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GENERALIZED DIRAC-OPERATORS WITH SEVERAL SINGULARITIES

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1. INTRODUCTION AND NOTATIONS

Jörgens generalizes in [6] the concept of Dirac-operators. But there he only allows singularities at infinity. For physical reasons — e.g. an electron possesses an anomalous magnetic moment [2; 3; 4; 8] — we have also to allow singularities localized at compact sets of measure zero [8, § 9]. In generalizing this concept one can also study multicenter operators of Jörgens-type (see [9; 12] for a physical nterpretation). Using a general decomposition principle we show in the following that the multicenter case can be reduced to the case of a single center.

In the second and the third section of our paper we determine a formula for the deficiency indices and the essential spectral kernel. Further, in all cases important for physical applications we also construct selfadjoint extensions and determine their essential spectrum. The techniques used are similar to those of Behncke [4]. Especially we generalize Theorem 1 in [4] in many ways.

Finally, we apply the results of the previous sections to Dirac-operators with several Coulomb-singularities. In particular, we generalize the results of Landgren and Rejto [12], Arai [1] and Klaus [9].

At the end we also discuss the problem of determining a physically distinguished selfadjoint extension and generalize some results of Klaus [9] and Nenciu [13]. The notation will be mostly standard. Especially $\overline{T}$ denotes the closure of the operator T, while $\mathscr{D}(T)$, $\mathscr{N}(T)$ and $\mathscr{R}(T)$ denote its domain, kernel and range respectively. For a closed symmetric operator T we denote the deficiency indices, the spectral kernel and the essential spectral kernel by def. T, $S(T)$ and $S_e(T)$ respectively (for a definition of these notions see [19, Chapter 8]). The spectrum and the essential spectrum of a selfadjoint operator T are denoted by $\sigma(T)$ and $\sigma_e(T)$ respectively.

In the following, all operators are defined in the Hilbert space

$$\mathfrak{H} = \mathscr{L}^2(\mathbf{R}^m) \otimes \mathbf{C}^p \quad (m, p \in \mathbf{N});$$

the scalar product and the norm in $\mathfrak{H}$ are given by

$$\langle u, v\rangle = \int \sum_{j=1}^{p} \bar{u}_j v_j \, dx \quad \text{and} \quad \|u\|^2 = \langle u, u\rangle.$$

Finally, we write $\mathscr{D}_0(\Omega)$ for $\mathscr{C}_0^\infty(\Omega) \otimes \mathbf{C}^p$ and $\mathscr{D}_0$ for $\mathscr{D}_0(\mathbf{R}^m)$ where $\Omega \subset \mathbf{R}^m$ is a non-void open set.

2. DEFICIENCY INDICES OF GENERALIZED DIRAC-OPERATORS

Let N_i $(i = 1, \ldots, n)$ be non-void disjoint compact subsets of $\mathbf{R}^m$ with measure zero. Then there exists a positive number δ such that $N_j \subset B_j(\delta) := \{y : d(y, N_j) < \delta\}$ and $B_i(\delta) \cap B_j(\delta) = \emptyset$ $(i \neq j)$ where $d(x, M)$ denotes the distance between the point x and the set M. In a manner analogous to Jörgens [6] let T_0 be an essentially seldfadjoint operator defined on $\mathscr{D}_0(N_0^c)$ with

$$N_0 := \bigcup_{i=1}^{n} N_i$$

and its closure T which satisfies the following conditions:

(1) For every $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$ the operator $A_0(\varphi)$ defined by $A_0(\varphi)u = T_0(\varphi u) - \varphi T_0 u$ for $u \in \mathscr{D}_0(N_0^c)$ is bounded; let $A(\varphi)$ be the uniquely determined bounded extension of $A_0(\varphi)$ to all of $\mathfrak{H}$.

(2) $\limsup_{n\to\infty} \|A_0(\psi_n)\| < 1$; the sequence (ψ_n) is defined by $\psi_n(x) := \varphi(x/n)$ where φ is a $\mathscr{C}_0^\infty(\mathbf{R}^m)$-function such that $\varphi(x) = 1$ for $|x| \leqslant 1$, $\varphi(x) = 0$ for $|x| \geqslant 2$ and $0 \leqslant \varphi \leqslant 1$.

(3) $$\mathscr{D}_0(N_j^c) \subset \mathscr{D}(T) \qquad \text{for all } j = 1, \ldots, n.$$

Further let V_i $(i = 1, \ldots, n)$ be a formally symmetric $p \times p$ matrix potential with $|V_i| \in \mathscr{L}^2_{\text{loc}}(N_i^c)$ having one of the following two propeıties ($|A|$ denotes the operator norm of the matrix A).

(4) For every non-void open bounded set $\Omega \subset \bar{\Omega} \subset N_i^c$ there are non-negative constants a_i and b_i with $b_0 := \sum_{i=1}^{n} b_i < 1/2$ such that

$$\|V_i u\| \leqslant b_i \|Tu\| + a_i \|u\| \quad \text{for all } u \in \mathscr{D}_0(\Omega)\ (i = 1, \ldots, n)$$

or

(5) $\mathscr{D}(T) \subset \mathscr{D}(V_i)$ and $|(1 - \chi_i)V_i| \in \mathscr{L}^\infty_{\text{loc}}(\mathbf{R}^m)$ $(i = 1, \ldots, n)$ with $\chi_i := \chi_{B_i(\delta/2)}$.

Frequently, we replace condition (5) by the stronger condition

(5′) $\mathscr{D}(T) \subset \mathscr{D}(V_i)$ and $|(1 - \chi_i)V_i| \in \mathscr{L}^\infty(\mathbf{R}^m)$ $(i = 1, \ldots, n)$.

Finally, let $\mathscr{H}_j$ $(j = 1, \ldots, n)$ be defined by

$$\mathscr{H}_j := T + V_j \quad \text{with } \mathscr{D}(\mathscr{H}_j) := \mathscr{D}_0(N_j^c).$$

The conditions (1), (3), (4) or (5) imply that $\mathscr{D}_0(N_0^c)$ is a core of $\overline{\mathscr{H}_j}$ $(j = 1, \ldots, n)$. The operator $\mathscr{H}_0$ is defined by

$$\mathscr{H}_0 := T_0 + V_0 \quad \text{with } \mathscr{D}(\mathscr{H}_0) = \mathscr{D}_0(N_0^c) \text{ and } V_0 := \sum_{j=1}^n V_j.$$

Operators of this type possess well localized properties. Especially, condition (1) implies for all $u \in \mathscr{D}(\mathscr{H}_j^*)$, $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$:

(6) $\varphi u \in \mathscr{D}(\mathscr{H}_j^*)$ and $\mathscr{H}_j^*(\varphi u) = \varphi \mathscr{H}_j^* u + A(\varphi)u$ $(j = 0, 1, \ldots, n)$.

The conclusion of (4) remains valid if one replaces $\mathscr{H}_j^*$ by $\overline{\mathscr{H}}_j$.

In all applications (see Section 4) T_0 will be the Dirac-operator for a free electron. This means $m = 3$, $p = 4$ and

$$T_0 = \alpha \cdot (-\mathrm{i} \nabla) + \beta,$$

where α_j $(j = 1,2,3)$ and $\alpha_0 := \beta$ are 4 by 4 Hermitean matrices satisfying the following anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \quad (j, k = 0, 1, 2, 3).$$

Especially, T_0 is essentially selfadjoint with $\mathscr{D}(T) = W^1(\mathbf{R}^3) \otimes \mathbf{C}^4$ ($W^1(\mathbf{R}^3)$ denotes the Sobolev space of order 1) and satisfies condition (3) if N_j $(j = 1, \ldots, n)$ is a point or a compact part of a line (see [16] for more general results). T_0 also satisfies the conditions (1) and (2) with $A(\varphi)u = \alpha \cdot (-\mathrm{i} \nabla \varphi)u$ for all $u \in \mathfrak{H}$ and $\varphi \in \mathbf{C}_0^\infty(\mathbf{R}^3)$. Further let V_j $(j = 1, \ldots, n)$ be a 4 by 4 matrix potential with $|V_j| \in \mathscr{L}^3_{\mathrm{loc}}(N_j^c)$ then the potentials V_j $(j = 1, \ldots, n)$ satisfy condition (4) for all $b_j > 0$ with a suitable $a_j > 0$. Clearly, we can allow more general potentials, e.g. Stummel-potentials (see [6] and [8, § 9]).

Finally, choosing $N_j = \{a_j\}$ with $a_j \neq a_k$ $(j \neq k)$, the Coulomb-potentials V_j localized in a_j satisfy the conditions (5) and (5′) (see Section 4).

To continue our abstract considerations we need the following perturbation lemma:

LEMMA 1. *Let T_0 and V_j $(j = 1, \ldots, n)$ satisfy the above assumptions then*

$$\operatorname{def.}\overline{\mathscr{H}_j} = \operatorname{def.}\overline{T_0 + \chi_j V_j} \quad (j = 0, 1, \ldots, n)$$

with $\chi_0 := \sum_{j=1}^{n} \chi_j$.

Proof. (i) First, we assume that the potentials V_j $(j = 1, \ldots, n)$ satisfy condition (4). For a fixed $\delta' \in (\delta/2, \delta)$ we decompose V_j $(j = 0, 1, \ldots, n)$ by $V_j = V_{j,1} + V_{j,2} + V_{j,3}$ with $V_{j,1} := \chi_j V_j$ and $V_{j,2} := \chi_{C_j(\delta)} V_j$ with $C_j(\delta) := \{y : \delta/2 \leqslant d(y, N_j) \leqslant \delta'\}$. Then there exist $\mathscr{C}_0^\infty(N_j^c)$-functions ψ_j and α_j with $\psi_j V_{j,2} = \psi_j$, $\alpha_j \psi_j = \psi_j$ and $0 \leqslant \alpha_j, \psi_j \leqslant 1$ and bounded open sets Ω_j with $\operatorname{supp}\psi_j$, $\operatorname{supp}\alpha_j \subset \Omega_j \subset \overline{\Omega}_j \subset B_j(\delta) \cap N_j^c$.

Further, we define the operators $\mathscr{H}_j'$ and $\mathscr{H}_j''$ by

$$\mathscr{H}_j' := T + \alpha_j(V_{j,1} + V_{j,3}) \text{ and } \mathscr{H}_j'' = T + V_{j,1} + V_{j,3} \text{ with } \mathscr{D}(\mathscr{H}_j') = \mathscr{D}(\mathscr{H}_j'') = \mathscr{D}(\mathscr{H}_j).$$

Because of $\mathscr{H}_j'(\psi_j u) = \alpha_j \mathscr{H}_j''(\psi_j u) + (1 - \alpha_j) A(\psi_j) u$ for all $u \in \mathscr{D}(\mathscr{H}_j)$ we get with suitable nonnegative constants a_j' and a_j'' for all $u \in \mathscr{D}(\mathscr{H}_j)$:

$$\|V_{j,2} u\| = \|V_{j,2} \psi_j u\| \leqslant b_j \|T(\psi_j u)\| + a_j \|u\| \leqslant$$

$$\leqslant b_j(1 - b_j)^{-1} \|\mathscr{H}_j'(\psi_j u)\| + a_j' \|u\| \leqslant b_j(1 - b_j)^{-1} \|\mathscr{H}_j'' u\| + a_j'' \|u\|.$$

Therefore, without loss of generality we may assume $V_{j,2} = 0$ (see Corollary 2 in [5]).

(ii) Arguing as in the proof of Theorem 4 in [2] or Satz 3 in [8, § 9] we can finish the proof.

REMARK. Without difficulty one can show that the stronger condition $\limsup_{n\to\infty} \|A_0(\psi_n)\| < 1/2$, which we assumed in Satz 3 in [8, § 9], can be replaced by Condition (2). Condition (2) corresponds to Condition (5.4) in [6] and can be omitted if the potentials V_j satisfy Condition (5').

In the following we frequently use functions φ_j, χ_j and α_j $(j = 1, \ldots, n)$ defined by the properties:

(i) $\varphi_j, \chi_j \in \mathscr{C}_0^\infty(B_j(\delta))$, $0 \leqslant \varphi_j \leqslant 1$, $\varphi_j(x) = 1$ for $x \in B_j(3\delta/4)$ and $\varphi_j \chi_j = \varphi_j$.

(ii) $\alpha_j \in \mathscr{C}_0^\infty(B_j(3\delta/4))$, $0 \leqslant \alpha_j \leqslant 1$ and $\alpha_j(x) = 1$ for $x \in B_j(5\delta/8)$.

Further we define α_0 and φ_0 by $\alpha_0 := \sum_{j=1}^{n} \alpha_j$ and $\varphi_0 := \sum_{j=1}^{n} \varphi_j$.

THEOREM 1. (a) *Under the above assumptions we have:*

$$\operatorname{def}.\bar{\mathscr{H}}_0 = \sum_{j=1}^{n} \operatorname{def}.\bar{\mathscr{H}}_j .$$

(b) *If the potentials V_j $(j = 1, \ldots, n)$ satisfy condition* (5′) *then we have:*

$$\mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_0) = \operatorname{span}\{\mathscr{D}(\bar{\mathscr{H}}_j) : j = 1, \ldots, n\}$$

and

$$\mathscr{D}(\mathscr{H}_0^*) = \operatorname{span}\{\mathscr{D}(\mathscr{H}_j^*) : j = 1, \ldots, n\}.$$

Proof. Because of Lemma 1 we may assume $V_j = \chi_j V_j$ $(j = 1, \ldots, n)$. Using induction we may also assume $n = 2$.

(a) We denote the deficiency indices of $\bar{\mathscr{H}}_j$ $(j = 0, 1, \ldots, n)$ by $\operatorname{def}.\bar{\mathscr{H}}_j = := (p_j, q_j) = (\dim \mathscr{N}(\mathscr{H}_j^* - \mathrm{i}),\ \dim \mathscr{N}(\mathscr{H}_j^* + \mathrm{i}))$.

STATEMENT 1. *Let $(w_{j,k})$ $(j = 1, 2;\ k = 1, \ldots, p_j + q_j)$ be a maximal set of vectors in $\mathscr{D}(\mathscr{H}_j^*)$ which are independent modulo $\mathscr{D}(\bar{\mathscr{H}}_j)$. Then all vectors $\tilde{w}_{j,k} := := \varphi_j w_{j,k}$ are in $\mathscr{D}(\mathscr{H}_0^*)$, independent modulo $\mathscr{D}(\bar{\mathscr{H}}_0)$ and we have $\mathscr{H}_j^* \tilde{w}_{j,k} = \mathscr{H}_0^* \tilde{w}_{j,k}$.*

Proof. Using Condition (1) and Property (6) we get for all $u \in \mathscr{D}(\mathscr{H}_0)$:

$$\langle \tilde{w}_{j,k}, \mathscr{H}_0 u \rangle = \langle w_{j,k}, \mathscr{H}_0(\varphi_j u) \rangle - \langle w_{j,k}, A_0(\varphi_j) u \rangle =$$

$$= \langle w_{j,k}, \mathscr{H}_j(\varphi_j u) \rangle + \langle A(\varphi_j) w_{j,k}, u \rangle = \langle \mathscr{H}_j^* w_{j,k}, u \rangle .$$

This means $w_{j,k} \in \mathscr{D}(\mathscr{H}_0^*)$ and $\mathscr{H}_j^* \tilde{w}_{j,k} = \mathscr{H}_0^* \tilde{w}_{j,k}$. Now let $n_j \leqslant p_j + q_j$ $(j = 1, 2)$ be arbitrary but finite and let $\lambda_{j,k}$ $(j = 1, 2;\ k = 1, \ldots, n_j)$ be complex numbers such that

$$\sum_{j=1}^{2} \tilde{f}_j \in \mathscr{D}(\bar{\mathscr{H}}_0) \qquad \text{with } \tilde{f}_j := \sum_{k=1}^{n_j} \lambda_{j,k} \tilde{w}_{j,k} .$$

Multiplication with χ_l $(l = 1, 2)$ yields: $\tilde{f}_j \in \mathscr{D}(\bar{\mathscr{H}}_0) \cap \mathscr{D}(\bar{\mathscr{H}}_j)$ $(j = 1, 2)$. If $f_j := := \sum_{k=1}^{n_j} \lambda_{j,k} w_{j,k}$ then we get for all $u \in \mathscr{D}_0(N_0^c)$

$$\langle \mathscr{H}_j^* (1 - \varphi_j) f_j, u \rangle = \langle f_j, \mathscr{H}_j (1 - \varphi_j) u + A_0(\varphi_j) u \rangle = \langle (1 - \varphi_j) f_j, T_0 u \rangle$$

and therefore $(1 - \varphi_j) f_j \in \mathscr{D}(T_0^*) = \mathscr{D}(T)$ with $T(1 - \varphi_j) f_j = \mathscr{H}_j^* (1 - \varphi_j) f_j$ $(j = 1, 2)$. Let $(u_n) \subset \mathscr{D}_0(N_0^c)$ be a sequence with $u_n \to (1 - \varphi_j) f_j$ in the graph norm of T. Then we get $(1 - \alpha_j) u_n \to (1 - \varphi_j) f_j$ and $\mathscr{H}_j((1 - \alpha_j) u_n) =$

$= T_0((1-\alpha_j)u_n) \to T(1-\varphi_j)f_j$ and therefore $(1-\varphi_j)f_j \in \mathscr{D}(\overline{\mathscr{H}}_j) \cap \mathscr{D}(T)$. Altogether we get $f_j \in \mathscr{D}(\overline{\mathscr{H}}_j)$ $(j=1,2)$ and hence $\lambda_{j,k}=0$ $(j=1,2;\ k=1,\ldots,n_j)$.

STATEMENT 2. $\text{def.}\overline{\mathscr{H}}_1 + \text{def.}\overline{\mathscr{H}}_2 \leqslant \text{def.}\dot{\mathscr{H}}_0$.

Proof. Let $(u_{j,k})_{k=1}^{p_j}$ respectively $(v_{j,k})_{k=1}^{q_j}$ $(j=1,2)$ be an orthogonal basis of $\mathscr{N}(\mathscr{H}_j^*-\mathrm{i})$ respectively $\mathscr{N}(\mathscr{H}_j^*+\mathrm{i})$ and $\tilde{u}_{j,k} := \varphi_j u_{j,k}$, $\tilde{v}_{j,k} := \varphi_j v_{j,k}$. Now we want to compute $I := \operatorname{Im}\langle H_0^*\tilde{u}, \tilde{u}\rangle$ for all $\tilde{u} \in \operatorname{span}\{\tilde{u}_{j,k} : j=1,2;\ k=1,\ldots,p_j\}$. For $\tilde{u} = \sum_{j=1}^{2}\sum_{k=1}^{n_j} \lambda_{j,k}\tilde{u}_{j,k}$ ($n_j \leqslant p_j$ and finite, $\lambda_{j,k} \in \mathbf{C}$) we obtain

$$I = \sum_{j,k,m} \operatorname{Im} \bar{\lambda}_{j,k}\lambda_{j,m}\langle \mathscr{H}_j^*(\varphi_j u_{j,k}), \varphi_j u_{j,m}\rangle +$$

$$+ \sum_{j=1}^{2}\sum_{k=1}^{n_j}\sum_{\substack{l=1\\ j\neq l}}^{2}\sum_{m=1}^{n_l} \operatorname{Im}\bar{\lambda}_{j,k}\lambda_{l,m}\langle A(\varphi_j)u_{j,k}, \varphi_l u_{l,m}\rangle = I_1 + I_2.$$

Because of $A(\varphi\psi) = A(\varphi)\psi + \varphi A(\psi)$ $(\varphi, \psi \in C_0^\infty(\mathbf{R}^m))$ and $A(0)=0$ we get $I_2=0$. As we have $(1-\varphi_j)u_{j,k} \in \mathscr{D}(\overline{\mathscr{H}}_j)$ (proof of Statement 1) we obtain:

$$I = I_1 = \sum_{j,k,m} \operatorname{Im}\bar{\lambda}_{j,k}\lambda_{j,m}\langle \mathscr{H}_j^*(u_{j,k} - (1-\varphi_j)u_{j,k}), u_{j,m} - (1-\varphi_j)u_{j,m}\rangle =$$

$$= \sum_{j,k} |\lambda_{j,k}|^2 \operatorname{Im}\langle \mathscr{H}_j^* u_{j,k}, u_{j,k}\rangle = -\sum_{j,k} |\lambda_{j,k}|^2.$$

But this means $\operatorname{span}\{\tilde{u}_{j,k} : j=1,2;\ k=1,\ldots,p_j\} \subset \Gamma_- \cup \{0\}$ with

$$\Gamma_\pm := \{u \in \mathscr{D}(\mathscr{H}_0^*) : \operatorname{sign}(\operatorname{Im}\langle \mathscr{H}_0^* u, u\rangle) = \pm 1\}.$$

Similarly we can prove that $\operatorname{span}\{\tilde{v}_{j,k} : j=1,2;\ k=1,\ldots,q_j\} \subset \Gamma_+ \cup \{0\}$.

Application of Statement 1 and Satz IV in [15, § 14.6] yields the above conclusion.

STATEMENT 3. $\text{def.}\overline{\mathscr{H}}_1 + \text{def.}\overline{\mathscr{H}}_2 \geqslant \text{def.}\dot{\mathscr{H}}_0$.

Proof. The proof is completed if we can show that $p_0 + q_0 \leqslant p_1 + q_1 + p_2 + q_2$. Therefore, without loss of generality we can assume $p_j, q_j < \infty$ for $j=1,2$. Now, in contrary, let us assume $p_0 + q_0 > p_1 + q_1 + p_2 + q_2$. Let $(w_{j,k})$ $(j=1,2;\ k=1,\ldots,p_j+q_j)$ be chosen as in Statement 1. Then there exists $u \in \mathscr{D}(\mathscr{H}_0^*)$ such that the vectors $\tilde{w}_{j,k} := \varphi_j w_{j,k}$ $(j=1,2;\ k=1,\ldots,p_j+q_j)$ and u are independent modulo $\mathscr{D}(\dot{\mathscr{H}}_0)$. Because of $\dim \mathscr{D}(\mathscr{H}_j^*)/\mathscr{D}(\overline{\mathscr{H}}_j) = p_j + q_j$ the proof of Statement 1 shows the existence of $\lambda_{j,k} \in \mathbf{C}$ and $u_j \in \mathscr{D}(\overline{\mathscr{H}}_j) \cap \mathscr{D}(\dot{\mathscr{H}}_0)$

such that

$$\varphi_j u = \sum_{k=1}^{p_j+q_j} \lambda_{j,k}\tilde{w}_{j,k} + u_j \quad (j = 1, 2).$$

Further arguing as in the proof of Statement 1 we can prove $(1 - \varphi_0)u \in \mathscr{D}(T) \cap \cap \mathscr{D}(\bar{\mathscr{H}}_0)$. Altogether we get $u = v + w$ with $v \in \text{span}\{\tilde{w}_{j,k} : j = 1, 2;\ k = 1, \ldots \ldots, p_j + q_j\}$ and $w \in \mathscr{D}(\bar{\mathscr{H}}_0)$. But this is a contradiction.

(b) By the closed-graph theorem the conditions $\mathscr{D}(T) \subset \mathscr{D}(V_i)$ $(i = 1,2)$ imply $\mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_j)$ $(j = 0, 1, 2)$. Let $u = u_1 + u_2$ such that $u_i \in \mathscr{D}(\mathscr{H}_i^*)$ $(i = 1,2)$ Then the proof of part (a) shows that we have $(1 - \varphi_j)\, u_j \in \mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_0)$ and $\varphi_j u_j \in \mathscr{D}(\mathscr{H}_0^*)$. This means $\text{span}\{\mathscr{D}(\mathscr{H}_1^*), \mathscr{D}(\mathscr{H}_2^*)\} \subset \mathscr{D}(\mathscr{H}_0^*)$.

On the contrary let $u \in \mathscr{D}(\mathscr{H}_0^*)$. Again using the proof of Part (a) we get $(1 - \varphi_0)u \in \mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_j)$ $(j = 1, 2)$ and $\varphi_j u \in \mathscr{D}(\mathscr{H}_j^*)$ $(j = 1, 2)$ and hence $u \in \text{span}\{\mathscr{D}(\mathscr{H}_1^*), \mathscr{D}(\mathscr{H}_2^*)\}$. The second conclusion can be proved analogously.

REMARK. In many ways Theorem 1(a) generalizes the first part of Theorem 1 in [4]. First Behncke only allows Dirac-operators such that each $\mathscr{H}_j$ possesses self-adjoint extensions. Further he only allows potentials which satisfy Condition (4). As Behncke uses in his proof the explicit representation of $\mathscr{H}_j^*$ it does not work if the potentials satisfy Condition (5).

Proceeding from the description of $\mathscr{D}(\bar{\mathscr{H}}_0)$ in Theorem 1(b), we finally construct closed symmetric extensions of $\mathscr{H}_0$.

THEOREM 2. *Assume that the potentials V_i $(i = 1, \ldots, n)$ satisfy Condition* (5′). *Further assume that each $\mathscr{H}_j$ $(j = 1, \ldots, n)$ possesses a closed symmetric extension $\tilde{\mathscr{H}}_j$ with $\varphi_j \mathscr{D}(\tilde{\mathscr{H}}_j) \subset \mathscr{D}(\tilde{\mathscr{H}}_j)$. Then $\tilde{\mathscr{H}}_0 := \mathscr{H}_0^* \restriction \mathscr{D}(\tilde{\mathscr{H}}_0)$ with $\mathscr{D}(\tilde{\mathscr{H}}_0) :=$ span$\{\mathscr{D}(\tilde{\mathscr{H}}_j) : j = 1, \ldots, n\}$ is a closed symmetric extension of $\mathscr{H}_0$ with $\varphi_j \mathscr{D}(\tilde{\mathscr{H}}_0) \subset \mathscr{D}(\tilde{\mathscr{H}}_j)$,*

$$\text{def.}\, \tilde{\mathscr{H}}_0 = \sum_{j=1}^{n} \text{def.}\, \tilde{\mathscr{H}}_j$$

and

$$\mathscr{D}(\tilde{\mathscr{H}}_0^*) = \text{span}\{\mathscr{D}(\tilde{\mathscr{H}}_j^*) : j = 1, \ldots, n\}.$$

Proof. Without loss of generality we can assume $V_j = \chi_j V_j$ $(j = 1, \ldots, n)$ and $n = 2$. Theorem 1(b) implies $\bar{\mathscr{H}}_0 \subset \tilde{\mathscr{H}}_0 \subset \mathscr{H}_0^*$. Further we note the following simple implications:

1. $u \in \mathscr{D}(\tilde{\mathscr{H}}_0) \Rightarrow \varphi_j u \in \mathscr{D}(\tilde{\mathscr{H}}_j)$ and $\tilde{\mathscr{H}}_j(\varphi_j u) = \tilde{\mathscr{H}}_0(\varphi_j u)$ $(j = 1, 2)$;

2. $u \in \mathscr{D}(\tilde{\mathscr{H}}_j^*)$ $(j = 1,2) \Rightarrow \varphi_j u \in \mathscr{D}(\tilde{\mathscr{H}}_0^*) \cap \mathscr{D}(\tilde{\mathscr{H}}_j^*)$ and $\tilde{\mathscr{H}}_j^*(\varphi_j u) = \tilde{\mathscr{H}}_0^*(\varphi_j u)$;

3. $u \in \mathscr{D}(\tilde{\mathscr{H}}_0^*) \Rightarrow \varphi_j u \in \mathscr{D}(\tilde{\mathscr{H}}_j^*) \cap \mathscr{D}(\tilde{\mathscr{H}}_0^*)$ and $\tilde{\mathscr{H}}_j^*(\varphi_j u) = \tilde{\mathscr{H}}_0^*(\varphi_j u)$ $(j = 1, 2)$, which one can prove analogously to the corresponding conclusions in the proof of Theorem 1.

STATEMENT 1. *$\tilde{\mathscr{H}}_0$ is a symmetric operator.*

Proof. Let $u := u_1 + u_2 \in \mathscr{D}(\tilde{\mathscr{H}}_0)$ with $u_j \in \mathscr{D}(\tilde{\mathscr{H}}_j)$ $(j = 1, 2)$ then we get

$$u = \sum_{j=1}^{2} \varphi_j u_j + w \quad \text{with } w = \sum_{j=1}^{2} (1 - \varphi_j) u_j \in \mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_0).$$

Using the symmetry of $\bar{\mathscr{H}}_0$ and $\tilde{\mathscr{H}}_k$ $(k = 1, 2)$ we obtain:

$$\operatorname{Im}\langle u, \tilde{\mathscr{H}}_0 u\rangle = \sum_{j,k} \operatorname{Im}\langle \varphi_j u_j, \tilde{\mathscr{H}}_0(\varphi_k u_k)\rangle = \sum_{j,k} \operatorname{Im}\langle \varphi_j u_j, \tilde{\mathscr{H}}_k(\varphi_k u_k)\rangle =$$

$$= \sum_{\substack{j,k \\ j\neq k}} \operatorname{Im}\langle \varphi_j u_j, \tilde{\mathscr{H}}_k(\varphi_k u_k)\rangle = \sum_{\substack{j,k \\ j\neq k}} \operatorname{Im}\langle \varphi_j u_j, A(\varphi_k) u_k\rangle = 0$$

and hence $\tilde{\mathscr{H}}_0$ is a symmetric operator.

STATEMENT 2. *$\tilde{\mathscr{H}}_0$ is a closed operator.*

Proof. Let $(u_n) \subset \mathscr{D}(\tilde{\mathscr{H}}_0)$ be a sequence such that $u_n \to u$, $\tilde{\mathscr{H}}_0 u_n \to v$. Because of the first implication we obtain $(\varphi_j u_n) \subset \mathscr{D}(\tilde{\mathscr{H}}_j)$ $(j = 1, 2)$ and $\varphi_j u_n \to \varphi_j u$, $\tilde{\mathscr{H}}_j(\varphi_j u_n) \to \varphi_j v + A(\varphi) u$. Further we get

$$((1 - \varphi_0) u_n) \subset \mathscr{D}(T) \subset \mathscr{D}(\bar{\mathscr{H}}_0) \quad \text{and} \quad (1 - \varphi_0) u_n \to (1 - \varphi_0) u,$$

$\bar{\mathscr{H}}_0(1 - \varphi_0) u_n \to (1 - \varphi_0) v - A(\varphi_0) u$. But the operators $\bar{\mathscr{H}}_0$ and $\tilde{\mathscr{H}}_k$ $(k = 1, 2)$ are closed and hence $u \in \mathscr{D}(\tilde{\mathscr{H}}_0)$ with $\tilde{\mathscr{H}}_0 u = v$.

Using the above mentioned implication we can repeat the proof of Theorem 1 and hence the remaining conclusions follow.

REMARK. If even $\varphi \mathscr{D}(\tilde{\mathscr{H}}_j) \subset \mathscr{D}(\tilde{\mathscr{H}}_j)$ $(j = 1, \ldots, n)$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$ then also

$$\varphi \mathscr{D}(\tilde{\mathscr{H}}_0) \subset \mathscr{D}(\tilde{\mathscr{H}}_0) \text{ for all } \varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m).$$

3. THE ESSENTIAL SPECTRAL KERNEL OF GENERALIZED DIRAC-OPERATORS

In the following we need a characterization of the essential spectral kernel $S_e(T) = \{\lambda \in \mathbf{C} : \mathscr{R}(T - \lambda) \text{ is closed and } \dim \mathscr{N}(T - \lambda) < \infty\}^c$ of a closed symmetric operator T by singular sequences.

LEMMA 2. *Let T be a closed symmetric operator then the following statements are equivalent:*

(a) $\lambda \in S_e(T)$;

(b) *there exists a sequence $(u_n) \subset \mathscr{D}$, where $\mathscr{D}$ is a core of T, such that $u_n \rightharpoonup 0$ $\liminf_{n\to\infty} \|u_n\| > 0$ and $(T - \lambda)u_n \to 0$.*

A sequence $(u_n) \subset \mathscr{D}(T)$ with the properties in (b) will be called a *λ-singular sequence.*

Proof. See [20, § 1] and [7, Theorem 1].

In this section we want to determine the essential spectral kernel of $\overline{\mathscr{H}}_0$ by the essential spectral kernels of the operators $\overline{\mathscr{H}}_j$. To do this, we have to sharpen the conditions of Section 2. Let T_0 be an essentially selfadjoint operator defined on $\mathscr{D}_0(N_0^c)$ which satisfies Conditions (1) and (3) (we do not need Condition (2)). Further, we suppose:

(7) The operators $u \mapsto \varphi u$, $u \mapsto A(\varphi)u$ $(u \in \mathfrak{H})$ are T_0-compact for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$.

(8) $A(\varphi)\psi u = 0$ $(u \in \mathfrak{H})$ for all $\varphi, \psi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$ such that φ is constant on all connected components of $\{x : \psi(x) \neq 0\}$ (it suffices only asking $A(\varphi_j)\alpha_k u = 0$ $(u \in \mathfrak{H})$ for $j = 1, \ldots, n$; $k = j, 0$).

Note, that the Dirac-operator for a free electron fulfils these properties (see [6, § 7]).

Mainly, the conditions (7) and (8) will allow us to localize a singular sequence near one N_j or near infinity. Finally, we suppose:

(9) The potentials V_i $(i = 1, \ldots, n)$ satisfy condition (4) for all $b_i > 0$ with a suitable $a_i > 0$. Further, $V_i(x) \to 0$ $(|x| \to \infty)$.

LEMMA 3. *Under the above assumptions the operator $u \mapsto \varphi u$ $(u \in \mathfrak{H})$ is $\mathscr{H}_j^*$-compact for all $\varphi \in \mathscr{C}_0^\infty(N_j^c)$ $(j = 0, 1, \ldots, n)$.*

Proof. Let $\varphi, \psi \in \mathscr{C}_0^\infty(N_j^c)$ $(j = 0, 1, \ldots, n)$ with $\varphi\psi = \varphi$. Let $(v_n) \subset \mathscr{D}(\mathscr{H}_j^*)$ be a sequence with $u_n \rightharpoonup 0$, $\mathscr{H}_j^* u_n \rightharpoonup 0$. Using Lemma 10 in [8, § 9.1] (see also Lemma 5.2 in [6]) we obtain $(\psi u_n) \subset \mathscr{D}(\overline{\mathscr{H}}_j)$, $\psi u_n \rightharpoonup 0$ and $\overline{\mathscr{H}}_j(\psi u_n) \rightharpoonup 0$. But the operator $u \mapsto \varphi u$ is $\overline{\mathscr{H}}_j$-compact (see Lemma 11 in [8, § 9.2] or Lemma 1 in [4]). Hence $\varphi u_n = \varphi\psi u_n \to 0$.

REMARK. Lemma 3 remains true if the potentials V_i $(i = 1, \ldots, n)$ satisfy Condition (4) with $a_i, b_i > 0$ and $b_0 < 1$.

COROLLARY. *Under the above assumptions $W_j := V_j(1 - \chi_j)$ is $\mathscr{H}_j^*$-compact $(j = 0, 1, \ldots, n)$.*

Proof. Let $K \subset N_j^c$ be a compact set; then we obtain, in a way similar to the proof of Lemma 3, that $\chi_K W_j$ is $\mathscr{H}_j^*$-compact (see Lemma 13 in [8, § 9.2]). But for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset N_j^c$ with $|W_j(x)| \leqslant \varepsilon$ for all $x \notin K_\varepsilon$. Now let $(u_n) \subset \mathscr{D}(\mathscr{H}_j^*)$ be a sequence with $u_n \rightharpoonup 0$, $\mathscr{H}_j^* u_n \rightharpoonup 0$ then we get for all $\varepsilon > 0$ $\|W_j u_n\| \leqslant \varepsilon \|u_n\| + \|\chi_{K_\varepsilon} W_j u_n\|$ and hence $\limsup_{n\to\infty} \|W_j u_n\| \leqslant \varepsilon \limsup_{n\to\infty} \|u_n\|$. But this means $W_j u_n \to 0$.

THEOREM 3. *Under the above assumptions we have*

$$\text{(a)} \qquad S_e(\bar{\mathscr{H}}_0) = \bigcup_{j=1}^{n} S_e(\bar{\mathscr{H}}_j); \quad \sigma_e(T) \subset S_e(\bar{\mathscr{H}}_j) \quad (j = 0, 1, \ldots, n).$$

If all operators $\bar{\mathscr{H}}_j$ *possess equal and finite deficiency indices then*

$$\sigma_e(\mathscr{H}_0') = \bigcup_{j=1}^{n} \sigma_e(\mathscr{H}_j').$$

Hereby, $\mathscr{H}_j'$ $(j = 0, 1, \ldots, n)$ *denotes an arbitrary selfadjoint extension of* $\mathscr{H}_j$.

(b) *In addition assume that also the conditions of Theorem* 2 *are fulfilled with* $\varphi\mathscr{D}(\tilde{\mathscr{H}}_j) \subset \mathscr{D}(\tilde{\mathscr{H}}_j)$ $(j = 1, \ldots, n)$ *for all* $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^m)$. *Then the conclusions of Part* (a) *remain true if one replaces* $\mathscr{H}_j$ *by* $\tilde{\mathscr{H}}_j$ $(j = 0, 1, \ldots, n)$.

Proof. Because relative compact perturbations do not change the essential spectral kernel (see [20, § 1] and Theorem 2 in [7]) we can assume $V_j = V_j\chi_j$ $(j = 0, 1, \ldots, n)$ using the Corollary to Lemma 3. By induction we may also assume $n = 2$.

(a) STATEMENT 1. $\sigma_e(T) \subset S_e(\bar{\mathscr{H}}_j)$ $(j = 0, 1, 2)$.

Proof. Let $\lambda \in \sigma_e(T)$ and let $(u_n) \subset \mathscr{D}_0(N_0^c)$ be a λ-singular sequence. Using condition (7) we get for $j = 0, 1, 2$: $\varphi_j u_n \to 0$, $(1 - \varphi_j)\, u_n \rightharpoonup 0$ and

$$(\mathscr{H}_j - \lambda)(1 - \varphi_j)u_n = (T_0 - \lambda)(1 - \varphi_j)\, u_n = (1 - \varphi_j)(T_0 - \lambda)u_n - A_0(\varphi_j)u_n \to 0$$

and hence $\lambda \in S_e(\bar{\mathscr{H}}_j)$.

STATEMENT 2. $S_e(\bar{\mathscr{H}}_j) \subset S_e(\bar{\mathscr{H}}_0)$ $(j = 1, 2)$.

Proof. Let $\lambda \in S_e(\bar{\mathscr{H}}_j)$ and let $(u_n) \subset \mathscr{D}_0(N_j^c)$ be a λ-singular sequence. Then we get $(1 - \alpha_j)u_n \rightharpoonup 0$ and $T_0(1 - \alpha_j)u_n = \mathscr{H}_j(1 - \alpha_j)u_n \rightharpoonup 0$. Using the conditions (7) and (8) we obtain $A_0(\varphi_j)u_n \to 0$. Therefore, we get

$$\varphi_j u_n \rightharpoonup 0, \quad (\mathscr{H}_0 - \lambda)\varphi_j u_n = (\mathscr{H}_j - \lambda)\varphi_j u_n \to 0$$

and

$$(T_0 - \lambda)(1 - \varphi_j)u_n = (\mathscr{H}_j - \lambda)(1 - \varphi_j)u_n \to 0$$

and hence $\lambda \in S_e(\bar{\mathscr{H}}_0) \cup \sigma_e(T) = S_e(\bar{\mathscr{H}}_0)$.

STATEMENT 3. $S_e(\overline{\mathscr{H}}_0) \subset \bigcup_{j=1}^{2} S_e(\overline{\mathscr{H}}_j)$.

Proof. Let $\lambda \in S_e(\overline{\mathscr{H}}_0)$ and let $(u_n) \subset \mathscr{D}_0(N_0^c)$ be a λ-singular sequence. With the same arguments as in the proof of Statement 2 (replace α_j by α_0) we obtain $A_0(\varphi_j)u_n \to 0$ and hence $\varphi_j u_n \rightharpoonup 0$,

$$(\mathscr{H}_j - \lambda)\varphi_j u_n = (\mathscr{H}_0 - \lambda)\varphi_j u_n \to 0 \quad (j = 1, 2)$$

and

$$(T_\theta - \lambda)(1 - \varphi_0)u_n = (\mathscr{H}_0 - \lambda)(1 - \varphi_0)u_n \to 0.$$

But this means

$$\lambda \in \bigcup_{j=1}^{2} S_e(\overline{\mathscr{H}}_j) \cup \sigma_e(T) = \bigcup_{j=1}^{2} S_e(\overline{\mathscr{H}}_j).$$

The other conclusion follows at once by Theorem 8.17 and Theorem 8.18 in [19].

(b) Using the techniques of Theorem 2 we can now completely copy the proof of Part (a).

REMARK 1. With a similar method Behncke [4, Theorem 1] proves Theorem 3 (a) for Dirac-operators where all operators $\mathscr{H}_j$ $(j = 1, \ldots, n)$ possess finite and equal deficiency indices.

REMARK 2. With the methods of Jörgens [6, §7] the condition $V_i(x) \to 0$ $(|x| \to \infty)$ can be replaced in Theorem 3 (a) by the following weaker condition: For all $\varepsilon > 0$ and all $j \in \{1, \ldots, n\}$ there exists a bounded open set $\Omega_{\varepsilon,j} \supset N_j$ with $\|V_j u\| \leqslant \varepsilon(\|Tu\| + \|u\|)$ for all $u \in \mathscr{D}_0(\overline{\Omega_{\varepsilon,j}}^c)$.

Therefore, in the case of Dirac-operators, we can also allow potentials V_j with $|V_j| \in \mathscr{L}^3_{\mathrm{loc}}(N_j^c) \cap \mathscr{L}^3(\overline{\Omega}_j^c)$ where Ω_j is a bounded open set with $N_j \subset \Omega_j$.

4. DIRAC-OPERATORS WITH SEVERAL COULOMB-SINGULARITIES

In this section we apply the results of the previous sections to multicenter Dirac-operators with singularities localized at $N_j := \{a_j\}$ with $a_j \neq a_k$ $(j \neq k)$. Especially, T_0 is the Dirac-operator for a free electron defined on $\mathscr{D}_0(N_0^c)$ with $N_0 := \{a_1, \ldots, a_n\}$. In addition we also define the operator T_{00} by $T_{00} := T \restriction \mathscr{D}_0$ and hence $\overline{T}_{00} = \overline{T}_0 = T$. Further let V_i $(i = 1, \ldots, n)$ be a formally symmetric 4 by 4 matrix potential which satisfies Condition (5'). Then we define the operator $\mathscr{H}_{j0}$ $(j = 1, \ldots, n)$ on $\mathscr{D}_0$ by $\mathscr{H}_{j0} := T_{00} + V_j$. Using Condition (5'), we obtain

$\bar{\mathscr{H}}_{j0} = \bar{\mathscr{H}}_j$. Finally, let V_{n+1} be a formally symmetric 4 by 4 matrix potential which satisfies the following condition:

(10) For all non-void open bounded sets Ω and for all $\varepsilon > 0$ there exists a positive number $M = M(\varepsilon, \Omega)$ with

$$\|V_{n+1}u\| \leqslant \varepsilon\|T_{00}u\| + M\|u\|$$

for all $u \in \mathscr{D}_0(\Omega)$.

Then we define the operators $\mathscr{H}_0$ and $\mathscr{H}_{00}$ by $\mathscr{H}_0 := T_0 + V_0 + V_{n+1}$ with $\mathscr{D}(\mathscr{H}_0) = \mathscr{D}_0(N_0^c)$ and $\mathscr{H}_{00}$ by $\mathscr{H}_{00} := T_{00} + V_0 + V_{n+1}$ with $\mathscr{D}(\mathscr{H}_{00}) := \mathscr{D}_0$. Using Conditions (1), (5′) and (10) we obtain $\bar{\mathscr{H}}_{00} = \bar{\mathscr{H}}_0$.

Studying the essential spectrum of $\bar{\mathscr{H}}_0$ we also need the following condition:

(11) For all $\varepsilon > 0$ there exists a non-void compact set K_ε such that

$$\|V_j u\| \leqslant \varepsilon(\|T_{00}u\| + \|u\|)$$

$(j = 1, \ldots, n+1)$ for all $u \in \mathscr{D}_0(K_\varepsilon^c)$.

THEOREM 4. *Suppose that the potentials V_j $(j = 1, \ldots, n+1)$ satisfy the above conditions except Condition* (11). *Suppose further, that each operator $\mathscr{H}_j$ $(j = 1, \ldots, n)$ is essentially selfadjoint with $\mathscr{D}(\bar{\mathscr{H}}_j) = \mathscr{D}(T)$. Then one obtains:*

(a) *$\mathscr{H}_0$ and $\mathscr{H}_{00}$ are essentially selfadjoint with*

$$\mathscr{D}(\bar{\mathscr{H}}_0) := \left\{u \in W^1_{\mathrm{loc}}(\mathbf{R}^3) \otimes \mathbf{C}^4 \cap \mathfrak{H} : \alpha\cdot(-\mathrm{i}\nabla u) + \sum_{j=1}^{n+1} V_j u \in \mathfrak{H}\right\}$$

and

$$\bar{\mathscr{H}}_0 u = \alpha\cdot(-\mathrm{i}\nabla)u + \beta u + \sum_{j=1}^{n+1} V_j u \quad \text{for all } u \in \mathscr{D}(\bar{\mathscr{H}}_0).$$

(b) *If Condition* (10) *is also valid for $\Omega := \mathbf{R}^3$ then*

$$\mathscr{D}(\bar{\mathscr{H}}_0) = \mathscr{D}(T) \quad \text{and} \quad \bar{\mathscr{H}}_0 u = Tu + \sum_{j=1}^{n+1} V_j u \quad \text{for all } u \in \mathscr{D}(\bar{\mathscr{H}}_0).$$

(c) *Suppose that the potentials V_j $(j = 1, \ldots, n+1)$ satisfy Condition* (11). *Then we have in addition to the conclusions of Parts* (a) *and* (b) *that*

$$\sigma_e(\bar{\mathscr{H}}_0) = \sigma_e(T) = (-1,1)^c.$$

Proof. First we assume $V_{n+1} = 0$. Then the conclusions in Parts (a) and (b) are trivial consequences of Theorem 1.

(a) This is an evident consequence of Theorem 5.3, Theorem 5.6 in [6] and the closed graph theorem if one chooses $T_{00} + V_0$ defined on $\mathscr{D}_0$ as the unperturbed operator.

(b) This conclusion is a trivial consequence of the closed graph theorem.

(c) If $V_{n+1} = 0$ the conclusion follows by application of Theorem 2.9 in [17] and Theorem 9.13 in [19]. In the general case V_{n+1} is T-compact (see Theorem 7.3.in [6]) and hence $T + V_0$-compact.

REMARK 1. In [6, § 6 and § 7] the conditions on V_{n+1} are verified for various classes of perturbations. As usual we choose the potential V_j $(j = 1, \ldots, n)$ to be of the form $V_j(x) = Q_j(x - a_j)$ with $|Q_j(x)| \leqslant c_j/|x|$. Then the Hardy-inequality (see Auxiliary Theorem 10.15 in [19]) and the translation invariance of T imply $\mathscr{D}(T) \subset \mathscr{D}(V_j)$. Especially, the potentials V_j $(j = 1, \ldots, n)$ satisfy the conditions of Theorem 4 if $0 \leqslant c_j < 1/2$ or $Q_j(x) = q_j(x)I$ where q_j is a real function with $|q_j(x)| \leqslant c_j/|x|$ and $0 \leqslant c_j < \sqrt{3}/2$ (see [1; 6]).

REMARK 2. Theorem 4 (b) corresponds to Theorem 2.1 and its Corollary 2.1 in [12]. Theorem 4 (a) generalizes Corollary 3.3 (2) in [1]. Further, Klaus handles the special case $V_j(x) = c_j/|x - a_j|$ $(j = 1, \ldots, n)$ with $|c_j| < \sqrt{3}/2$ and $V_{n+1} = 0$ [9, Theorem 4.1]. The proofs in these papers are all based upon resolvent methods in connection with localization of the singularities.

More generally, we even can handle multicenter Dirac-operators with arbitrary Coulomb-potentials.

THEOREM 5. *Assume that* $V_j = c_j/|x - a_j|$ $(j = 1, \ldots, n)$ *with* $c_j \in \mathbf{R}$ *and* $V_{n+1} = 0$. *Then one obtains:*

(a) *Let* l_j *be the uniquely determined nonnegative integer with* $l_j^2 < c_j^2 + 1/4 \leqslant (l_j + 1)^2$. *Then*

$$\operatorname{def}.\overline{\mathscr{H}}_0 = (p, p) \quad \textit{with } p := \sum_{j=1}^{n} 2l_j(l_j + 1).$$

(b) $\mathscr{H}_0$ *is essentially selfadjoint (with* $\mathscr{D}(\overline{\mathscr{H}}_0) = \mathscr{D}(T)$*) iff* $|c_j| \leqslant \sqrt{3}/2$ *(*$|c_j| < \sqrt{3}/2$*) for all* $j = 1, \ldots, n$. *If* $|c_j| \leqslant \sqrt{3}/2$ *for all* $j = 1, \ldots, n$ *and* $|c_j| = \sqrt{3}/2$ *for at least one* j *then* $\mathscr{D}(T)$ *is properly contained in* $\mathscr{D}(\overline{\mathscr{H}}_0)$.

(c) *Let* $\tilde{\mathscr{H}}_0$ *be an arbitrary selfadjoint extension of* $\mathscr{H}_0$ *then*

$$\sigma_e(\tilde{\mathscr{H}}_0) = (-1, 1)^c.$$

Proof. (a) Using the formula for $\operatorname{def}.\overline{\mathscr{H}}_j$ (see Theorem 1 (d) in [2] or Satz 4 in [8, §10]) the conclusion follows by application of Theorem 1 (a).

(b) Because of Remark 1 concerning Theorem 4 we only have to discuss the case $|c_j| \leqslant \sqrt{3}/2$ $(j = 1, \ldots, n)$ and $|c_j| = \sqrt{3}/2$ for at least one j. Assuming $\mathscr{D}(T) = \mathscr{D}(\mathscr{H}_0)$ we get by Theorem 1(b) $\mathscr{D}(\mathscr{H}_j^*) = \mathscr{D}(T)$ $(j = 1, \ldots, n)$. Thus we may reduce the problem to the case of a single singularity localized at 0. Then the application of a well-known perturbation result [19, Theorem 5.29] yields the existence of a c with $|c| > \sqrt{3}/2$ such that $\mathscr{H} := T_{00} + c/|x|$ is essentially selfadjoint.

(c) Because of $V_j \in \mathscr{L}^3_{\mathrm{loc}}(\{a_j\}^c)$ we can apply Theorem 3(a). But in the case of a single singularity the conclusion follows by Satz 11 in [8, § 11].

Finally, we discuss in analogy to the case of one center [1; 10; 11; 14; 21; 22] the problem of determining a distinguished selfadjoint extension of $\mathscr{H}_0$.

THEOREM 6. *Assume that* $V_j = Q_j(x - a_j)$ *with* $\sup_{x \in \mathbf{R}^3} |x|\,|Q_j(x)| = c_j \in (0, 1)$ *for* $j = 1, \ldots, n$ *and* $V_{n+1} = 0$. *Then* $\mathscr{H}_0$ *possesses a selfadjoint extension* $\tilde{\mathscr{H}}_0$ *with the following properties:*

(i)
$$\tilde{\mathscr{H}}_0 = \mathscr{H}_0^* \upharpoonright \mathscr{D}(\tilde{\mathscr{H}}_0)$$

with

$$\mathscr{D}(\tilde{\mathscr{H}}_0) = \mathscr{D}(\mathscr{H}_0^*) \cap \mathscr{D}(|T|^{1/2}) =$$
$$= \mathscr{D}(\mathscr{H}_0^*) \cap \mathscr{D}(W^{1/2}) \quad \textit{and} \quad W := \sum_{j=1}^{n} |x - a_j|^{-1}.$$

(ii) $\tilde{\mathscr{H}}_0$ *is the uniquely determined selfadjoint extension with*

$$\mathscr{D}(\tilde{\mathscr{H}}_0) \subset \mathscr{D}(W^{1/2}) \quad \textit{or} \quad \mathscr{D}(\tilde{\mathscr{H}}_0) \subset \mathscr{D}(|T|^{1/2}).$$

(iii)
$$\sigma_e(\tilde{\mathscr{H}}_0) = (-1, 1)^c.$$

Proof. Using the translation invariance of T, Theorem (4.1) in [1] yields that the operators $\mathscr{H}_j$ $(j = 1, \ldots, n)$ possess a selfadjoint extension with $\tilde{\mathscr{H}}_j = \mathscr{H}_j^* \upharpoonright \mathscr{D}(\tilde{\mathscr{H}}_j)$, $\mathscr{D}(\tilde{\mathscr{H}}_j) = \mathscr{D}(\mathscr{H}_j^*) \cap \mathscr{D}(|x - a_j|^{-1/2})$ and $\sigma_e(\tilde{\mathscr{H}}_j) = (-1, 1)^c$. It is a trivial fact that $\varphi\mathscr{D}(|x - a_j|^{-1/2}) \subset \mathscr{D}(|x - a_j|^{-1/2})$ and $\varphi\mathscr{D}(\mathscr{H}_j^*) \subset \mathscr{D}(\mathscr{H}_j^*)$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^3)$. Let $\mathscr{F}$ be the Fourier transformation in $\mathfrak{H}$ then we get using the spectral theorem

$$|T|^{1/2} = (|T|^2)^{1/4} = (T^2)^{1/4} = (-\Delta + 1)^{1/4} = \mathscr{F}^*(1 + |\xi|^2)^{1/4}\mathscr{F}$$

and therefore $\mathscr{D}(|T|^{1/2}) = W^{1/2}(\mathbf{R}^3) \otimes \mathbf{C}^4$. Hence it follows by Theorem 1.1 in [18] that $\varphi\mathscr{D}(|T|^{1/2}) \subset \mathscr{D}(|T|^{1/2})$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^3)$. Especially, all conditions of Theorem 2 are fulfilled and we have

$$\operatorname{span}\{\mathscr{D}(\tilde{\mathscr{H}}_j) : j = 1, \ldots, n\} = \mathscr{D}(\mathscr{H}_0^*) \cap \mathscr{D}(|T|^{1/2}) = \mathscr{D}(\mathscr{H}_0^*) \cap \mathscr{D}(W^{1/2}).$$

Property (ii) is a trivial consequence of the first one. Finally, Property (iii) follows by Theorem 3(b).

Especially the conditions of Theorem 6 are fulfilled if $Q_j(x) = q_j(x)I$ with a scalar function $q_j \in \mathscr{L}^2_{\text{loc}}(\mathbf{R}^3)$ with

$$\sup_{x \in \mathbf{R}^3_+} |xq_j(x)| = c_j \in (0, 1)$$

for all $j = 1, \ldots, n$. Following the notation of [11] we introduce the potential classes

$$\mathscr{M} := \{q \in \mathscr{L}^2_{\text{loc}}(\mathbf{R}^3) : \sup_{x \in \mathbf{R}^3_+} |xq_j(x)| = c \text{ with } c \in 0,1)\}$$

$$\mathscr{M}^+ \ (\mathscr{M}^-) := \{q \in \mathscr{M} : q \geqslant 0 \ (q \leqslant 0)\}.$$

For potentials of this type we want to show that $\tilde{\mathscr{H}}_0$ can also be constructed by means of a cut-off procedure. To do this we need a probably well-known perturbation result.

LEMMA 4. *Let A_n, B_n, A and B ($b \in \mathbf{N}$) be selfadjoint operators in a Hilbert space $\mathfrak{H}$ such that B_n ($n \in \mathbf{N}$) and B are bounded with $\|B_n - B\| \to 0$. If $A_n \to A$ in the norm resolvent sense then also $A_n + B_n \to A + B$ in the norm resolvent sense.*

THEOREM 7. *Assume $V_j(x) = q_j(x - a_j)I$ with $q_j \in \mathscr{M}^+ \cup \mathscr{M}^-$ ($j = 1, \ldots, n$) and $V_{n+1} = 0$. Let V_+ (V_-) be the sum of all V_j with $q_j \in \mathscr{M}^+$ ($\mathscr{M}^-$). Let $\mathscr{H}_{0,l} := := T + V_{+,l} + V_{-,l}$ defined on $\mathscr{D}(T)$ ($l \in \mathbf{N}$) where $V_{+,l} := \min(V_+, l)$ and $V_{-,l} := := \max(V_-, -l)$. Then $\mathscr{H}_{0,l}$ converges in the norm resolvent sense to $\tilde{\mathscr{H}}_0$.*

Proof. First, we define the potentials V'_j for $q_j \in \mathscr{M}^+$ ($\mathscr{M}^-$) by

$$V'_j(x) := \begin{cases} V_+(x) \ (V_-(x)) & |x - a_j| < \delta \\ 0 & |x - a_k| < \delta, \ k \neq j \text{ and } q_k \in \mathscr{M}^- \ (\mathscr{M}^+) \\ V_j(x) & |x - a_k| < \delta, \ k \neq j \text{ and } q_k \in \mathscr{M}^- \ (\mathscr{M}^+) \\ V_j(x) & \text{else.} \end{cases}$$

For $\delta > 0$ sufficiently small we see that $q'_j(x) := V'_j(x + a_j) \in \mathscr{M}^+$ ($\mathscr{M}^-$) for $q_j \in \mathscr{M}^+$ ($\mathscr{M}^-$). Further, we get $\sum_{j=1}^{n} V'_j = \sum_{j=1}^{n} V_j$. Let $V'_{j,l} := \min(V'_j, l)$ ($\max(V'_j, -l)$) for $q_j \in \mathscr{M}^+$ ($\mathscr{M}^-$) ($l \in \mathbf{N}$). Then $\mathscr{H}_{j,l} := T + V'_{j,l}$, defined on $\mathscr{D}(T)$, converges in the norm resolvent sense to $\tilde{\mathscr{H}}_j$, where $\tilde{\mathscr{H}}_j$ is the uniquely determined selfadjoint extension of $\mathscr{H}_j := T \dotplus V'_j$, defined on $\mathscr{D}_0(\{a_j\}^c)$ with

$$\mathscr{D}(\tilde{\mathscr{H}}_j) = \mathscr{D}(\mathscr{H}_j^*) \cap \mathscr{D}(|x - a_j|^{-1/2}) = \mathscr{D}(\mathscr{H}_j^*) \cap \mathscr{D}(|T|^{1/2})$$

(Theorem II.2 in [11]). Further, there exists $N \in \mathbf{N}$ with $\sum_{j=1}^{n} V'_{j,l} =: V_{+,l} \dotplus$ $\dotplus V_{-,l}$ for all $l \geqslant N$. Because of Lemma 4 we may assume, without loss of generality, $V'_j = V'_j \chi_j$ $(j = 1, \ldots, n)$. Now we fix a real $c \neq 0$ with $|c| > 2 \sum_{j=1}^{n} \|A(\varphi_j)\|$. Then we get for all $u \in \mathfrak{H}$, $j = 1, \ldots, n$ and $l \geqslant N$:

$$\varphi_j(\tilde{\mathscr{H}}_0 - ci)^{-1} u = (\tilde{\mathscr{H}}_j - ci)^{-1}(\varphi_j u + A(\varphi_j)(\tilde{\mathscr{H}}_0 - ci)^{-1} u)$$

and

$$\varphi_j(\mathscr{H}_{0,l} - ci)^{-1} u = (\mathscr{H}_{j,l} - ci)^{-1}(\varphi_j u + A(\varphi_j)(\mathscr{H}_{0,l} - ci)^{-1} u)$$

and hence

$$\|\varphi_j(\mathscr{H}_{0,l} - ci)^{-1} - \varphi_j(\tilde{\mathscr{H}}_0 - ci)^{-1}\| \leqslant \|(\mathscr{H}_{j,l} - ci)^{-1} - (\tilde{\mathscr{H}}_j - ci)^{-1}\| +$$

$$+ \|((\mathscr{H}_{j,l} - ci)^{-1} - (\tilde{\mathscr{H}}_j - ci)^{-1}) A(\varphi_j)(\mathscr{H}_{0,l} - ci)^{-1}\| +$$

$$+ \|(\tilde{\mathscr{H}}_j - ci)^{-1} A(\varphi_j)((\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1})\| \leqslant$$

$$\leqslant \|(\mathscr{H}_{j,l} - ci)^{-1} - (\tilde{\mathscr{H}}_j - ci)^{-1}\| (1 + |c|^{-1}\|A(\varphi_j)\|) +$$

$$+ |c|^{-1}\|A(\varphi_j)\| \, \|(\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1}\|.$$

Analogously we get for all $l \geqslant N$:

$$\|(1 - \varphi_0)((\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1})\| = \|(T - ci)^{-1} A(\varphi_0)((\mathscr{H}_{0,l} - ci)^{-1} -$$

$$- (\tilde{\mathscr{H}}_0 - ci)^{-1})\| \leqslant |c|^{-1}\|A(\varphi_0)\| \, \|(\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1}\|.$$

Addition of these inequalities gives:

$$\|(\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1}\| \leqslant \sum_{j=1}^{n} (1 + |c|^{-1}\| A(\varphi_j)\|) \|(\mathscr{H}_{j,l} - ci)^{-1} -$$

$$- (\tilde{\mathscr{H}}_j - ci)^{-1}\| + 2|c|^{-1} \sum_{j=1}^{n} \|A(\varphi_j)\| \, \|(\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1}\|$$

and hence

$$\|(\mathscr{H}_{0,l} - ci)^{-1} - (\tilde{\mathscr{H}}_0 - ci)^{-1}\| \to 0 \quad (l \to \infty).$$

REMARK. Theorem 6 generalizes Theorem 1 in [13]. Nenciu only shows the existence of a uniquely determined selfadjoint extension $\tilde{\mathscr{H}}_0$ of $\mathscr{H}_0$ with $\mathscr{D}(\tilde{\mathscr{H}}_0)\subset \subset\mathscr{D}(|T|^{1/2})$ and $\sigma_e(\tilde{\mathscr{H}}_0)\subset\sigma_e(T)$. Klaus [9, Theorem 4.2] proves Theorem 6 and Theorem 7 only for the case $V_j=c_j/|x-a_j|$ with $c_j<0$ for all $j=1,\ldots,n$. Although Klaus and Nenciu localize the singularities they do not reduce the problem to the case of a single singularity but they carry over the methods of [10; 11; 14; 21; 22] to the general case.

REFERENCES

1. ARAI, M., On essential selfadjointness, distinguished selfadjoint extensions and essential spectrum of Dirac operators with matrix valued potentials, *Publ. Res. Inst. Math. Sci.*, to appear.
2. BEHNCKE, H., The Dirac equation with an anomalous magnetic moment, *Math. Z.*, **174**(1980) 213–225.
3. BEHNCKE, H., The Dirac equation with anomalous magnetic moment. II, in *Proc. of the* 7^{th} *Conference on Ordinary and Partial Differential Equations*, Lecture Notes in Mathematics, **964**(1982), Berlin—Heidelberg—New York, Springer-Verlag.
4. BEHNCKE, H., Spectral properties of the Dirac equation with anomalous magnetic moment, Preprint, 1983.
5. BEHNCKE, H.; FOCKE, H., Stability of deficiency indices, *Proc. Roy. Soc. Edinburgh Sect. A.*, **78**(1977), 119–127.
6. JÖRGENS, K., Perturbations of the Dirac operator, in *Conference on the Theory of Ordinary and Partial Differential Equations (Dundee* 1972*)*, pp. 87–102, Lecture Notes in Mathematics, **280**(1972), Berlin—Heidelberg—New York, Springer-Verlag.
7. KARNARSKI, B., Contributions to the spectral theory of symmetric operators, preprint, 1983.
8. KARNARSKI, B., Matrix-Differentialoperatoren erster Ordnung und ihre Anwendung auf die Dirac-Gleichung mit anomalem magnetischen Moment, Dissertation, Universität Osnabrück, 1982.
9. KLAUS, M., Dirac operators with several Coulomb singularities, *Helv. Phys. Acta*, **53**(1980), 463–482.
10. KLAUS, M.; WÜST, R., Characterization and uniqueness of distinguished selfadjoint extensions of Dirac operators, *Comm. Math. Phys.*, **64**(1979), 171–176.
11. KLAUS, M.; WÜST, R., Spectral properties of Dirac operators with singular potentials, *J. Math. Anal. Appl.*, **72**(1979), 206–214.
12. LANDGREN, J. H.; REJTO, P. A., On a theorem of Jörgens and Chernoff concerning essential selfadjointness of Dirac operators, *J. Reine Angew. Math.*, 322(1981), 1–14.
13. NENCIU, G., Distinguished self-adjoint extension for Dirac operator with potential dominated by multicenter Coulomb potentials, *Helv. Phys. Acta*, **50**(1977), 1–3.
14. NENCIU, G., Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms, *Comm. Math. Phys.*, **48**(1976), 235–247.
15. NEUMARK, M. A., *Lineare Differentialoperatoren*, Akademie-Verlag, Berlin, 1967.
16. SVENDSEN, E. C., The effect of submanifolds upon essential self-adjointness and deficiency indices, *J. Math. Anal. Appl.*, **80**(1981), 551–565.
17. WEIDMANN, J., Perturbations of self-adjoint operators in $L_2(G)$ with applications to differential operators, *Lecture Notes in Mathematics*, **415**(1974), pp. 268–281, Berlin—Heidelberg — New York, Springer-Verlag.

18. WEIDMANN, J., Spectral theory of partial differential operators, *Lecture Notes in Mathematics* **448**(1974), pp. 71–111, Berlin–Heidelberg–New York, Springer-Verlag.
19. WEIDMANN, J., *Linear operators in Hilbert spaces*, Springer-Verlag, New York–Heidelberg–Berlin, 1980.
20. WOLF, F., On the essential spectrum of partial differential boundary problems, *Comm. Pure Appl. Math.*, **12**(1959), 211–228.
21. WÜST, R., Distinguished self-adjoint extensions of Dirac operators constructed by means of cut-off potentials, *Math. Z.*, **141**(1975), 93–98.
22. WÜST, R., Dirac operators with strongly singular potentials, *Math. Z.*, **151**(1977), 259–271.

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