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ON THE REDUCTION AND TRIANGULARIZATION OF SEMIGROUPS OF OPERATORS

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1. INTRODUCTION

Let $\mathscr{S}$ be a multiplicative semigroup of operators on a complex Hilbert space $\mathscr{H}$. We shall be interested in sufficient conditions under which $\mathscr{S}$ can be "reduced", i.e. there exists a (closed) subspace $\mathscr{M}$ of $\mathscr{H}$, other than $\{0\}$ and $\mathscr{H}$, which is invariant for every member of $\mathscr{S}$. Some of these conditions are strong enough to give (simultaneous) triangularizability. This means the existence of a chain $\mathscr{C}$ of subspaces of $\mathscr{H}$ such that

(a) $\mathscr{C}$ is maximal, and

(b) every member of $\mathscr{C}$ is invariant for $\mathscr{S}$.

The maximality requirement for $\mathscr{C}$ implies that if $\mathscr{M}$ is in $\mathscr{C}$, and if $\mathscr{M}_-$ is the closed linear span of $\{\mathscr{N} \in \mathscr{C} : \mathscr{N} \subsetneq \mathscr{M}\}$, then $\mathscr{M} \ominus \mathscr{M}_-$ has dimension 0 or 1.

Kaplansky [4] gives triangularizability results for semigroups of operators on finite-dimensional spaces. One of his results is that if all the members of $\mathscr{S}$ have the same trace, then $\mathscr{S}$ is triangularizable. It was conjectured in [7] that this should hold for trace-class operators on $\mathscr{H}$. We prove it in Section 3 along with the infinite-dimensional analog of its companion in [4]: if the spectrum, $\sigma(A)$, of every A in $\mathscr{S}$ is a subset of $\{0,1\}$ and the algebraic multiplicity of 1 in $\sigma(A)$ is r, where r is a fixed nonnegative integer, then $\mathscr{S}$ is triangularizable. (It should be noted here that Kaplansky's results are valid for very general fields.) We also show the triangularizability of a semigroup of compact (not necessarily orthogonal) projections. Other triangularizablity results for semigroups are given in [7]. In Section 2 stronger results on reducibility are given. In particular, it is proved that if $\mathscr{S}$ is a semigroup of trace-class operators and if 1 is the unique element of maximal modulus in $\sigma(A)$ for every A in $\mathscr{S}$, then $\mathscr{S}$ has an invariant subspace. The algebraic multiplicity of 1 as an eigenvalue of A is allowed to vary, even without bound.

In what follows, "operator" means bounded linear operator on $\mathscr{H}$. The notation $\mathscr{C}_1$ is used for trace-class operators, $|A|$ denotes the trace norm of A in $\mathscr{C}_1$, i.e., $|A| = \operatorname{tr}((A^*A)^{1/2})$. For properties of the trace, trace norm, and the Banach algebra $\mathscr{C}_1$ see, e.g., [1, pp. 1088—1119]. The operator norm of A will be denoted by $\|A\|$ as usual. We shall denote by $\bigvee \mathscr{S}$ the (not necessarily closed) linear span of the set $\mathscr{S}$ of operators.

2. EXISTENCE OF INVARIANT SUBSPACES

It is convenient to begin with some simple lemmas.

LEMMA 1. *Let N be a nilpotent operator on a finite-dimensional space with $N^k \neq 0$, $N^{k+1} = 0$. Then*

$$\lim_{n\to\infty} \frac{(1+N)^n}{|(1+N)^n|} = \frac{N^k}{|N^k|}.$$

Proof. The equation $(1+N)^n = \sum_{j=0}^{k} \binom{n}{j} N^j$ implies that $\lim_{n\to\infty} (1+N)^n \Big/ \binom{n}{k} = N^k$. Also,

$$\binom{n}{k}|N^k| - \sum_{j=0}^{k-1}\binom{n}{j}|N^j| \leqslant |(1+N)^n| \leqslant \binom{n}{k}|N^k| + \sum_{j=0}^{k-1}\binom{n}{j}|N^j|.$$

This yields

$$\lim_{n\to\infty} \frac{|(1+N)^n|}{\binom{n}{k}} = |N^k|.$$

LEMMA 2. *Let $\mathscr{S}$ be a semigroup in $\mathscr{C}_1$ and B a nonzero operator. If* $\operatorname{tr}(AB) = 0$ *for all A in $\mathscr{S}$, then $\mathscr{S}$ has a nontrivial invariant subspace.*

Proof. By linearity and continuity, the equation $\operatorname{tr}(AB) = 0$ holds for every A belonging to the closed linear span $\mathfrak{A}$ of $\mathscr{S}$ in $\mathscr{C}_1$. If $\mathscr{S}$ had only trivial invariant subspaces, then so would $\mathfrak{A}$, and since $\mathfrak{A}$ is a closed subalgebra of $\mathscr{C}_1$, it would follow that $\mathfrak{A} = \mathscr{C}_1$. This is a consequence of Lomonosov's result [6], given in [9]. Thus $\operatorname{tr}(AB)$ would vanish for all A in $\mathscr{C}_1$, contradicting the hypothesis $B \neq 0$.

LEMMA 3. *Let $\mathscr{S}$ be any semigroup of operators on $\mathscr{H}$, and let $\mathscr{S}_1$ be a nonzero ideal in $\mathscr{S}$, i.e. $\mathscr{S}_1 \neq \{0\}$, $\mathscr{S}\mathscr{S}_1 \subseteq \mathscr{S}_1$, $\mathscr{S}_1\mathscr{S} \subseteq \mathscr{S}_1$. If $\mathscr{S}_1$ has a nontrivial invariant subspace, so does $\mathscr{S}$.*

Proof. The linear span, $\bigvee \mathscr{S}$, of $\mathscr{S}$ is an algebra on $\mathscr{H}$, and the condition on $\mathscr{S}_1$ implies that $\bigvee \mathscr{S}_1$ is a nonzero (two-sided) ideal in this algebra. The lemma then follows from the well-known fact that if $\mathscr{J}$ is an nonzero ideal of the algebra $\mathfrak{A}$ of operators on $\mathscr{H}$, and if $\mathscr{J}$ has a nontrivial invariant subspace, so does $\mathfrak{A}$. (Short proof:

let $\mathcal{M}$ be such an invariant subspace for $\mathcal{I}$, and let $\mathcal{N}$ be the intersection of the null spaces of all the members of $\mathcal{I}$. Then $\overline{\bigvee \mathcal{I}\mathcal{M}}$ and $\mathcal{N}$ are both invariant under $\mathfrak{A}$, and at least one of them has to be nontrivial.)

THEOREM 1. *Let $\mathcal{S}$ be a semigroup of trace-class operators. For each A in $\mathcal{S}$ assume $1 \in \sigma(A)$ and $|\lambda| < 1$ for $1 \neq \lambda \in \sigma(A)$. Then $\mathcal{S}$ has a nontrivial invariant subspace.*

Proof. For each A in $\mathcal{S}$, let $r(A)$ denote the algebraic multiplicity of the eigenvalue 1 of A. We distinguish two cases.

1) Assume for each A in $\mathcal{S}$, $r(A)$ equals the nullity of $A-I$. Pick A_0 with minimal r, $r_0 = r(A_0)$. We can clearly assume that r_0 is less than the dimension of the space. Letting P be the Riesz projection [8, p. 31] of A_0 corresponding to the eigenvalue 1, we write $A_0 = P + C$, where C has spectral radius less than 1. We can assume, by transforming every member of $\mathcal{S}$ by a similarity, that P is self-adjoint and that $PC = CP = 0$. Also, using Rota's Theorem [3, p. 81], we assume $\|C\| < 1$. Now

$$\lim_{n\to\infty} |A_0^n - P| = \lim_{n\to\infty} |C^n| \leqslant \lim_{n\to\infty} \|C\|^{n-1}\cdot|C| = 0,$$

and for arbitrary A in $\mathcal{S}$,

$$\lim_{n\to\infty} |A_0^n A A_0^n - PAP| = 0.$$

Thus the finite-rank operator PAP is the limit of operators each of which has an eigenspace corresponding to 1 of dimension at least r_0; it follows that PAP has this property. Since the rank of PAP is at most r_0, we must have $PAP = P$. The proof can be finished here if $r_0 \geqslant 2$. (Just pick unit vectors x and y in the range of P with $x \perp y$ and observe that $PAP = P$ implies $Ax \perp y$ for all A in $\mathcal{S}$. Thus the closed span of $\mathcal{S}x$ is a proper invariant subspace.) However, since $r_0 = 1$ is a possibility, we proceed as follows.

If $P\mathcal{H}$ is invariant for $\mathcal{S}$, we are done; otherwise there is B in $\mathcal{S}$ with $(1 - P)BP \neq 0$. It follows from the above paragraph that for every A in $\mathcal{S}$,

$$P = PABP = (PAP)(PBP) + PA(1 - P)BP =$$

$$= P + PA(1 - P)BP,$$

or $PA(1 - P)BP = 0$. Thus

$$\operatorname{tr}(A(1 - P)BP) = \operatorname{tr}(PA(1 - P)BP) = 0$$

for all A in $\mathcal{S}$. Since $(1 - P)BP \neq 0$, $\mathcal{S}$ has a nontrivial invariant subspace by Lemma 2.

2) We can now assume the existence of an A_0 in $\mathscr{S}$ with $r_0 = r(A_0)$ greater than the nullity of $A_0 - I$. We argue as in (1) to assume a decomposition

$$A_0 = (P + N) + C,$$

where C and P are as before, $N^k \neq 0$, $N^{k+1} = 0$ for some k, and $PN = NP = N$. Then for arbitrary A in $\mathscr{S}$

$$A_0^n A = (P + N)^n A + C^n A,$$

so that $|A_0^n A - (P + N)^n A| = |C^n A| \leqslant \|C^n\| \cdot |A|$, and thus

$$(*) \qquad \lim_{n\to\infty} |A_0^n A - (P + N)^n A| = 0.$$

The remainder of the proof consists in showing that $\mathrm{tr}(N^k A) = 0$ for all A in $\mathscr{S}$. Lemma 2 would then imply the existence of a nontrivial invariant subspace.

Pick A in $\mathscr{S}$. If $\mathrm{tr}(A_0^n A)$ is bounded, then so is

$$\mathrm{tr}((P + N)^n A) = \mathrm{tr}(PA) + \sum_{j=1}^{k} \binom{n}{j} \mathrm{tr}(N^j A),$$

because $\lim_{n\to\infty} |\mathrm{tr}(A_0^n A) - \mathrm{tr}((P + N)^n A)| = 0$. Thus the dominant term on the right side is zero: $\mathrm{tr}(N^k A) = 0$.

We can now assume $\mathrm{tr}(A_0^n A)$ is unbounded; then there is a sequence $n(i)$ of intergers such that $|A_0^{n(i)} A| \to \infty$ as $i \to \infty$. It follows that

$$\lim_{i\to\infty} |(P + N)^{n(i)} A| = \infty.$$

Since A is fixed and P has finite rank r_0, by passing to a subsequence we can assume

$$\lim_{i\to\infty} \frac{(P + N)^{n(i)} A}{|(P + N)^{n(i)} A|} = F$$

in the $\mathscr{C}_1$ norm, where $F = PF$, $|F| = 1$. It follows from $(*)$ that

$$\lim_{i\to\infty} \frac{A_0^{n(i)} A}{|A_0^{n(i)} A|} = F$$

also. Since the spectral radius of $A_0^{n(i)} A$ is always 1, and since $\lim_{i\to\infty} |A_0^{n(i)} A| = \infty$, the operator F has to be nilpotent: $F^{r_0} = 0$.

Now the inequality

$$\left| \left(\frac{(P+N)^{n(i)}A}{|(P+N)^{n(i)}|\cdot|A|} \right)^{r_0} \right| \leqslant \left| \left(\frac{(P+N)^{n(i)}A}{|(P+N)^{n(i)}A|} \right)^{r_0} \right|$$

implies

$$\lim_{i\to\infty} \left(\frac{(P+N)^{n(i)}A}{|(P+N)^{n(i)}|} \right)^{r_0} = 0.$$

Since

$$\lim_{i\to\infty} \frac{(P+N)^{n(i)}A}{|(P+N)^{n(i)}|} = \frac{N^k A}{|N^k|}$$

by Lemma 1, we conclude that $(N^k A)^{r_0} = 0$, i.e., $N^k A$ is nilpotent. Thus $\operatorname{tr}(N^k A) = 0$ for every A in $\mathscr{S}$.

COROLLARY 1. *Let $\mathscr{S}$ be a semigroup in $\mathscr{C}_1$. If $\sigma(A) = \{0,1\}$ for every A in $\mathscr{S}$, then $\mathscr{S}$ has a nontrivial invariant subspace.*

Note that the condition $\sigma(A) = \{0,1\}$ can be weakened to $1 \in \sigma(A) \subseteq \{0,1\}$, which would amount to the same thing in infinite dimensions. In the finite-dimensional case, however, this allows $\sigma(A)$ to be $\{1\}$ for some members of $\mathscr{S}$.

COROLLARY 2. *Let $\mathscr{S}$ be a semigroup of operators on a finite-dimensional space. If $1 \in \sigma(A) \subseteq \{0,1\}$ for every A in $\mathscr{S}$, then $\mathscr{S}$ has a nontrivial invariant subspace.*

This corollary weakens both the hypothesis and the conclusion of the result in [4]. Simpe 3×3 examples of $\mathscr{S}$ can be constructed which satisfy the above condition (and hence have invariant subspaces), but are not triangularizable. Also, as shown in [4], the above condition cannot be weakened to $\sigma(A) \subseteq \{0,1\}$ in general. Here is a case in which it can.

THEOREM 2. *A semigroup $\mathscr{S}$ of compact idempotents has a nontrivial invariant subspace.*

Proof. We can assume $\mathscr{S} \neq \{0\}$. If $1 \in \sigma(A)$ for all A in $\mathscr{S}$, the assertion follows from Theorem 1. Thus assume $0 \in \mathscr{S}$. Again, if the nonzero members of $\mathscr{S}$ form a semigroup, we are done by Corollary 1. So assume the existence of a nonzero left divisor A_0 of zero. Note that for A and B in $\mathscr{S}$ we have $AB = 0$ if and only if $BA = 0$, because AB and BA are idempotents with equal trace.

Consider $\mathscr{S}_1 = \{B \in \mathscr{S} : A_0 B = 0\}$. Then $A\mathscr{S}_1 \subseteq \mathscr{S}_1$ and $\mathscr{S}_1 A \subseteq \mathscr{S}_1$ for all A in $\mathscr{S}$, and $\mathscr{S}_1$ is a nonzero semigroup, which has a nontrivial invariant subspace by Lemma 2. So does $\mathscr{S}$, by Lemma 3.

This result can be strengthened. (See also Theorem 6 below.)

THEOREM 3. *Let $\mathscr{S}$ be a semigroup of idempotents. If $\mathscr{S}$ contains a nonzero compact operator, then it has a nontrivial invariant subspace.*

Proof. Let $\mathscr{S}_1$ be the semigroup of compact operators in $\mathscr{S}$. Then $\mathscr{S}_1$ has a nontrivial invariant subspace by Theorem 2, and $\mathscr{S}_1 \neq \{0\}$ by hypothesis. The assertion then follows from Lemma 3.

3. TRIANGULARIZABILITY

We start with the generalization of Kaplansky's result [4] in the case where the spectrum of every operator in the semigroup is contained in $\{0,1\}$.

THEOREM 4. *Let $\mathscr{S}$ be a semigroup in $\mathscr{C}_1$ and r a fixed nonnegative integer. Assume that, for every A in $\mathscr{S}$, $\sigma(A) \subseteq \{0,1\}$ and the algebraic multiplicity of 1 in $\sigma(A)$ is r. Then $\mathscr{S}$ is triangularizable.*

The proof is an adaptation of Kaplansky's method to the infinite-dimensional case, but, as may be expected, it is complicated by topological considerations. We start with a lemma.

LEMMA 4. *Let $\mathscr{S}$ and r be as in the statement of Theorem 4. Let $\mathscr{M}$ be an invariant subspace of $\mathscr{S}$, and $\mathscr{S}_1$ the restriction of $\mathscr{S}$ to $\mathscr{M}$:*

$$\mathscr{S}_1 = \{A|\mathscr{M} : A \in \mathscr{S}\}.$$

There exists an integer $r_1 \leqslant r$ such that, for every B in $\mathscr{S}_1$, the algebraic multiplicity of 1 in $\sigma(B)$ is r_1 (and $\sigma(B) \subseteq \{0,1\}$).

Proof. The assertion is clearly true if $r = 0$. Thus assume $r \geqslant 1$. Let E denote the orthogonal projection on $\mathscr{M}$. Let A and B be arbitrary members of $\mathscr{S}$, and denote by s and t the multiplicities of 1 in $\sigma(EAE)$ and in $\sigma(EBE)$ respectively. We must show that $s = t$. Assume, with no loss of generality, that $s \leqslant t$. Denote by p_n the multiplicity of 1 in $\sigma(EA^nB^nE)$. We first prove that $p_n \leqslant s$ for sufficiently large n.

Let $A_1 = EAE$, $B_1 = EBE$, and let P be the Riesz projection of A_1 corresponding to the eigenvalue 1. Denoting PA_1P by F, we write $A_1 = F + Q$, where F has rank s, Q is quasinilpotent, and $FQ = QF = 0$. Let n be so large that

$$\|Q^n\| \cdot \|B_1\|^n < 1 \quad \text{and} \quad \|Q^n\|(2\|B_1\| \cdot \|F\|)^{rn} < 1.$$

If, for any n, $p_n = 0$, there is nothing to prove. Otherwise, there is an orthogonal projection P_n of rank p_n whose range is invariant under $A_1^nB_1^n = EA^nB^nE$ and such that the restriction of $A_1^nB_1^n$ to $P_n\mathscr{H}$ has determinant 1. We restrict both sides of the equation

$$P_nA_1^nB_1^nP_n = P_nF^nB_1^nP_n + P_nQ^nB_1^nP_n$$

to $P_n\mathscr{H}$ and rewrite it, for computational purposes, as

$$T = S + R,$$

viewing the operators as $p_n \times p_n$ matrices. Note that the rank of S cannot exceed s, because $F^n B_1^n$ has rank $\leqslant s$.

Now assume, contrary to our claim, that $s < p_n$. Applying a unitary transformation, we can assume that S has at least one zero column. Let S_i and R_i denote the columns of S and R respectively, where $S_1 = 0$. By Hadamard's inequality [2, p. 202]

$$1 = \det T = |\det T| \leqslant \prod_{i=1}^{p_n} \|S_i + R_i\| \leqslant$$

$$\leqslant \|R_1\| \cdot \prod_{i=2}^{p_n} (\|S_i\| + \|R_i\|),$$

and, since $\|S_i\| \leqslant \|F^n B_1^n\|$ and $\|R_i\| \leqslant \|Q^n B_1^n\|$ for every i,

$$1 \leqslant \|Q^n\| \cdot \|B_1\|^{np_n} \cdot (\|F\|^n + \|Q^n\|)^{p_n - 1}.$$

If $\|F\| = 0$, this implies $1 \leqslant \|Q^n\| \cdot \|B_1\|^n$; if $\|F\| \neq 0$, then

$$1 \leqslant \|Q^n\| \cdot \|B_1\|^{nr} \cdot (2\|F\|^n)^r \leqslant$$

$$\leqslant \|Q^n\| \cdot (2\|B_1\| \cdot \|F\|)^{rn}.$$

This contradicts the assumption on n. Thus we have shown $p_n \leqslant s$ for sufficiently large n.

Next letting $A_2 = (1 - E)A(1 - E)$ and $B_2 = (1 - E)B(1 - E)$, we observe that the multiplicity of 1 is $r - s$ in $\sigma(A_2)$, $r - t$ in $\sigma(B_2)$, and $r - p_n$ in $\sigma(A_2^n B_2^n)$ for every n. Now $r - t \leqslant r - s$, and an argument similar to the one given above proves $r - p_n \leqslant r - t$ for large enough n. Hence

$$r = p_n + (r - p_n) \leqslant s + (r - t),$$

or $t \leqslant s$. It follows that $s = t$, as asserted.

Proof of Theorem 4. The existence of a nontrivial invariant subspace for $\mathscr{S}$ is proved by Corollary 1 if $r \geqslant 1$ and by [5] if $r = 0$. By Zorn's lemma there is a maximal chain $\mathscr{C}$ of invariant subspaces for $\mathscr{S}$, and we claim that it is a maximal subspace chain, i.e., for every $\mathscr{M}$ in $\mathscr{C}$ with $\mathscr{M} \neq \mathscr{M}_-$, the subspace $\mathscr{M} \ominus \mathscr{M}_-$ has dimension 1. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be the restrictions of $\mathscr{S}$ to $\mathscr{M}$ and to $\mathscr{M}_-$ respectively.

Let r_1 and r_2 be the respective constant multiplicities corresponding to $\mathcal{S}_1$ and $\mathcal{S}_2$ given by Lemma 4. Let P denote the orthogonal projection on $\mathcal{M} \ominus \mathcal{M}_-$. Then, considering $\mathcal{S}_2$ as a restriction of $\mathcal{S}_1$, it is easily seen that $r_2 \leqslant r_1$, and that $P\mathcal{S}P = \{PAP : A \in \mathcal{S}\}$ is a semigroup on $\mathcal{M} \ominus \mathcal{M}_-$ with the corresponding multiplicity $r_2 - r_1$. This would imply the existence of a nontrivial invariant subspace $\mathcal{N}$ for $P\mathcal{S}P$ if $\mathcal{M} \ominus \mathcal{M}_-$ had dimension greater than 1, giving an invariant subspace $\mathcal{M}_- \oplus \mathcal{N}$ for $\mathcal{S}$, strictly between $\mathcal{M}_-$ and $\mathcal{M}$. This contradicts the assumption on $\mathcal{C}$.

THEOREM 5. *Every semigroup in $\mathcal{C}_1$ with constant trace is triangularizable.*

Proof. As in [4], we reduce this to Theorem 4 by showing that if A is any operator in $\mathcal{C}_1$ and if $\operatorname{tr}(A^n) = c$ for all n, then $\sigma(A) \subseteq \{0,1\}$ and c is the multiplicity of 1 in $\sigma(A)$. Thus let $\{a_i\}$ be an enumeration of the eigenvalues of A (repeated according to multiplicty) such that $|a_1| \geqslant |a_2| \geqslant \dots$. We must verify that $\sum_{i=1}^{\infty} a_1^n = c$ for all n implies that c is a nonnegative integer, $a_1 = \dots = a_c = 1$ and $a_i = 0$ for $i \geqslant c+1$. Elementary calculus and algebra can be used to prove this, as follows.

(i) First show that if $\{z_1, \dots, z_k\}$ is any set of complex numbers of modulus 1, and if $\lim_{n\to\infty} (z_1^n + \dots + z_k^n) = b$, then $b = k$. To see this just take a subsequence n_j of integers so that the sequence $\{z_i^{n_j}\}$ is convergent, say to b_i, for $i = 1, \dots, k$. Since, for every m, $z_i^{mn_j}$ converges to b_i^m, we have $b_1^m + \dots + b_k^m = b$ for all m. As in [4], this implies $b_1 = \dots = b_k = 1$, $b = k$.

(ii) Next we do the special case $c = 0$: we must show that if $\sum_{i=1}^{\infty} a_i^n = 0$ for all n, then $a_i = 0$ for every i. Assume, on the contrary, that $a_1 \neq 0$. Dividing each a_i by a_1 we can assume $1 = |a_1| = \dots = |a_k| > |a_{k+1}| \geqslant \dots$. Thus $\sum |a_i| < \infty$ implies that $\lim_n \sum_{i=k+1}^{\infty} a_i^n = 0$. It follows that $\lim_n (a_1^n + \dots + a_k^n) = 0$, which is a contradiction by (i).

(iii) Finally, assume $c \neq 0$, so that $|a_1| \geqslant 1$. Let $p = |a_1| = \dots = |a_k| > |a_{k+1}| \geqslant \geqslant \dots$. Then

$$\lim_n \left(\frac{c}{p^n} - \sum_{i=1}^{k} \left(\frac{a_i}{p} \right)^n \right) = 0.$$

Now if p exceeded 1, this equation would give $k = 0$ by (i), which is a contradiction. Thus $p = 1$ and $\lim_n \sum_{i=1}^{k} a_i^n = c$. It follows from (i) that $c = k$. We must show that $a_1 = \dots = a_k = 1$. Assume, on the contrary, that $a_r \neq 1$ for some $r \leqslant k$. Pick a subsequence $\{n(j)\}$ of integers such that $\lim_j a_r^{n(j)} = b \neq 1$. Then

$$\lim_j \sum_{\substack{1 \leqslant i \leqslant k \\ i \neq r}} a_i^{n(j)} = k - b,$$

which is a contradiction, because the left hand side has modulus $\leqslant k - 1$, and $k - 1 < |k - b|$.

THEOREM 6. *A semigroup $\mathscr{S}$ of compact idempotents is triangularizable.*

Proof. Use Theorem 2 and Zorn's lemma to get a maximal chain of invariant subspaces. If $\mathscr{M}$ and $\mathscr{M}_-$ are subspaces as in the proof of Theorem 4, and P the orthogonal projection on $\mathscr{M} \ominus \mathscr{M}_-$, then $P\mathscr{S}P$ is a semigroup of compact idempotents. Theorem 2 would apply again if P had rank greater than 1. This completes the proof.

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REFERENCES

1. DUNFORD, N.; SCHWARTZ, J. T., *Linear operators, Part II: Spectral theory*, Interscience, New York, 1963.
2. GREUB, W., *Linear algebra*, Springer-Verlag, New York, 1975.
3. HALMOS, P. R., *A Hilbert space problem book*, Springer-Verlag, New York, 1982.
4. KAPLANSKY, I., The Engel-Kolchin theorem revisited, in *Contributions to algebra*, edited by Bass, Cassidy, and Kovacik, Academic Press, New York, 1977.
5. LAURIE, C; NORDGREN, E.; RADJAVI, H.; ROSENTHAL, P., On triangularization of algebras of operators, *J. Reine Angew. Math.*, **327**(1981), 143–155.
6. LOMONOSOV, V. I., Invariant subspaces for operators commuting with compact operators, *Functional Anal. Appl.*, 7(1973), 213–214.
7. NORDGREN, E.; RADJAVI, H.; ROSENTHAL, P., Triangularizing semigroups of compact operators, *Indiana Univ. Math. J.*, to appear.
8. RADJAVI, H.; ROSENTHAL, P., *Invariant subspaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
9. RADJAVI, H.; ROSENTHAL, P., On transitive and reductive operator algebras, *Math Ann.*, **209**(1974), 43–56.

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