{ran} \tilde{\Phi}(e^{it}) = \ker \tilde{\Theta}_T(e^{it}) \quad \text{a.e. } t. \tag{7}$$

Proof. Put $\alpha = 0$ in Lemma 3, and consider an operator-function $\Psi \in H^\infty(\mathscr{D}_{T^*})$ defined by

$$\Psi = \delta_0 I_{\mathscr{D}_{T^*}} - \Theta_T \Omega_0 V_0^* . \tag{8}$$

Then since $\delta_0(e^{it}) \neq 0$ a.e. t, it follows from (4) and (8) that

$$\ker \Psi(e^{it}) = \operatorname{ran} \Theta_T(e^{it}) \quad \text{and} \quad \operatorname{ran} \Psi(e^{it}) = \ker V_0^* \quad \text{a.e. } t. \tag{9}$$

We first see that $\Psi \neq 0$, consequently $\ker V_0^* \neq \{0\}$ and $\operatorname{ind} T < 0$. In fact, if $\Psi = 0$, then δ_0 is a scalar multiple of Θ_T by (4) and (8), and since Θ_T is *-outer, it is also outer (see [5, Chapter V, Theorem 6.2]). Then, as remarked earlier, T is of class C_{11}; this is a contradiction.

Let $\Psi = \Psi_2 \Psi_1$ be the *-canonical factorization of Ψ, that is, $\Psi_1 \in H^\infty(\mathscr{D}_{T^*}, \mathscr{F})$ is *-inner and $\Psi_2 \in H^\infty(\mathscr{F}, \mathscr{D}_{T^*})$ is *-outer. Then since $\Psi_2(e^{it})$ is injective a.e. t,

$$\ker \Psi(e^{it}) = \ker \Psi_1(e^{it}) \tag{10}$$

and

(11) $$\dim \operatorname{ran} \Psi_1(e^{it}) = \dim \operatorname{ran} \Psi(e^{it}) = \dim \ker V_0^*.$$

Finally let $\Psi_1 = \Psi_{12}\Psi_{11}$ be the canonical factorization of Ψ_1, that is, $\Psi_{11} \in H^\infty(\mathscr{D}_{T^*}, \mathscr{E})$ is outer and $\Psi_{12} \in H^\infty(\mathscr{E}, \mathscr{F})$ is inner. Put $\Phi = \Psi_{11}$. Then clearly Φ is *-inner and outer, and for almost every t,

(12) $$\ker \Phi(e^{it}) = \ker \Psi_1(e^{it})$$

and

(13) $$\dim \mathscr{E} = \dim \operatorname{ran} \Phi(e^{it}) = \dim \operatorname{ran} \Psi_1(e^{it}).$$

It follows from (11), (13) and (3) that

$$0 < \dim \mathscr{E} = \dim \ker \Theta_T(0)^* = \dim \ker T^* = -\operatorname{ind} T$$

while (9), (10) and (12) imply (6). Then (7) follows from (6) by taking ortho-complements because Φ is *-inner, hence for almost every t, $\tilde{\Phi}(e^{it})$ is isometric and $\operatorname{ran} \tilde{\Phi}(e^{it})$ is closed. ▨

COROLLARY 1. *The operator-function Φ in Lemma 4 possesses the following properties:*

(14) $$\Phi(\lambda)\Theta_T(\lambda) = 0 \quad \textit{for all } \lambda \textit{ with } |\lambda| < 1$$

and

(15) $$\ker \Phi = \operatorname{ran} \Theta_2 \quad \textit{and} \quad \operatorname{ran} \tilde{\Phi} = \ker \tilde{\Theta}_T,$$

where Θ_2 is the inner part of Θ_T.

Proof. (6) implies

$$\Phi(e^{it})\Theta_T(e^{it}) = 0 \quad \text{a.e. } t.$$

Since both $\Phi(\lambda)$ and $\Theta_T(\lambda)$ are analytic functions of λ, this relation on the boundary yields (14). Also since the outer part Θ_1 of Θ_T has a scalar multiple by Lemma 2, $\Theta_1(e^{it})$ is invertible a.e. t, and therefore (6) implies

$$\ker \Phi(e^{it}) = \operatorname{ran} \Theta_2(e^{it}) \quad \text{a.e. } t.$$

Now (15) follows from this and (7) by using the isometric property of Θ_2 and $\tilde{\Phi}$.

The following lemma shows the relation $S_{\mathscr{E}} \oplus U_T \overset{\text{ci}}{\prec} T$ in Theorem 1, and it is also used in the subsequent discussion.

LEMMA 5. *Let T be a c.n.u. $C_{1\cdot}$-contraction with Hilbert-Schmidt defect operator. If T is not of class C_{11}, then there are a sequence $\{J_n : n = 1, 2, \dots\}$ of injections from $H(\Theta_T)$ to $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ and a sequence $\{K_n : n = 1, 2, \dots\}$ of injections from $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ to $H(\Theta_T)$, where $\mathscr{E}$ is a Hilbert space of dimension $-\operatorname{ind} T$, which satisfy the following conditions:*

$$(S_{\mathscr{E}} \oplus U_T)J_n = J_n S(\Theta_T) \quad \text{and} \quad K_n(S_{\mathscr{E}} \oplus U_T) = S(\Theta_T)K_n, \tag{16}$$

$$J_n K_n = \delta_n(S_{\mathscr{E}} \oplus U_T) \quad \text{and} \quad K_n J_n = \delta_n(S(\Theta_T)) \tag{17}$$

where δ_n are in H^∞, $\delta_n(S_{\mathscr{E}} \oplus U_T)$ and $\delta_n(S(\Theta_T))$ are operators obtained by the H^∞-functional calculus of Sz.-Nagy and Foiaş for all n, and

$$\bigvee_n \operatorname{ran} K_n = H(\Theta_T). \tag{18}$$

If T is of class C_{11}, then there exist injections J and K with dense range such that

$$U_T J = JT \quad \text{and} \quad KU_T = TK, \tag{16$'$}$$

$$JK = \delta(U_T) \quad \text{and} \quad KJ = \delta(T), \tag{17$'$}$$

where δ is an outer function in H^∞.

Proof. Suppose that T is not of class C_{11}. Fix a sequence of distinct complex numbers $\{\alpha_n\}$ such that $|\alpha_n| < 1$ and $\lim \alpha_n = 0$. Since $\ker \Theta_T(\alpha_n)^*$ has the same dimension $-\operatorname{ind} T$ for every n, it is possible to take an isometry W_n from $\mathscr{E}$ to $\mathscr{D}_{T^*}$ such that $\operatorname{ran} W_n = \ker \Theta_T(\alpha_n)^*$. Write, for simplicity, $V_n = V_{\alpha_n}$, $\Omega_n = \Omega_{\alpha_n}$ and $\delta_n = \delta_{\alpha_n}$ in Lemma 3. First define an operator $\hat{J}_n$ from $K(\Theta_T)$ to $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ and one $\hat{K}_n$ from $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ to $K(\Theta_T)$ by

$$\hat{J}_n = \begin{bmatrix} W_n^*(\delta_n I - \Theta_T \Omega_n V_n^*) & 0 \\ -\Delta_T \Omega_n V_n^* & \delta_n I \end{bmatrix}$$

and

$$\hat{K}_n = \begin{bmatrix} W_n & 0 \\ 0 & I \end{bmatrix},$$

respectively, where W_n also denotes a constant function in $H^\infty(\mathscr{E}, \mathscr{D}_{T^*})$ whose value is W_n. Then obviously

$$\hat{J}_n(S_{T^*} \oplus U_T) = (S_{\mathscr{E}} \oplus U_T)\hat{J}_n \quad \text{and} \quad (S_{T^*} \oplus U_T)\hat{K}_n = \hat{K}_n(S_{\mathscr{E}} \oplus U_T). \tag{19}$$

Since $W_n W_n^* = I - V_n V_n^*$ by (3), it follows from (4) that

$$\begin{aligned} W_n W_n^*(\delta_n I - \Theta_T \Omega_n V_n^*) &= \delta_n(I - V_n V_n^*) - \Theta_T \Omega_n V_n^* + V_n V_n^* \Theta_T \Omega_n V_n^* = \\ &= \delta_n I - \Theta_T \Omega_n V_n^* . \end{aligned} \tag{20}$$

Therefore

$$\hat{K}_n \hat{J}_n = \begin{bmatrix} \delta_n I & 0 \\ 0 & \delta_n I \end{bmatrix} - \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} [\Omega_n V_n^* \quad 0] . \tag{21}$$

Since ran $W_n = \ker V_n^*$ by (3),

$$\hat{J}_n\hat{K}_n = \begin{bmatrix} \delta_n I & 0 \\ 0 & \delta_n I \end{bmatrix}. \tag{22}$$

Next we claim that

$$\ker \hat{J}_n = K(\Theta_T) \ominus H(\Theta_T). \tag{23}$$

That the right side is included in the left side follows immediately from (4). Take $f \oplus g$ in $H(\Theta_T) \cap \ker \hat{J}_n$. Then firstly $f \oplus g \in H(\Theta_T)$ implies that $\Theta_T^* f + \Delta_T g \in L^2(\mathscr{D}_T) \ominus H^2(\mathscr{D}_T)$. Next $f \oplus g \in \ker \hat{J}_n$ implies, via (21), that $\delta_n f = \Theta_T \Omega_n V_n^* f$ and $\delta_n g = \Delta_T \Omega_n V_n^* f$, hence

$$\delta_n \cdot (\Theta_T^* f + \Delta_T g) = \Omega_n V_n^* f.$$

Therefore $\delta_n^{-1}\Omega_n V_n^* f$ belongs to $L^2(\mathscr{D}_T) \ominus H^2(\mathscr{D}_T)$. But since $\Theta_T \delta_n^{-1}\Omega_n V_n^* f = f$ and Θ_T is *-outer, by Lemma 1 this is possible only when $\delta_n^{-1}\Omega_n V_n^* f = 0$, hence $f = 0$ and $g = 0$. This establish (23).

Now let $J_n = \hat{J}_n | H(\Theta_T)$ and $K_n = P\hat{K}_n$. Then (16) and (17) follow from (19), (21), (22), (23) and the identity $P(S_{T*} \oplus U_T) = P(S_{T*} \oplus U_T)P$. Further J_n is injective by (23). Since obviously $\delta_n(S_{\mathscr{E}} \oplus U_T)$ is injective, so is K_n by (17). It remains to prove (18). It suffices to show that $f \oplus g$ in $H(\Theta_T)$ can be orthogonal to all ran $\hat{K}_n$ ($n = 1, 2, \ldots$) only if $f = 0$ and $g = 0$. First of all, $g = 0$ follows immediately from the orthogonality. Then since $f \oplus g \in H(\Theta_T)$ means $\Theta_T^* f + \Delta_T g \in L^2(\mathscr{D}_T) \ominus H^2(\mathscr{D}_T)$, we have $\Theta_T^* f \in L^2(\mathscr{D}_T) \ominus H^2(\mathscr{D}_T)$. Let $\Theta_T = \Theta_2\Theta_1$ be the canonical factorization of Θ_T ($\Theta_1 \in H^\infty(\mathscr{D}_T, \mathscr{F})$ and $\Theta_2 \in H^\infty(\mathscr{F}, \mathscr{D}_{T*})$). Then since Θ_1 is outer, it follows that

$$\Theta_2^* f \in L^2(\mathscr{F}) \ominus H^2(\mathscr{F}). \tag{24}$$

Next the orthogonality of f to ran W_n implies that $f(\alpha_n)$ is orthogonal to ran W_n. Since ran $W_n = \ker \Theta_T(\alpha_n)^*$, it follows from Corollary 1 that $\Phi(\alpha_n) f(\alpha_n) = 0$ ($n = 1, 2, \ldots$). Then by the uniqueness theorem for an operator analytic function $\Phi(\lambda) f(\lambda) = 0$ for all $|\lambda| < 1$, hence $f \in \ker \Phi$. Then again by Corollary 1, $f \in$ ran Θ_2. Since Θ_2 is inner and *-outer by Lemma 2, it follows from Lemma 1 that $\Theta_2^* f$ belongs to $H^2(\mathscr{F})$. When combined with (24), this yields $\Theta_2^* f = 0$, and finally $f = \Theta_2\Theta_2^* f = 0$. This completes the proof for the case T is not of class C_{11}.

If T is of class C_{11}, then it is a weak contraction, hence Θ_T has a scalar multiple δ that is outer (see [5, Chapter VIII]); $\Omega\Theta_T = \delta I$ and $\Theta_T\Omega = \delta I$ for some $\Omega \in H^\infty(\mathscr{D}_{T*}, \mathscr{D}_T)$. We define the operators J and K by

$$J = [-\Delta_T\Omega \quad \delta I] \,|\, H(\Theta_T) : H(\Theta_T) \mapsto \overline{\Delta_T L^2(\mathscr{D}_T)}, \tag{25}$$

$$K = P\begin{bmatrix} 0 \\ I \end{bmatrix} : \overline{\Delta_T L^2(\mathscr{D}_T)} \mapsto H(\Theta_T). \tag{26}$$

Then the identites (16)′ and (17)′ are clear. Since δ is outer, $\delta(S(\Theta_T))$ is an injection with dense range (see (5, Chapter III, Proposition 3.1]). Also obviously $\delta(U_T)$ is injective and has dense range. Therefore it follows from (17)′ that J and K are injective and have dense range. ▨

Proof of Theorem 1. It is known that a c.n.u. C_{11}-contraction T is quasi-similar to U_T, that is, $T \prec U_T$ and $U_T \prec T$ (see [5, pp. 71—72]) and it is also proved in Lemma 5. So suppose that T is not of class C_{11}. Consider an operator $\hat{X}$ from $K(\Theta_T)$ to $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ defined by

$$\hat{X} = \begin{bmatrix} \Phi & 0 \\ -\Delta_T \Theta_T^* & \Theta_T^* \Theta_T \end{bmatrix},$$

where Φ is the outer function in Lemma 4. Obviously $\hat{X}$ intertwines $S_{\mathscr{E}} \oplus U_T$ and $S_{T^*} \oplus U_T$, that is, $(S_{\mathscr{E}} \oplus U_T)\hat{X} = \hat{X}(S_{T^*} \oplus U_T)$.

First we claim that $\hat{X}$ has dense range. In fact, since, Θ_T being $*$-outer, $\Theta_T^*\Theta_T$ has dense range and commutes with Δ_T, $\Theta_T^*\Theta_T$ maps $\overline{\Delta_T L^2(\mathscr{D}_T)}$ to a dense set of $\overline{\Delta_T L^2(\mathscr{D}_T)}$. This implies that $\{0\} \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$ is contained in the closure of ran $\hat{X}$. Further since Φ is outer, it maps $H^2(\mathscr{D}_{T^*})$ to a dense set of $H^2(\mathscr{E})$. Therefore the closure of ran $\hat{X}$ must contain $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$.

Next we claim that ker $\hat{X}$ coincides with $K(\Theta_T) \ominus H(\Theta_T)$. Since $\Phi\Theta_T = 0$ by (6), and Δ_T commutes with $\Theta_T^*\Theta_T$,

$$\begin{bmatrix} \Phi & 0 \\ -\Delta_T \Theta_T^* & \Theta_T^* \Theta_T \end{bmatrix} \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which shows that ker $\hat{X}$ contains $K(\Theta_T) \ominus H(\Theta_T)$. Suppose that $f \oplus g$ is in $H(\Theta_T) \cap \cap \ker \hat{X}$, or equivalently

$$\Theta_T^* f + \Delta_T g \in L^2(\mathscr{D}_T) \ominus H^2(\mathscr{D}_T), \tag{27}$$

$$\Phi f = 0 \quad \text{and} \quad -\Delta_T \Theta_T^* f + \Theta_T^* \Theta_T g = 0. \tag{28}$$

Let $\Theta_T = \Theta_2 \Theta_1$ be the canonical factorization of Θ_T; $\Theta_1 \in H^\infty(\mathscr{D}_T, \mathscr{F})$ is outer and $\Theta_2 \in H^\infty(\mathscr{F}, \mathscr{D}_{T^*})$ is inner. Let $\Delta_1 = (I - \Theta_1^* \Theta_1)^{1/2}$ and $\Delta_{*1} = (I - \Theta_1 \Theta_1^*)^{1/2}$. Then since Θ_2 is inner, it follows from (28) and (15) that $f = \Theta_2 \Theta_2^* f$ and

$$0 = -\Delta_T \Theta_T^* f + \Theta_T^* \Theta_T g = -\Delta_1 \Theta_1^* \Theta_2^* f + \Theta_1^* \Theta_1 g = \Theta_1^*(-\Delta_{*1} \Theta_2^* f + \Theta_1 g).$$

Since Θ_1 is outer, $-\Delta_{*1}\Theta_2^* f + \Theta_1 g = 0$, hence $\Delta_{*1}^2 \Theta_2^* f = \Delta_{*1} \Theta_1 g$. Thus we have

$$\Theta_2^* f = \Theta_1(\Theta_T^* f + \Delta_T g),$$

which implies

$$f = \Theta_2 \Theta_2^* f = \Theta_T(\Theta_T^* f + \Delta_T g). \tag{29}$$

Since $f \in H^2(\mathscr{D}_{T*})$ and Θ_T is $*$-outer, it follows by Lemma 1 and (29) that

$$\Theta_T^* f + \Delta_T g \in H^2(\mathscr{D}_T).$$

When combined with (27), this yields $\Theta_T^* f + \Delta_T g = 0$. Then $f = 0$ follows from (29), hence $\Delta_T g = 0$. Since g is in $\overline{\Delta_T L^2(\mathscr{D}_T)}$, $g = 0$. This establishes the claim.

Now let X be the restriction of $\hat{X}$ to $H(\Theta_T)$. Then $\ker X = \{0\}$ and $\operatorname{ran} X$ is dense in $H^2(\mathscr{E}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}$, and

$$(S_{\mathscr{E}} \oplus U_T)X = XS(\Theta_T),$$

which proves $T \prec S_{\mathscr{E}} \oplus U_T$.

The relation $S_{\mathscr{E}} \oplus U_T \overset{\text{ci}}{\prec} T$ follows immediately from Lemma 5. ▨

4. DOUBLE COMMUTANT

In this section we consider the double commutant $\{T\}''$ of a c.n.u. contraction T of class $C_{1\cdot}$ with Hilbert-Schmidt defect operator D_T.

The minimal isometric dilation of $T = S(\Theta_T)$ on $H(\Theta_T)$ is $S_{T*} \oplus U_T$ on $K(\Theta_T)$. Consider the class $\mathscr{L}$ of operators $\hat{X}$ on $K(\Theta_T)$ that commute with $S_{T*} \oplus U_T$ and make $K(\Theta_T) \ominus H(\Theta_T)$ invariant. More explicitly, $\hat{X}$ belongs to $\mathscr{L}$ if and only if firstly it admits a representation

$$\hat{X} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \tag{30}$$

where $A \in H^\infty(\mathscr{D}_{T*})$, $B \in L^\infty(\mathscr{D}_{T*}, \mathscr{D}_T)$ and $C \in L^\infty(\mathscr{D}_T)$ such that B maps $H^2(\mathscr{D}_{T*})$ into $\overline{\Delta_T L^2(\mathscr{D}_T)}$ and C maps $\overline{\Delta_T L^2(\mathscr{D}_T)}$ into itself, and secondly there exists $K \in H^\infty(\mathscr{D}_T)$ such that

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} = \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} K. \tag{31}$$

According to the lifting theorem of Sz.-Nagy and Foiaş (see [5, Chapter II, Theorem 2.3] or [6]) the correspondence π that assigns to $\hat{X}$ its compression to $H(\Theta_T)$, i.e. $\pi(\hat{X}) = P\hat{X}|H(\Theta_T)$, maps $\mathscr{L}$ onto the commutant $\{T\}'$. Obviously π is multiplicative.

In case Θ_T is $*$-outer, if $\hat{X}$ maps $H^2(\mathscr{D}_{T*}) \oplus \{0\}$ into $K(\Theta_T) \ominus H(\Theta_T)$, then $\pi(\hat{X}) = 0$. In fact, if $H^2(\mathscr{D}_{T*}) \oplus \{0\}$ is mapped into $K(\Theta_T) \ominus H(\Theta_T)$, there exists $L \in H^\infty(\mathscr{D}_{T*}, \mathscr{D}_T)$ such that

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} L.$$

When combined with (31), this yields that

$$\Theta_T(L\Theta_T - K) = 0 \quad \text{and} \quad \Delta_T L \Theta_T + C\Delta_T = \Delta_T K.$$

Since, Θ_T being *-outer, Θ_T is injective, it follows that $L\Theta_T = K$ and

$$C\Delta_T = \Delta_T K - \Delta_T L\Theta_T = 0.$$

Then clearly C vanishes on the whole $\overline{\Delta_T L^2(\mathscr{D}_T)}$, hence $\hat{X}$ maps the whole $K(\Theta_T)$ into $K(\Theta_T) \ominus H(\Theta_T)$, or equivalently $\pi(\hat{X}) = 0$.

THEOREM 2. *If a c.n.u. $C_{1\cdot}$-contraction T with Hilbert-Schmidt defect operator is not of class C_{11}, then*

$$\{T\}'' = \{\varphi(T) : \varphi \in H^\infty\},$$

and in particular

$$\{T\}'' = \operatorname{Alg} T.$$

Proof. It is clear that $\varphi(T)$ is in $\{T\}''$ for every $\varphi \in H^\infty$. Therefore let us prove that for each $\hat{X}$ in $\mathscr{L}$ for which $\pi(\hat{X})$ is in the double commutant $\{T\}''$, there is $\varphi \in H^\infty$ such that $\pi(\hat{X}) = \varphi(T)$. Suppose that $\hat{X}$ admits a representation (30) with (31). Take the operator-function $\Phi \in H^\infty(\mathscr{D}_{T^*}, \mathscr{E})$ in Lemma 4 that is outer and *-inner. Since $\Phi\Theta_T = 0$ by (6), the relation (31) implies $\Phi A\Theta_T = 0$ or equivalently $\tilde{\Theta}_T(\Phi A)^\sim = 0$, hence by (15)

$$\operatorname{ran}(\Phi A)^\sim \subset \ker \tilde{\Theta}_T = \operatorname{ran} \tilde{\Phi}.$$

Since $\tilde{\Phi}$ is inner and *-outer, it follows that $\tilde{\Phi}^*(\Phi A)^\sim$ belongs to $H^\infty(\mathscr{E})$. Let $A_1 = (\tilde{\Phi}^*(\Phi A)^\sim)^\sim$. Then $A_1 \in H^\infty(\mathscr{E})$ and

$$\Phi A = A_1 \Phi. \tag{32}$$

We claim that there is a function $\varphi \in H^\infty$ such that $A_1 = \varphi I_{\mathscr{E}}$. For this purpose, take any $F \in H^\infty(\mathscr{E}, \mathscr{D}_{T^*})$. Since $F\Phi\Theta_T = 0$ by (6), the operator

$$\hat{Y} = \begin{bmatrix} F\Phi & 0 \\ 0 & 0 \end{bmatrix}$$

belongs to $\mathscr{L}$. Then the assumption $\pi(\hat{X}) \in \{T\}''$ implies, via the multiplicativity of π, that

$$\pi(\hat{X}\hat{Y} - \hat{Y}\hat{X}) = \pi(\hat{X})\pi(\hat{Y}) - \pi(\hat{Y})\pi(\hat{X}) = 0,$$

hence the operator

$$\hat{X}\hat{Y} - \hat{Y}\hat{X} = \begin{bmatrix} AF\Phi - F\Phi A & 0 \\ BF\Phi & 0 \end{bmatrix}$$

maps $H^2(\mathscr{D}_{T^*}) \oplus \{0\}$ into $K(\Theta_T) \ominus H(\Theta_T)$. Then it follows from (32), Φ being outer, that $\begin{bmatrix} AF - FA_1 \\ BF \end{bmatrix}$ maps $H^2(\mathscr{E})$ into $K(\Theta_T) \ominus H(\Theta_T)$, the range of $\begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix}$. Then again $\Phi\Theta_T = 0$ implies that

$$A_1\Phi F - \Phi FA_1 = \Phi(AF - FA_1) = 0,$$

and therefore $A_1\Phi\cdot(\chi^{-n}F) = \Phi\cdot(\chi^{-n}F)A_1$ for $n = 1,2, \ldots$, where $\chi(e^{it}) = e^{it}$. The set $\{\chi^{-n}F : F \in H^\infty(\mathscr{E}, \mathscr{D}_{T^*})$ and $n=1,2, \ldots\}$ is operator-weakly dense in $L^\infty(\mathscr{E}, \mathscr{D}_{T^*})$ and since Φ is $*$-inner, $\Phi L^\infty(\mathscr{E}, \mathscr{D}_{T^*}) = L^\infty(\mathscr{E})$. Therefore it follows that A_1 in $H^\infty(\mathscr{E})$ commutes with all of $L^\infty(\mathscr{E})$, which is possible only when $A_1 = \varphi I$ for some function $\varphi \in H^\infty$, establishing the claim.

Now since for every $F \in H^\infty(\mathscr{E}, \mathscr{D}_{T^*})$ the operator

$$\begin{bmatrix} A - \varphi I \\ B \end{bmatrix} F = \begin{bmatrix} AF - FA_1 \\ BF \end{bmatrix}$$

maps $H^2(\mathscr{E})$ into $K(\Theta_T) \ominus H(\Theta_T)$ and obviously

$$H^2(\mathscr{D}_{T^*}) = \bigvee \{\operatorname{ran} F : F \in H^\infty(\mathscr{E}, \mathscr{D}_{T^*})\},$$

it follows that the operator $\begin{bmatrix} A - \varphi I \\ B \end{bmatrix}$ maps $H^2(\mathscr{D}_{T^*})$ into $K(\Theta_T) \ominus H(\Theta_T)$. Finally the operator $\hat{Z} = \begin{bmatrix} \varphi I & 0 \\ 0 & \varphi I \end{bmatrix}$ belongs to $\mathscr{L}$ and $\pi(\hat{Z}) = \varphi(T)$, and

$$\hat{X} - \hat{Z} = \begin{bmatrix} A - \varphi I & 0 \\ B & C - \varphi I \end{bmatrix}$$

maps $H^2(\mathscr{D}_{T^*}) \oplus \{0\}$ into $K(\Theta_T) \ominus H(\Theta_T)$. Then since Θ_T is $*$-outer, as remarked in the front of this theorem, $\pi(\hat{X} - \hat{Z}) = 0$, or equivalently $\pi(\hat{X}) = \varphi(T)$. ▨

For contractions whose defect operators are of finite rank, Theorem 2 was proved in [16]. The theorem of the same type was proved in [10] and [11] for $C_{\cdot 0}$-contractions not of class C_{00} whose defect operators are of finite rank.

We next characterize C_{11}-contractions which satisfy the bicommutant property. This characterization was obtained in [14] when the defect operators are of finite rank.

LEMMA 6. *Let T be a c.n.u. C_{11}-contraction with Hilbert-Schmidt defect operator and let J and K be the operators satisfying the conditions* (16)′ *and* (17)′ *in Lemma 5. Then*

$$\operatorname{Alg} T = \{A : JAK \in \operatorname{Alg} U_T\}.$$

We use the following celebrated result of Sarason (see [8, Theorem 7.1]): Let T and A be bounded linear operators on a Hilbert space. Then $A \in \operatorname{Alg} T$ if and only if

$$\operatorname{Lat} T^{(n)} \subseteq \operatorname{Lat} A^{(n)}$$

for every positive integer n, where for an operator X, $X^{(n)}$ denotes the direct sum of n copies of X.

Proof of Lemma 6. Assume $A \in \operatorname{Alg} T$. We shall show that $\operatorname{Lat} U_T^{(n)} \subseteq \operatorname{Lat}(JAK)^{(n)}$ for all n. Then it will follow from the above result that $JAK \in \operatorname{Alg} U_T$. Let $\mathcal{M} \in \operatorname{Lat} U_T^{(n)}$. Since $T^{(n)}K^{(n)} = K^{(n)}U_T^{(n)}$ by (16)$'$, $\overline{K^{(n)}\mathcal{M}} \in \operatorname{Lat} T^{(n)}$. Then, since $A^{(n)} \in \operatorname{Alg} T^{(n)}$, $(JAK)^{(n)} = J^{(n)}A^{(n)}K^{(n)}$ and $J^{(n)}K^{(n)} = \delta(U_T^{(n)})$ by (17)$'$, we have

$$(JAK)^{(n)}\mathcal{M} = J^{(n)}A^{(n)}K^{(n)}\mathcal{M} \subseteq \overline{J^{(n)}\overline{K^{(n)}\mathcal{M}}} = \overline{\delta(U_T^{(n)})\mathcal{M}} \subseteq \mathcal{M}.$$

This shows that $\operatorname{Lat} U_T^{(n)} \subseteq \operatorname{Lat}(JAK)^{(n)}$.

Next we assume that $JAK \in \operatorname{Alg} U_T$. To prove $A \in \operatorname{Alg} T$, let us show that $\operatorname{Lat} T^{(n)} \subseteq \operatorname{Lat} A^{(n)}$ for all n. Since, from $JAK \in \operatorname{Alg} U_T$ and (16)$'$,

$$JTAK = U_T JAK = JAKU_T = JATK$$

and J is injective and K has dense range, it follows that A belongs to the commutant $\{T\}'$ of T. Let $\mathcal{N} \in \operatorname{Lat} T^{(n)}$. Since $\overline{J^{(n)}\mathcal{N}} \in \operatorname{Lat} U_T^{(n)}$ by (16)$'$ and the relation $JAK \in \operatorname{Alg} U_T$ implies $J^{(n)}A^{(n)}K^{(n)} = (JAK)^{(n)} \in \operatorname{Alg} U_T^{(n)}$,

$$K^{(n)}J^{(n)}A^{(n)}K^{(n)}J^{(n)}\mathcal{N} \subseteq \overline{K^{(n)}J^{(n)}\mathcal{N}} = \overline{\delta(T^{(n)})\mathcal{N}} \subseteq \mathcal{N}.$$

On the other hand, since $A \in \{T\}'$,

$$K^{(n)}J^{(n)}A^{(n)}K^{(n)}J^{(n)} = \delta(T^{(n)})A^{(n)}\delta(T^{(n)}) = A^{(n)}\delta(T^{(n)})^2$$

and it follows that $A^{(n)}\delta(T^{(n)})^2\mathcal{N} \subseteq \mathcal{N}$. But since δ is outer,

$$\overline{\delta(T^{(n)})^2\mathcal{N}} = \overline{\operatorname{ran} \delta(T^{(n)}|\mathcal{N})^2} = \mathcal{N},$$

hence $A^{(n)}\mathcal{N} \subseteq \mathcal{N}$. This shows that $\operatorname{Lat} T^{(n)} \subseteq \operatorname{Alg} A^{(n)}$ for all n. ▨

THEOREM 3. *Let T be a c.n.u. C_{11}-contraction with Hilbert-Schmidt defect operator. Then $\{T\}'' = \operatorname{Alg} T$ if and only if $\Theta_T(e^{it})$ is isometric on a set of t's of positive Lebesgue measure.*

Proof. Suppose that $\Theta_T(e^{it})$ is isometric on a set of positive Lebesgue measure, and let us prove the bicommutant property for T. Since $\operatorname{Alg} T \subseteq \{T\}''$, we shall show that $\{T\}'' \subseteq \operatorname{Alg} T$.

We claim that U_T has the bicommutant property. In fact, if U_T has not the bicommutant property, then, as shown in [9], it follows from the reflexivity of U_T and the double commutant theorem for von Neumann algebras that U_T is not reductive; i.e., there exists a subspace $\mathcal{M}$ of $\overline{\Delta_T L^2(\mathcal{D}_T)}$ that is U_T-invariant but not U_T-reducing. The subspace $\mathcal{M}$, which is a subspace of $L^2(\mathcal{D}_T)$, is invariant under the bilateral shift on $L^2(\mathcal{D}_T)$ but not reducing. Therefore it follows from the invariant subspace theorem of shifts (see for example, [8, Theorem 3.25]) that $\mathcal{M}$ contains the subspace $\Psi H^2(\mathcal{G})$, where $\mathcal{G}$ is a Hilbert space and Ψ is an operator-function in $L^\infty(\mathcal{G}, \mathcal{D}_T)$ whose value is isometric a.e., which implies that $\overline{\Delta_T L^2(\mathcal{D}_T)}$ contains a function which does not vanish almost everywhere. This contradicts the assumption that $\Delta_T(e^{it}) = 0$ on some set of positive Lebesgue measure, proving the claim.

Now by Lemma 6 and the bicommutant property of U_T, to prove $\{T\}'' \subseteq \operatorname{Alg} T$, it suffices to show that $JAK \in \{U_T\}''$ for each $A \in \{T\}''$. Let $A \in \{T\}''$ and $B \in \{U_T\}'$. Then (16)′ implies $JAK \in \{U_T\}'$ and $KBJ \in \{T\}'$, hence we use (17)′ to have

$$JKJAKB = JAKBJK = JKBJAK.$$

Since JK is injective, $JAKB = BJAK$, and therefore $JAK \in \{U_T\}''$.

We next assume that $\Theta_T(e^{it})$ is non-isometric for almost every t, i.e. $\Delta_T(e^{it}) \neq 0$ a.e. t. Then there exists a function $E \in \overline{\Delta_T L^2(\mathcal{D}_T)}$ such that $\|E(e^{it})\| = 1$ a.e. t. Let $\mathcal{N}$ denote the closure of $\{P(0 \oplus fE) : f \in H^2\}$, which is a subspace of $H(\Theta_T)$. Obviously $\mathcal{N}$ is invariant for $T = S(\Theta_T)$ and $\mathcal{N} \neq \{0\}$ by the injectivity of K defined by (26) in the proof of Lemma 5. Using the injection J defined by (25), we have

$$(J|\mathcal{N})(T|\mathcal{N}) = (U_T|\overline{J\mathcal{N}})(J|\mathcal{N}).$$

Since J is injective and the inclusion

$$J\mathcal{N} \subseteq \overline{\{\delta fE : f \in H^2\}} \subseteq \{fE : f \in H^2\}$$

implies that $U_T|\overline{J\mathcal{N}}$ is a unilateral shift, $T|\mathcal{N}$ is of class $C_{\cdot 0}$. Then it follows that $\{T\}'' \neq \operatorname{Alg} T$, which completes the proof. Indeed, if $\{T\}'' = \operatorname{Alg} T$, then $\mathcal{N} \in \operatorname{Lat} A$ for all $A \in \{T\}''$. Since particularly $\mathcal{N} \in \operatorname{Lat}(\lambda I - T)^{-1}$ for all $\lambda \notin \sigma(T)$ (= the spectrum of T), we have $\sigma(T|\mathcal{N}) \subseteq \sigma(T)$, hence $T|\mathcal{N}$ as well as T is a weak contraction. Then the $C_{1\cdot}$-contraction $T|\mathcal{N}$ is of class C_{11}, as remarked earlier. This contradicts $T|\mathcal{N} \in C_{\cdot 0}$. ▨

5. REFLEXIVITY

It was proved in [15] and [17] that a $C_{1\cdot}$-contraction whose defect operator is of finite rank is reflexive. We obtain the following theorem.

THEOREM 4. *A c.n.u. $C_{1\cdot}$-contraction T with Hilbert-Schmidt defect operator is reflexive.*

Proof. Let $A \in \operatorname{Alg}\operatorname{Lat} T$, and let us prove $A \in \operatorname{Alg} T$. If T is of class C_{11}, then by Lemma 6 it suffices to show that $JAK \in \operatorname{Alg} U_T$, where J and K are the operators in Lemma 5. But since it easily follows from (16)′ and (17)′ that $JAK \in \operatorname{Alg}\operatorname{Lat} U_T$, the reflexivity of the unitary operator U_T implies that $JAK \in \operatorname{Alg} U_T$. So we assume that T is not of class C_{11}. By Theorem 2 it suffices to show that $A \in \{T\}''$. Let $\{J_n\}$ and $\{K_n\}$ be the sequences of injections in Lemma 5. From (16) and (17) we have $J_nAK_n \in \operatorname{Alg}\operatorname{Lat}(S_{\mathcal{E}} \oplus U_T)$, so the reflexivity of the isometry $S_{\mathcal{E}} \oplus U_T$ implies $J_nAK_n \in \operatorname{Alg}(S_{\mathcal{E}} \oplus U_T)$, and therefore

$$J_nTAK_n = (S_{\mathcal{E}} \oplus U_T)J_nAK_n = J_nAK_n(S_{\mathcal{E}} \oplus U_T) = J_nATK_n$$

for all n. Then, since the injectivity of J_n implies $TAK_n = ATK_n$ for all n and $\bigvee_n \operatorname{ran} K_n$ is the whole space by (18), it follows that $TA = AT$. Let $B \in \{T\}'$. Since $J_nBK_n \in \{S_{\mathcal{E}} \oplus U_T\}'$, $J_nAK_n \in \operatorname{Alg}(S_{\mathcal{E}} \oplus U_T)$, $A \in \{T\}'$ and $K_nJ_n = \delta_n(T)$,

$$J_nK_nJ_nBAK_n = J_nBK_nJ_nAK_n = J_nAK_nJ_nBK_n = J_nK_nJ_nABK_n.$$

Using the injectivity of $J_nK_nJ_n$ and the condition (18) of K_n again, we have $BA = AB$, hence $A \in \{T\}''$. ▨

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