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HOMOTOPY INVARIANCE OF THE ANALYTIC INDEX OF SIGNATURE OPERATORS OVER C^*-ALGEBRAS

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INTRODUCTION

A proof of the Novikov conjecture for finitely generated free abelian groups was given by Lusztig in [6] using the Atiyah-Singer Index Theorem for families of operators. It was observed by Miscenko [11] that a family of elliptic operators parametrized by a compact space X can be viewed as a single operator over the C^*-algebra $C(X)$. In [6] the space X was the n-dimensional torus, $\mathbf{T}^n$, and the operators were signature operators. Since $C(\mathbf{T}^n) = C^*_r(\mathbf{Z}^n)$ Miscenko was led to consider signature operators over the algebras $C^*_r(\Gamma)$, where Γ is a countable discrete group.

In this paper we generalize some of the results in [6] to the non-commutative case. The methods used are based on [6] and [14] and on the theory of Hilbert modules as developed by several people (e.g. [10], [8]). This latter theory allows great simplification of previous work on these questions. Our goal was to prove the homotopy invariance of the signature operator (considered as an element of K-homology) on a closed manifold. This result is expressed in the form of a commuting diagram which, in current terminology, relates the Novikov Conjecture to the Strong Novikov Conjecture [13]. Our main result, Theorem 4.1, corresponds roughly to Theorem 3 (p. 24, Part 2) of Kasparov's Conspectus [7]. In Section 6 we indicate the changes necessary to handle algebras without unit. This has applications to homotopy invariance properties of the signature operator along the leaves of a foliation of a compact manifold, as suggested to us by Paul Baum and Alain Connes. We briefly sketch how a Theorem of theirs also follows from our results.

1. PRELIMINARIES ON HILBERT MODULES

DEFINITION 1.1. Let A be a C^*-algebra with or without unit. A Hilbert A-module is a complex vector space, M, which is a right A-module provided with an A-valued inner product,

$$(\, , \,) : M \times M \to A$$

satisfying

i) $(\, , \,)$ is sesquilinear over $\mathbf{C}$;
ii) $(x, ya) = (x, y)a$;
iii) $(x, y)^* = (y, x)$;
iv) $(x, x) \geqslant 0$ and $(x, x) = 0$ if and only if $x = 0$;
v) M is complete in the norm $\|(x, x)\|^{1/2} = \|x\|$.

REMARK. We assume that A is separable and all modules are countably generated. If A has a unit then unitary modules are to be understood.

DEFINITION 1.2. Let M and N be Hilbert A-modules. Then $L(M, N)$ is the set of linear A-module maps, T, such that there is a $T^* : N \to M$ satisfying $(Tx, y) = (x, T^*y)$ for all $x \in M$ and $y \in N$.

Every $T \in L(M, N)$ is bounded. The set $L(M, M) = L(M)$ is a C^*-algebra. Even if not explicitly stated, all maps between A-modules are required to have adjoints.

DEFINITION 1.3. If $x \in N$, $y \in M$, define $\theta_{x,y} \in L(M, N)$ by $\theta_{x,y}(z) = x(y, z)$. The *compact maps* are the closed linear span of the $\theta_{x,y}$. They are denoted $K(M, N)$.

The set $K(M, N)$ is a $L(M)$-$L(N)$ bimodule and $K(M, M) = K(M)$ is a closed two-sided ideal in $L(M)$. A map which factors through a finitely generated module is compact.

Let $H_A = \ell_2 \otimes A$, the Hilbert space of A. The following Stabilization Theorem of Kasparov will be needed later.

PROPOSITION 1.4. [8]. *If M is countably generated then $M \oplus H_A \cong H_A$.*

We will have need of the following theorem of Miscenko.

PROPOSITION 1.5. [11]. *If $f \in L(M, N)$ is surjective then there is a self-adjoint projection $p : M \to \ker(f)$. Moreover, there is a map $j : N \to M$ satisfying $fj = 1_N$.*

Miscenko's proof appears to have a gap but Bill Phillips, Larry Brown and Maurice Dupré have provided a correct version.

DEFINITION 1.6. If M is an A-module then the *dual module* is $M' = L(M, A)$, with the A-module structure provided by $(fa)(x) = a^*f(x)$.

In the unital case the inner product, $(\, , \,)$, induces an isomorphism $\psi : M \to M'$, where $\psi(x)(y) = (x, y)$. The inverse is given by $\psi^{-1}(f) = f^*(1)$.

PROPOSITION 1.7. *The sets $L(M, N)$ and $K(M, N)$ are independent of the A-valued inner products on M and N, provided that the inner products give equivalent norms.*

Proof. We may assume that A has a unit. If not consider modules over A as modules over A^+, with $1 \in A^+$ acting as the identity. The sets $L(M, N)$ and $K(M, N)$ remain the same.

Let $T : M \to N$ be given. Suppose that N has inner products ψ_1 and ψ_2 and T has an adjoint, T^*, with respect to ψ_1. Then the adjoint of T with respect to ψ_2 is $T^*(\psi_1)^{-1}\psi_2$. Similarly, if φ_1 and φ_2 are metrics on M and T^* is the adjoint of T with respect to φ_1, then $(\varphi_2)^{-1}\varphi_1 T^*$ is the adjoint with respect to φ_2. For the compacts, the map $\theta_{x,y}(z)$ defined using φ_1 becomes $\theta_{x,\varphi_1^{-1}\varphi_2(y)}(z)$ with respect to φ_2.

A consequence of Proposition 1.7 is that projections onto and inclusions of topological direct summands have adjoints. Further, a topological direct summand has an orthogonal complement with respect to any metric. This follows by applying Proposition 1.5 to the projection onto a topological complement.

DEFINITION 1.8. If $T \in L(M, N)$ is invertible modulo $K(M, N)$ in the sense that there is an $S \in L(M, N)$ satisfying $ST - I \in K(M)$ and $TS - I \in K(N)$, then T is *Fredholm*. The Fredholm maps are denoted $F(M, N)$.

There is a map, called the analytic index, Ind: $F(M, N) \to \mathrm{K}_0(A)$ satisfying
(i) $\mathrm{Ind}(T) = 0$ if T is an isomorphism;
ii) $\mathrm{Ind}(T_1 \oplus T_2) = \mathrm{Ind}(T_1) + \mathrm{Ind}(T_2)$;
iii) If $T_1 \simeq T_2$ then $\mathrm{Ind}(T_1) = \mathrm{Ind}(T_2)$, where "$\simeq$" denotes homotopy;
iv) If A has a unit, and M, N are finitely generated, then $\mathrm{Ind}(T) = [M] - [N]$;
v) If $f : A \to B$ is a homomorphism, then there is a map $f_\#$, defined by sending T to $T \otimes_A I$, making the following diagram commute

$$\begin{array}{ccc} F(M, N) & \xrightarrow{f_\#} & F(M \otimes_A B, N \otimes_A B) \\ \downarrow{\scriptstyle \mathrm{Ind}} & & \downarrow{\scriptstyle \mathrm{Ind}} \\ \mathrm{K}_0(A) & \xrightarrow{f_*} & \mathrm{K}_0(B) \end{array}$$

In (iii), "homotopy" means a norm continuous path.

2. HERMITIAN COMPLEXES

From this section on A is assumed to have a unit unless expicitely stated otherwise.

DEFINITION 2.1. A *Hermitian complex* is a bounded co-chain complex of Hilbert A-modules $\mathscr{C} = \{\mathscr{C}^i, d^i\}_{|i|<n}$ with a Hermitian product $\langle\ ,\ \rangle : \mathscr{C}^i \times \mathscr{C}^{-i} \to A$, satisfying
i) $\langle x, ya\rangle = \langle x, y\rangle a$;
ii) $\langle x, y\rangle^* = \langle y, x\rangle$;
iii) $\langle d^i x, y\rangle = \langle x, d^{-i-1} y\rangle$.

Moreover, if $\mathscr{C}'$ is the complex with $(\mathscr{C}')^i = (\mathscr{C}^{-i})'$ and $(d')^i = (d^{-i-1})'$, then $\langle\ ,\ \rangle$ induces a chain map $\varphi : \mathscr{C} \to \mathscr{C}'$ which is required to be a chain homotopy equivalence.

If one is given a chain homotopy equivalence of complexes of A-modules, $\varphi : \mathscr{C} \to \mathscr{C}'$ which is symmetric in the sense that

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ \varphi\ } & \mathscr{C}' \\ & \searrow \quad \nearrow{\scriptstyle \varphi'} & \\ & \mathscr{C}'' & \end{array}$$

commutes, then setting $\langle x, y\rangle = \varphi(x)(y)$ yields a Hermitian complex, where φ' is the transpose of φ. In general we denote a Hermitian complex by $(\mathscr{C}, \varphi)$.

DEFINITION 2.2. A *homotopy equivalence* of Hermitian complexes, $h : (\mathscr{C}, \varphi) \to$ $\to (\mathscr{D}, \psi)$, is a chain homotopy equivalence for which the following diagram commutes up to chain homotopy

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ \varphi\ } & \mathscr{C}' \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle h'} \\ \mathscr{D} & \xrightarrow{\ \psi\ } & \mathscr{D}' \end{array}.$$

DEFINITION 2.3. A Hermitian complex $(\mathscr{C}, \varphi)$ is *regular* if $\varphi : \mathscr{C} \to \mathscr{C}'$ is an isomorphism.

We will need a non-commutative analog of a theorem of Lusztig [6].

PROPOSITION 2.4. *Let $(\mathscr{C}, \varphi)$ be an Hermitian complex. Then there exists a regular complex $(\tilde{\mathscr{C}}, \tilde{\varphi})$ and an adjointable injection $i : \mathscr{C} \to \tilde{\mathscr{C}}$ which is a chain homotopy equivalence admitting a left inverse and which satisfies $\varphi = i'\tilde{\varphi}i$.*

Proof. The proof in [6, Proposition 1.3] carries over if one uses Proposition 1.5 at several points.

DEFINITION 2.5. A *grading* of a Hermitian complex $(\mathscr{C}, \varphi)$ is an involution $\tau : \mathscr{C} \to \mathscr{C}$ satisfying

i) $\tau^i : \mathscr{C}^i \to \mathscr{C}^{-i}$;

ii) $\tau^2 = I$;

iii) $\tau'\varphi = \varphi\tau$;

iv) $\varphi(\tau(x))(x) > 0$ if $x \neq 0$.

Then, denoting $(x, y) = \langle\tau(x), x\rangle$, we have $(\tau(x), y) = (x, \tau(y))$ and $(d(x), y) =$ $= (x, \tau d\tau(y))$. That is, with respect to $(\ ,\ )$, τ is self-adjoint and the adjoint of d is $\tau d\tau$.

PROPOSITION 2.6. *A regular Hermitian complex admits a grading. Any two gradings are homotopic through gradings.*

Proof. We follow Karoubi [5]. Let φ be the Hermitian structure and let ψ be the A-valued inner product on $\mathscr{C}^0$. Let $k = \psi^{-1}\varphi$. Considered as an element of the C^*-algebra $L(\mathscr{C}^0)$, k is self-adjoint and 0 is not in its spectrum. Define p_+ and p_- by the formula

$$p_{\pm} = \frac{1}{2\pi i}\int_{\gamma_{\pm}} \frac{dz}{z-k}$$

where γ_+ and γ_- are curves surrounding the positive and negative parts of the spectrum. Set $\tau = p_+ - p_-$ on $\mathscr{C}^0$. Let $(\mathscr{C}^0)_+$ and $(\mathscr{C}^0)_-$ be the $+i$ and $-i$ eigenspaces for τ. Note that φ is positive definite on one and negative definite on the other, and that they provide an orthogonal decomposition of $\mathscr{C}^0$ with respect to ψ. It is then readily checked that τ is the required grading on $\mathscr{C}^0$. Outside the middle dimension the grading is determined by $\tau = \psi^{-1}\varphi : \mathscr{C}^i \to \mathscr{C}^{-i}$ for $i > 0$.

For the uniqueness up to homotopy, note first that every grading τ arises from the above constructions, for if they are applied to the inner product $\varphi\tau$ we recover τ. Since the constructions depend continuously on the inner product it is enough to show that any two of them, ψ_1 and ψ_2, are homotopic. Applying Kasparov's Stability Theorem reduces one to the case where the module is H_A. The element $\psi_1^{-1}\psi_2$ is self-adjoint with respect to ψ_1 with positive spectrum. Let s be its positive square root. Then $\psi_1(s(x))(s(y)) = \psi_2(x)(y)$. Since s is invertible, the contractibility of the invertibles in $L(H_A)$, [9], yields the result.

Another non-commutative analog of a theorem of Lusztig [6] which we will have need of is the following.

PROPOSITION 2.7. *Let $h : \mathscr{C} \to \mathscr{D}$ be a chain map which induces isomorphisms on homology. Then h is a chain homotopy equivalence.*

Proof. The standard proof works if one makes use of Proposition 1.5.

3. FREDHOLM COMPLEXES

DEFINITION 3.1. A co-chain complex of Hilbert modules, $\mathscr{C}$, is a *Fredholm complex* if it is chain contractible modulo the compacts. That is, there is a degree -1 map $h : \mathscr{C} \to \mathscr{C}$ satisfying $h^{i+1}d^i + d^{i-1}h^i = I + k$ where k is compact.

For convenience we will write $f \approx g$ if $f - g$ is compact. Note that if $\mathscr{C}$ is finitely generated then it is Fredholm. Conversely we have the analog of a theorem of Segal [14, Theorem 3.3].

PROPOSITION 3.2. *A Fredholm complex $\mathscr{C}$ over a C^*-algebra with unit is chain equivalent to a finitely generated complex.*

Proof. Assume inductively that $\mathscr{C}^i$ is finitely generated for $i < n$ and consider the chain map

$$\begin{array}{ccccccccc}
\longrightarrow \mathscr{C}^{n-1} & \xrightarrow{d^{n-1}} & \mathscr{C}^{n} & \xrightarrow{d^{n}} & \mathscr{C}^{n+1} & \xrightarrow{d^{n+1}} & \mathscr{C}^{n+2} \longrightarrow \\
\downarrow & & \downarrow{\scriptstyle h^{n+1}d^{n}} & & \downarrow{\scriptstyle h^{n+1}} & & \downarrow \\
0 & \longrightarrow & \mathscr{C}^{n} & \xrightarrow{I} & \mathscr{C}^{n} & \longrightarrow & 0
\end{array}$$

Since $d^{n-1}h^n$ factors through a finitely generated module it is compact, and $h^{n+1}d^n \approx \approx h^{n+1}d^n + d^{n-1}h^n \approx I$ is Fredholm. By taking the sum of each row with the elementary complex $0 \to H_A \to H_A \to 0$ we may assume that $\mathscr{C}^n \cong H_A$. According to Miscenko and Fomenko [11], hd has a decomposition $(hd)_1 \oplus (hd)_2 : M^n \oplus N^n \to P \oplus Q$, where $(hd)_1$ is an isomorphism and N^n and Q are finitely generated. Let p be the projection of $\mathscr{C}^n$ onto P. Then $\ker(phd) = N^n$. Let M^{n+1} be the image of M^n under d and $N^{n+1} = \ker(ph)$. Then $\mathscr{C}^{n+1} = M^{n+1} \oplus N^{n+1}$, since if i is the inclusion of P, $d((hd)_1^{-1} \oplus 0)i$ splits ph and has image M^{n+1}. It follows that $d : M^n \oplus N^n \to \to M^{n+1} \oplus N^{n+1}$ is in diagonal form and an isomorphism on the first summands. $\mathscr{C}$ is thus homotopy equivalent to

$$\longrightarrow \mathscr{C}^{n-1} \longrightarrow N^n \longrightarrow N^{n+1} \longrightarrow \mathscr{C}^{n+2} \longrightarrow$$

which is finitely generated in degree n.

DEFINITION 3.2. Let $(\mathscr{C}, \varphi)$ be a regular Hermitian Fredholm complex and choose a grading τ. The *signature operator of* $\mathscr{C}$ is $D_{\mathscr{C}} = (d - \tau d\tau)|\mathscr{C}^+$. Then $D_{\mathscr{C}} : \mathscr{C}^+ \to \mathscr{C}^-$, where $\mathscr{C}^+$ and $\mathscr{C}^-$ are the $+1$ and -1 eigenspaces for τ.

THEOREM 3.3. *If $(\mathscr{C}, \varphi)$ is a graded regular Hermitian Fredholm complex, then $D_{\mathscr{C}}$ is a Fredholm operator.*

Proof. We have a chain homotopy, h, with $hd + dh \approx I$. From this it follows that dh and hd are idempotents modulo $K(\mathscr{C})$, the compacts. By the argument of [1, p. 149] applied to the image of dh in $L(\mathscr{C})/K(\mathscr{C})$ we may obtain a self-adjoint projection $\bar{p}$ whose lift back to $L(\mathscr{C})$, p, satisfies $p^2 \approx p, p^* \approx p, pdh \approx dh$, and $dhp \approx p$. Define a new chain homotopy by $\bar{h} = (I - p)hp$. We will show that $\bar{h} - \tau\bar{h}\tau$ is a parametrix for $d - \tau d\tau$. The computation requires a number of identities.

i) $dp \approx 0$ and $d \approx d(I - p)$.

ii) $(I - p)hd \approx (I - p)$.

iii) $(I - p)d \approx 0$ and $d \approx pd$.

iv) $\bar{h}d + d\bar{h} \approx I$.

This follows from (i), (ii), and (iii).

v) $d\tau\bar{h} \approx 0$ and $\bar{h}\tau d \approx 0$.
This will follow from $p\tau d \approx 0$ and $d\tau(I-p) \approx 0$. For the former $p\tau d\tau \approx p^* d^* = (dp)^* \approx (d(I-p)p)^* \approx 0$ and the result follows since τ is an isomorphism. The second part is similar.

vi) $p\tau p \approx 0$ and $(I-p)\tau(I-p) \approx 0$.
For the first $p\tau p \approx p\tau dhp \approx 0$ and the second is similar.

vii) $\tau p \approx (I-p)\tau$.
This follows from (vi).

viii) $\tau d\bar{h} \approx \bar{h}d\tau$.
For this $\tau d(I-p)hp \approx \tau dhp \approx \tau p \approx (I-p)\tau \approx (I-p)hd\tau \approx (I-p)hpd\tau$.

Now we compute using (iv), (v), and (viii) $(d-\tau d\tau)(\bar{h}-\tau\bar{h}\tau) = d\bar{h}-(d\tau\bar{h})\tau-\tau(d\tau\bar{h})+\tau d\bar{h}\tau \approx d\bar{h}+\bar{h}d\tau^2 = d\bar{h}+\bar{h}d \approx I$. Similarly $(\bar{h}-\tau\bar{h}\tau)(d-\tau h\tau) \approx I$. Finally note that $D_{\mathscr{C}}$ is the composition of the inclusion of $\mathscr{C}^+$ into $\mathscr{C}$, $d-\tau d\tau$, and the projection of $\mathscr{C}$ onto $\mathscr{C}^-$ and it readily follows that $D_{\mathscr{C}}$ is Fredholm.

COROLLARY 3.4. *Let $(\mathscr{C}, \varphi)$ be a regular Hermitian graded Fredholm complex. If $(\mathscr{C}, \varphi)$ is contractible, then* $\operatorname{Ind}(D_{\mathscr{C}}) = 0$.

Proof. Since $\mathscr{C}$ is contractible there is a chain homotopy h with $dh + hd = I$. The proof of Proposition 3.3 may now be executed with equality, "$=$", replacing "$\approx$". The conclusion is that $D_{\mathscr{C}}$ is an isomorphism and hence $\operatorname{Ind}(D_{\mathscr{C}}) = 0$.

The next step is to show that $\operatorname{Ind}(D_{\mathscr{C}})$ is independent of the choice of the grading τ. To accomplish this requires some naturality properties of the index. Let $f: B \to A$ be a homomorphism of C^*-algebras and let $\mathscr{E}$ be a complex over B. Let $f_\#(\mathscr{E})$ be the induced complex over A, with $f_\#(\mathscr{E})^i = \mathscr{E}^i \otimes_B A$. If $\mathscr{E}$ is a graded Hermitian Fredholm complex, then it is a direct check that $f_\#(\mathscr{E})$ has these same properties. If $f_\#: F_B(\mathscr{E}^+, \mathscr{E}^-) \to F_A(f_\#(\mathscr{E}^+), f_\#(\mathscr{E}^-))$ is the induced maps on the Fredholms, then $f_\#(D_{\mathscr{E}}) = D_{f_\#(\mathscr{E})}$. Hence, by 1.8(v), we obtain $f_\#(\operatorname{Ind}(D_{\mathscr{E}})) = \operatorname{Ind}(D_{f_\#(\mathscr{E})})$.

DEFINITION 3.5. Let $(\mathscr{C}, \varphi)$ be a regular Fredholm complex. Choose any grading τ, and let $D_{\mathscr{C}}$ be the associated signature operator. Define the *signature* of $(\mathscr{C}, \varphi)$ to be $\operatorname{Sign}(\mathscr{C}, \varphi) = \operatorname{Ind}(D_{\mathscr{C}})$. We will usually drop the φ and denote this by $\operatorname{Sign}(\mathscr{C})$.

PROPOSITION 3.6. $\operatorname{Sign}(\mathscr{C})$ *is independent of the choice of τ.*

Proof. By Proposition 2.6 any two gradings τ_0 and τ_1 are connected by a path of gradings τ_t. Let $\tilde{\mathscr{C}}$ be the complex over $\tilde{A} = C(I, A)$ with $\tilde{\mathscr{C}} = C(I, \mathscr{C}^i)$. One defines $\tilde{d}$, $\tilde{h}$, and $\tilde{\varphi}$ in the obvious ways (e.g. $\tilde{d}^i(\tilde{x})(t) = d^i(\tilde{x}(t))$). Then $\tilde{\mathscr{C}}$ is a regular Hermitian complex. Moreover, $\tilde{d}\tilde{h} + \tilde{h}\tilde{d} = \tilde{I} + \tilde{k}$, with $\tilde{k}$ a constant path in

$C(I, K(\mathscr{C})) = K(\tilde{\mathscr{C}})$ so that $\tilde{\mathscr{C}}$ is Fredholm. By Proposition 3.3, $\mathrm{Ind}(D_{\tilde{\mathscr{C}}})$ is defined as an element of $K_0(\tilde{A})$. There are evaluation maps $e_0, e_1 : \tilde{A} \to A$. Since e_0 is homotopic to e_1 we have $(e_0)_*(\mathrm{Ind}(D_{\tilde{\mathscr{C}}})) = (e_1)_*(\mathrm{Ind}(D_{\tilde{\mathscr{C}}}))$ in $K_0(A)$. Since $(e_t)_*(\tilde{\mathscr{C}})$ is $\mathscr{C}$ with the grading τ_t we obtain the desired result.

Following Lusztig, [6], we will associate an element $\sigma(\mathscr{C}, \varphi) \in K_0(A)$ to a regular, finitely generated complex. It is the image of the symmetric signature of Miscenko [10] and Ranicki [12] under a natural map which will be discussed in more detail in Section 5.

Using the argument of Proposition 2.6 we may express $\mathscr{C}^0 = (\mathscr{C}^0)_+ \oplus (\mathscr{C}^0)_-$, where φ is positive definite on $(\mathscr{C}^0)_+$ and negative definite on $(\mathscr{C}^0)_-$.

DEFINITION 3.7. If $\mathscr{C}$ is a finitely generated regular Hermitian complex then $\sigma(\mathscr{C}, \varphi) = [(\mathscr{C}^0)_+] - [(\mathscr{C}^0)_-]$.

That $\sigma(\mathscr{C}, \varphi)$ is independent of the decomposition follows from the next proposition.

PROPOSITION 3.8. *If $(\mathscr{C}, \varphi)$ is a finitely generated Hermitian complex, then* $\sigma(\mathscr{C}, \varphi) = \mathrm{Sign}(\mathscr{C})$.

Proof. Define a grading on $\mathscr{C}^0$ to be multiplication by 1 on $(\mathscr{C}^0)_+$ and multiplication by -1 on $(\mathscr{C}^0)_-$ and extend it to $\mathscr{C}$. Since $\mathscr{C}$ is finitely generated $\mathrm{Sign}(\mathscr{C}) = [\mathscr{C}^+] - [\mathscr{C}^-]$. If M is the sum of the $\mathscr{C}^i$ in non-zero degrees, one may check that $M^+ \cong M^-$ so that $\mathrm{Sign}(\mathscr{C}, \varphi) = [(\mathscr{C}^0)_+] - [(\mathscr{C}^0)_-]$.

We note for the next section the following fact.

PROPOSITION 3.9. *If $f: \mathscr{C} \to \mathscr{D}$ is a chain equivalence and $\mathscr{C}$ is Fredholm, then $\mathscr{D}$ is Fredholm also.*

Proof. Let $g: \mathscr{C} \to \mathscr{D}$ be a homotopy inverse to f. Note that $\mathscr{C}$ is Fredholm if and only if $I_{\mathscr{C}} \simeq k$, k compact. (Here "$\simeq$" denotes chain homotopy.) Thus, $I_{\mathscr{D}} \simeq fg = f(I_{\mathscr{C}})g \simeq fkg$, which is compact, so $\mathscr{D}$ is Fredholm.

4. DEFORMATIONS OF FREDHOLM COMPLEXES

In this section we prove the main theorem. This result and Proposition 2.4 allow the signature to be defined as a homotopy invariant of non-regular complexes, as observed by Lusztig.

THEOREM 4.1. *Let $(\mathscr{C}, \varphi)$ and $(\mathscr{C}, \psi)$ be regular Hermitian Fredholm complexes. Let $f: \mathscr{C} \to \mathscr{D}$ be a homotopy equivalence of Hermitian complexes. Then* $\mathrm{Sign}(\mathscr{C}) = \mathrm{Sign}(\mathscr{D})$.

Proof. For each $t, 0 \leqslant t \leqslant 1$, we construct a Fredholm complex, $(\mathscr{E}_t, \varphi_t)$. The modules $\mathscr{E}_t$ will not change, but the differentials and the Hermitian structure be deformed. Given $f : \mathscr{C} \to \mathscr{D}$ consider the maps, for $0 \leqslant t \leqslant 1/2$, $g_t : \mathscr{C} \to \mathscr{C} \oplus \mathscr{D}$, defined by $g_t(x) = ((\cos \pi t)x, (\sin \pi t)f(x))$. The complex $\mathscr{E}_t$ is the mapping cone of g_t. Thus

$$\mathscr{E}_t^i = \mathscr{C}^i \oplus \mathscr{C}^{i-1} \oplus \mathscr{D}^{i-1} \qquad d_t^i = \begin{bmatrix} -d_{\mathscr{C}}^i & 0 & 0 \\ (\cos \pi t)I & d_{\mathscr{C}}^{i-1} & 0 \\ (\sin \pi t)I & 0 & d_{\mathscr{D}}^{i-1} \end{bmatrix}.$$

We will define a Hermitian structure on $\mathscr{E}_t$ by the following device [14]. Consider each row of the diagram

$$\begin{array}{ccccc} \mathscr{C} & \xrightarrow{g_t} & \mathscr{C} \oplus \mathscr{D} & \longrightarrow & 0 \\ \downarrow & & \downarrow T_t & & \downarrow \\ 0 & \longrightarrow & \mathscr{C}' \oplus \mathscr{D}' & \xrightarrow{g_t'} & \mathscr{C}' \end{array}$$

to be a double complex. The map T_t is defined by

$$T_t = \begin{bmatrix} (\sin^2 \pi t) f' \psi f & -(\sin \pi t)(\cos \pi t)\psi f \\ -(\sin \pi t)(\cos \pi t)\psi f & (\cos^2 \pi t)\psi \end{bmatrix}.$$

This induces a map of double complexes. The mapping cones of g_t and g_t' are the ordinary complexes obtained from these double complexes. Thus, T_t induces a map $\theta_t : \mathscr{E}_t \to \mathscr{E}_t'$. One checks directly that θ_t is symmetric. It remains to show that θ_t is a chain equivalence. For this consider

$$\begin{array}{ccccc} \mathscr{D} & \xrightarrow{h_t} & \mathscr{D} \oplus \mathscr{D} & \longrightarrow & 0 \\ \downarrow & & \downarrow S_t & & \downarrow \\ 0 & \longrightarrow & \mathscr{D}' \oplus \mathscr{D}' & \xrightarrow{h_t'} & \mathscr{D}' \end{array}$$

where $h_t(x) = ((\cos \pi t)x, (\sin \pi t)x)$, and

$$S_t = \begin{bmatrix} (\sin^2 \pi t)\psi & -(\sin \pi t)(\cos \pi t)\psi \\ -(\sin \pi t)(\cos \pi t)\psi & (\cos^2 \pi t)\psi \end{bmatrix}.$$

Let $\mathscr{F}_t$ be obtained in the same way that $\mathscr{E}_t$ was above. Let $P : \mathscr{F}_t \to \mathscr{F}_t$ be the chain map defined by

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \psi^{-1} & 0 \\ 0 & 0 & \psi^{-1} \end{bmatrix}$$

and set

$$H_t = \begin{bmatrix} 0 & \cos \pi t & \sin \pi t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $d_t H_t + H_t d_t = I - P\bar{S}_t$, where

$$\bar{S}_t = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{S_t}} \\ 0 & & \end{array}\right].$$

Similarly, one gets G_t so that $d_t G_t + G_t d_t = I - \bar{S}_t P$. Thus $\bar{S}_t$ is a chain equivalence. Since $\theta_t = (f' \oplus I) S_t (f \oplus I)$ and f is an equivalence we obtain that θ_t is one also.

Define $(\mathscr{E}_t, \theta_t)$ for $1/2 \leqslant t \leqslant 1$ by $\mathscr{E}_t = \mathscr{E}_{1/2}$ and $\theta_t = (2 - 2t) f'\psi f \oplus (2t - 1)\varphi$. Since $f'\psi f$ is chain homotopic to φ, it follows that θ_t is a continuous family of chain equivalences which will yield the required family of Hermitian forms.

We have constructed a continuous family of Hermitian complexes where the Hermitian forms and differentials change, but the modules remain fixed. Define a single complex, $\tilde{\mathscr{E}}$, over $\tilde{A} = C(I, A)$ by $\tilde{\mathscr{E}}^i = C(I, \mathscr{E}^i)$, and $\tilde{d}^i(\tilde{x})(t) = d^i(\tilde{x}(t))$. A Hermitian structure is defined by $\tilde{\theta}(\tilde{x})(\tilde{y})(t) = \theta_t(\tilde{x}(t), \tilde{y}(t))$. We check that $\tilde{\mathscr{E}}$ is a Fredholm complex. For this it is enough to show that $\tilde{\mathscr{E}}$ is homotopy equivalent to a Fredholm complex. But, letting $Q_t = P\bar{S}^t$ for $0 \leqslant t \leqslant 1/2$ and $Q_t = Q_{1/2}$, for $1/2 \leqslant t \leqslant 1$ we may obtain, as above, a map $\tilde{Q} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}$ satisfying $\tilde{Q} \simeq I_{\tilde{\mathscr{E}}}$. Now, define $j : C(I, \mathscr{D}) \to \tilde{\mathscr{E}}$ by $j(\alpha)(t) = (0, -(\sin \pi t)\alpha(t), (\cos \pi t)\alpha(t))$, where $C(I, \mathscr{D})$ is a Hermitian $\tilde{A}$ complex in the natural way. Then one checks that j is an isomorphism onto the image of $\tilde{Q}$. It follows from this that $j : C(I, \mathscr{D}) \to \tilde{\mathscr{E}}$ is a homotopy equivalence with homotopy inverse $j^{-1}\tilde{Q}$. Since $C(I, \mathscr{D})$ is Fredholm, we are done.

We now have a Fredholm complex, $(\tilde{\mathscr{E}}, \tilde{\theta})$, over $\tilde{A}$. Let $(e_t)_\#(\tilde{\mathscr{E}})$, $t = 0, 1$, be its restrictions. These may be identified with the mapping cones of $g_0(x) = (x, 0)$ and $g_1(x) = (0, f(x))$ respectively. Since the mapping cone of an equivalence is contractible, they may be expressed as $\mathscr{D} \oplus V$ and $W \oplus \mathscr{C}$ where V and W are contractible. Let $(\tilde{R}, \tilde{\chi})$ be a regularization of $(\tilde{\mathscr{E}}, \tilde{\theta})$. By Proposition 3.3 $\operatorname{Sign}(\tilde{R}) \in K_0(\tilde{A})$, and $\operatorname{Sign}((e_0)_\#(\tilde{R})) = \operatorname{Sign}((e_1)_\#(\tilde{R}))$.

For each t, $(e_t)_\#(\tilde{R})$ is a regularization of $(e_t)_\#(\tilde{\mathscr{E}})$ so we may write $(e_0)_\#(\tilde{R}) = \mathscr{D} \oplus V'$ and $(e_1)_\#(\tilde{R}) = W' \oplus \mathscr{C}$, with V' and W' contractible. Let the induced gradings and forms on these be τ_t and χ_t, $t = 0, 1$. There exists an orthogonal complement to $\mathscr{D}$ in $(e_0)_\#(\tilde{R})$ with respect to the inner product $\chi_0\tau_0$. Applying τ_0 to this gives $\mathscr{D}^\perp$, the orthogonal complement with respect to χ_0, which is a complex. Define a new grading on $(e_0)_\#(\tilde{R})$ to be the sum of any gradings on $\mathscr{D}$ and $\mathscr{D}^\perp$. Since $\mathscr{D}^\perp$ is contractible we have $\mathrm{Sign}(\mathscr{D}^\perp) = 0$, and by evident additivity $\mathrm{Sign}((e_0)_\#(\tilde{R})) = \mathrm{Sign}(\mathscr{D})$. Similarly $\mathrm{Sign}((e_1)_\#(\tilde{R})) = \mathrm{Sign}(\mathscr{C})$, so $\mathrm{Sign}(\mathscr{C}) = \mathrm{Sign}(\mathscr{D})$.

REMARK. The conclusion of Theorem 4.1 still holds when A is non-unital provided the complexes considered are quasi-regular. (See Section 6 for the notion of quasi-regular.)

5. APPLICATIONS

In this section we outline some of the applications of the theory of the previous sections. Let M^{2k} be a smooth closed manifold, oriented and of even dimension, with fundamental group $\pi_1(M)$. Given a homomorphism of $\pi_1(M)$ into a discrete group Γ let $p : \tilde{M} \to M$ be the corresponding covering. We will denote by ψ^Γ the associated flat bundle with fiber $C_r^*(\Gamma)$, the reduced C^*-algebra of Γ. We will define below the signature operator of M with coefficients in ψ^Γ, D^Γ. It is an elliptic $C_r^*(\Gamma)$--operator in the sense of Miscenko and Fomenko. Theorem 4.1 will be used to identify its analytic index.

To this end fix a smooth triangulation of M. Lusztig [6, p. 247] gives the simplicial cochains of M with coefficients in the bundle ψ^Γ the structure of a finitely generated Hermitian complex over $C_r^*(\Gamma)$. This requires symmetrizing the cup product and multiplying the product and differentials by i in odd degrees. Call the result $C^*(M, C_r^*(\Gamma))$.

Miscenko and Ranicki [12] have associated to the simplicial cochains of M an invariant $\bar{\sigma}$, the *symmetric signature*, lying in the symmetric surgery group $L^{2k}(\mathbf{Z}[\Gamma])$. We sketch its relationship to the σ-invariant of 3.7. The inclusion $\mathbf{Z}(\Gamma) \to C_r^*(\Gamma)$ defines a map $L^{2k}(\mathbf{Z}[\Gamma]) \to L^{2k}(C_r^*(\Gamma))$. Since 2 is invertible in $C_r^*(\Gamma)$ there are isomorphisms $L^{2k}(C_r^*(\Gamma)) \approx L^i(C_r^*(\Gamma))$, where $i = 0$ for k even and $i = 2$ for k odd. The latter groups are respectively the Witt groups of Hermitian and skew-Hermitian $C_r^*(\Gamma)$-valued forms on finitely generated free $C_r^*(\Gamma)$-modules F. There are isomorphisms $L^i(C_r^*(\Gamma)) \approx \mathrm{K}_0(C_r^*(\Gamma))$. For $i = 0$ this is $\sigma(F, \varphi)$, for $i = 2$ $\sigma(F, i\varphi)$. If we denote the composition of the above homomorphisms by $m : L^{2k}(\mathbf{Z}[\Gamma]) \to \mathrm{K}_0(C_r^*(\Gamma))$ then $m\bar{\sigma}(C^*(M, \mathbf{Z}[\Gamma])) = \sigma(C^*(M, C_r^*(\Gamma)))$. We will refer to this element as the surgery obstruction of Miscenko and Ranicki.

Let $\Omega^*(M, \psi^\Gamma)$ denote the smooth forms on M with values in ψ^Γ, modified as above. Complete $\Omega^p(M, \psi^\Gamma)$ with respect to the norm $\|f\|_s = \|\langle (I + \triangle)^s f, f\rangle\|_{C_r^*(\Gamma)}$, $s \in \mathbf{Z}^+$, where $\triangle$ is the Laplacian. Denote the resulting Hilbert module by $H^{p,s}(M)$. The differentials extend as adjointable maps and we have a Hermitian Fredholm complex H^*,

$$\longrightarrow H^{p,s+1}(M) \longrightarrow H^{p+1,s}(M) \longrightarrow H^{p+2,s-1}(M) \longrightarrow \quad .$$

Let D^Γ be the signature operator of H^* as defined in 3.5.

A version of the following theorem is stated in the literature, but only modulo torsion [11].

THEOREM 5.1. *The analytic index of D^Γ is the surgery obstruction of Miscenko and Ranicki.*

Proof. By Proposition 3.7, $\sigma(C^*(M, C_r^*(\Gamma))) = \mathrm{Sign}(C^*(M, C_r^*(\Gamma)))$. Thus it suffices to show that $\mathrm{Sign}(C^*(M, C_r^*(\Gamma))) = \mathrm{Ind}(D^\Gamma)$. Consider the maps of complexes

$$C^*(M, C_r^*(\Gamma)) \xrightarrow{i} \Omega^*(M, \psi^\Gamma) \xrightarrow{j} H^*(M).$$

The appropriate Poincaré lemma holds for each complex, so i and j induce isomorphisms on homology by the classical de Rham Theorem [2]. Hence, by Proposition 2.7, ji is a chain homotopy equivalence. Now i is the "Whitney map", and a direct check shows that ji satisfies the homotopy commutativity condition of Definition 2.2. Thus $C^*(M, C_r^*(\Gamma))$ and $H^*(M)$ are chain homotopy equivalent Hermitian Fredholm complexes, and by Theorem 4.1 $\mathrm{Sign}(C^*(M, C_r^*(\Gamma))) = \mathrm{Sign}(H^*(M))$. But $\mathrm{Sign}(H^*(M)) = \mathrm{Ind}(D^\Gamma)$ by definition. This completes the proof.

It is convenient to assemble this into a diagram. We represent K-homology tensored with $\mathbf{Z}[1/2]$ of a space X in a manner analogous to that used by P. Baum and R. G. Douglas [1]. This version is based on unpublished work of P. Haskell. The groups are formed from suitable equivalence classes of triples (M, E, f) where M is a compact, oriented manifold, E is a vector bundle over M and $f: M \to X$ is a continuous map. The equivalence relation called "vector bundle modification" is changed so that one uses the $\mathrm{K}\otimes\mathbf{Z}[1/2]$ orientation associated to the signature operator rather than the Dirac operator. If the bundles, E, are real we obtain $\mathrm{KO}_*(X)\otimes\mathbf{Z}[1/2]$, and if E is required to be complex we get $\mathrm{K}_*(X)\otimes\mathbf{Z}[1/2]$. There is a complexification map $c_*: \mathrm{KO}_*(X)\otimes\mathbf{Z}[1/2] \to \mathrm{K}_*(X)\otimes\mathbf{Z}[1/2]$ defined by $c_*([M, E, f]) = [M, E\otimes C, f]$. Note that $\mathrm{KO}_*\otimes\mathbf{Z}[1/2]$ is $\mathbf{Z}/4$ graded and $\mathrm{K}_*\otimes\mathbf{Z}[1/2]$ is $\mathbf{Z}/2$ graded. It follows from results of Conner and Floyd that it is sufficient to consider triples of the form $(M', 1, f)$, where 1 is the trivial line bundle. Define a map $\beta: \mathrm{K}_0(B\Gamma) \to \mathrm{K}_0(C_r^*(\Gamma))$ by $\beta([M, E, f]) = \mathrm{Ind}(D^{f^*(\psi^\Gamma)})$. It can be checked that β respects the equivalence relation and, hence, is a well defined homomorphism.

Let $L_i(\mathbf{Z}[\Gamma])$ be Wall's surgery groups, [15], which are periodic mod 4. By [12] there are isomorphisms $L^{4k+i}(\mathbf{Z}[\Gamma]\otimes\mathbf{Z}[1/2] \approx L_i(\mathbf{Z}[\Gamma])\otimes\mathbf{Z}[1/2]$. Wall's map $l_\Gamma: \mathrm{KO}_i(B\Gamma)\otimes\mathbf{Z}[1/2] \to L_i(\mathbf{Z}[\Gamma])\otimes\mathbf{Z}[1/2]$ may be defined by sending $[M, 1, f]$ to the image of $\bar{\sigma}(C^*(M, \mathbf{Z}[\Gamma]))$ under the above identification. We now have

$$m\otimes\mathbf{Z}[1/2]: \{L_0(\mathbf{Z}[\Gamma])\oplus L_2(\mathbf{Z}[\Gamma])\}\otimes\mathbf{Z}[1/2] \to \mathrm{K}_0(C_r^*(\Gamma))\otimes\mathbf{Z}[1/2].$$

These maps fit together into the following diagram.

THEOREM 5.2. *The following diagram commutes,*

$$\begin{array}{ccc} K_0(B\Gamma)\otimes \mathbf{Z}[1/2] & \xrightarrow{\beta\otimes \mathbf{Z}[1/2]} & K_0(C_r^*(\Gamma))\otimes \mathbf{Z}[1/2] \\ \uparrow c_* & & \uparrow m \\ \{KO_0(B\Gamma)\oplus KO_2(B\Gamma)\}\otimes \mathbf{Z}[1/2] & \xrightarrow{l_\Gamma} & \{L_0(\mathbf{Z}[\Gamma])\oplus L_2(\mathbf{Z}[\Gamma])\}\otimes \mathbf{Z}[1/2]. \end{array}$$

Proof. Let ψ^Γ be associated to $f_* : \pi_1(M) \to \Gamma$. We must show that $(m \otimes \mathbf{Z}[1/2])l_\Gamma([M, 1, f]) = \mathrm{Ind}(D^\Gamma) \otimes \mathbf{Z}[1/2]$. By Theorem 5.1 $\mathrm{Ind}(D^\Gamma) = m\bar{\sigma}(C^*(M, \mathbf{Z}[\Gamma]))$. The result follows from the definition of l_Γ.

COROLLARY 5.3. *Assume Γ is torsion free. If $\beta\otimes\mathbf{Z}[1/2]$ is injective, then the Novikov Conjecture holds for Γ, modulo $\mathbf{Z}[1/2]$.*

Proof. The Novikov Conjecture, modulo $\mathbf{Z}[1/2]$ is equivalent to l_Γ being injective. Since c_* is injective, the result follows.

If Γ is not torsion free (e.g. $\Gamma = \mathbf{Z}_3\oplus\mathbf{Z}_3$) then $\beta\otimes\mathbf{Z}[1/2]$ need not be injective. It is possible, however, that β is always an isomorphism (without any tensoring) if Γ is torsion free.

There is an analogous diagram for the odd case. Note that $l_\Gamma\otimes Q$ being injective is equivalent to the Novikov Conjecture and $\beta\otimes Q$ being injective is the Strong Novikov Conjecture, [13]. The relation between these notions depends on whether $m\otimes Q$ is injective, about which little seems to be known. Our formulations are based on the ideas of Miscenko and Kasparov. Note, however, that the diagram commutes after tensoring with merely $\mathbf{Z}[1/2]$ rather than Q, hence contains odd torsion information.

6. THE NON-UNITAL CASE AND APPLICATIONS TO FOLIATIONS

In this section we merely indicate the adjustments necessary to handle the case when A does not have a unit. One can no longer expect that a Hilbert A-module M will satisfy $M \cong M'$. One defines Hermitian and Fredholm complexes as before, except that $\varphi : \mathscr{C} \to \mathscr{C}'$ is not required to be a chain homotopy equivalence. The notion of regularity must be changed to require that $\varphi : \mathscr{C} \to \mathscr{C}'$ is injective with range the space of functionals representable as $(x, \;)$ where $(\;,\;)$ is the Hilbert A-module inner product. We will call this being *quasi-regular*. This is exactly the condition that $(\mathscr{C}, \varphi)$ become regular when extended to be a complex over A^+, the unitalization of A, by letting $1 \in A^+$ act as the identity. Then, if $\mathscr{C}$ is Fredholm, $D_\mathscr{C}$ is Fredholm over A^+. It follows from Proposition 1.7 that $D_\mathscr{C}$ is also Fredholm over A and $\mathrm{Ind}(D_\mathscr{C}) \in K_0(A^+)$ comes from $\mathrm{Ind}(D_\mathscr{C}) \in K_0(A)$. Thus one may work with A^+ modules throughout and the conclusion of Theorem 4.1 still holds when A does not have a unit.

Now, let M and M' be compact manifolds foliated by $\mathscr{F}$ and $\mathscr{F}'$ with the leaves of both oriented and of even dimension. Let $h : M \to M'$ be a leaf preserving homotopy equivalence. Fix Riemannian metrics on M, M'. Let $\Omega\mathscr{F}^*(M)$ be the complex of differential forms along the leaves. One may complete these to be a complex of Hilbert $C^*(M, \mathscr{F})$-modules with a quasi-regular Hermitian structure provided by the $*$-operation and the Hilbert module structure, [3]. It is a Fredholm complex via the parametrix for the signature operator and thus has a well-defined signature in $K_0(C^*(M, \mathscr{F}))$.

Next, following Baum and Connes, one uses the homotopy equivalence to construct a bimodule which provides a Morita equivalence between $C^*(M, \mathscr{F})$ and $C^*(M', \mathscr{F}')$ which are then stably isomorphic, hence, being stable C^*-algebras, are isomorphic. Using this isomorphism we may consider both complexes to be over $C^*(M, \mathscr{F})$. The given homotopy equivalence yields a chain homotopy equivalence of the complexes and our general theory applies. Thus the signatures of the complexes are equal. This being the same as the index of the respective signature operators we obtain the following theorem of Baum and Connes.

THEOREM 6.1. [0]. *Let M and M' be compact manifolds with foliations $\mathscr{F}$ and $\mathscr{F}'$. Assume that the leaves are oriented and even dimensional. Let $h : M \to M'$ be a leaf preserving homotopy equivalence. Let $D_{\mathscr{F}}$ and $D_{\mathscr{F}'}$ be the signature operators along the leaves. Then* $\mathrm{Ind}(D_{\mathscr{F}}) = \mathrm{Ind}(D_{\mathscr{F}'})$ *in* $K_0(C^*(M,\mathscr{F})) \cong K_0(C^*(M', \mathscr{F}'))$.

The results of [0] also handle the case of odd-dimensional leaves.

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